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REDUCIBILITY OF KAMPÉ DE FÉRIET'S HYPERGEOMETRIC SERIES OF HIGHER ORDER

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The main object of this paper is to obtain a reduction formula of Kampé de Fériet's hypergeometric series. As special cases of our main result, we obtain the correct form of three erroneous reduction formulae of Pathan [10; pp. 1115(3.2), 1116(3.3) and (3.4)].

1. Introduction. From time to time by the concept of separation of a power series into its even and odd terms, many research workers such as Barr [4, p. 591(1)], Carlson [5, p. 234(10)], MacRobert [7, p. 95(8)], Manocha [9, p. 43(3)], Sharma [13, p. 145(2)], [14, pp. 95(1), 99 (line 6)] and Srivastava [16, p. 191(3)], etc. have developed and used different types of hypergeometric series identities. Indeed, the useful nature of these identities had an impressive influence in the development of the transformations and reductions of hypergeometric functions of one and more variables. Using the fact, we apply here a series identity together with Dixon's theorem to obtain a reduction formula of Kampé de Fériet's double hypergeometric function of higher order into a combination of two generalized hypergeometric series ${}_pF_q$'s. By suitable adjustment of parameters, variables and by process of confluence in the main reduction formula, we obtain some known reduction formulae of Bailey [2, p, 245(2.05)], [3, p. 239(4.6)] and Preece [11, pp. 375(8), 378 (11)]. Subsequently some incorrect reduction formulae of Pathan [10, pp. 1115 (3.2), 1116(3.3) and (3.4)] are also corrected here.

2. Main reduction formula. Consider the double hypergeometric series of Kampé de Fériet in the notation of Srivastava and Panda [17, p. 423 (26)] in the form

$$S=F^{p:\,1;\,1}_{q:\,1;\,1}\left[\begin{matrix}(a_p):\ d-2e+1\ ;\ d\ ; \\ (b_q):\ 2-2e\ ;\quad 2e\ ;\end{matrix}\ x,\ -x\right], \tag{2.1}$$

where (a_p) etc. are abbreviated for the array of p parameters $a_1, a_2, \dots, a_p$ and (a_0) denotes the absence of the parameter. (2.1) can be written in power series form

$$S=\sum_{m,\,n=0}^{\infty}\frac{[(a_p)]_{m+n}(d-2e+1)_m(d)_n(-1)^n x^{m+n}}{[(b_q)]_{m+n}(2-2e)_m(2e)_n\ m!\ n!}, \tag{2.2}$$

where $[(a_p)]_s$ etc. denote

$$\prod_{i=1}^{p}(a_i)_s=\prod_{i=1}^{p}\frac{(\Gamma a_i+s)}{\Gamma a_i}.$$

On using Lemma [12, p. 56(1)] and a result [12, p. 32(8)] in (2.2), we get

$$S=\sum_{m=0}^{\infty}\frac{[(a_p)]_m(d-2e+1)_m x^m}{[(b_q)]_m(2-2e)_m\, m!}\,{}_3F_2\left[\begin{matrix}-m,\ d,\ 2e-m-1\,;\\ 2e,\ 2e-d-m\,;\end{matrix}\ 1\right]. \tag{2.3}$$

Now using Dixon's summation theorem [15, p. 52(2.3.3.6)] for well poised generalized hypergeometric polynomial ${}_3F_2(1)$ [12, pp. 73(2), 92(§ 53)] in (2.3), we have

$$S=\sum_{m=0}^{\infty}\frac{[(a_p)]_m(d-2e+1)_m\left(\frac{1}{2}+e-d-\frac{m}{2}\right)_m(2e-m)_m x^m}{[(b_q)]_m(2-2e)_m(2e-d-m)_m\left(\frac{1}{2}+e-\frac{m}{2}\right)_m m!} \tag{2.4}$$

which on using the series identity

$$\sum_{m=0}^{\infty}A(m)=\sum_{m=0}^{\infty}A(2m)+\sum_{m=0}^{\infty}A(2m+1) \tag{2.5}$$

and Lemma 5 in [12, p. 22], reduces in the form of our required reduction formula

$$F^{p:\,1:\,1}_{q:\,1:\,1}\left[\begin{matrix}(a_p):\ d-2e+1\,;\ d\,;\\ (b_q):\ 2-2e\,;\ 2e\,;\end{matrix}\ x,\ -x\right] \tag{2.6}$$

$$={}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p),\ \frac{1}{2}(a_p)+\frac{1}{2},\ \frac{1}{2}+(e-d),\ \frac{1}{2}+(d-e)\,;\\ \frac{1}{2}(b_q),\ \frac{1}{2}(b_q)+\frac{1}{2},\ \frac{1}{2},\ \frac{1+2e}{2},\ \frac{3-2e}{2}\,;\end{matrix}\ 4^{p-q-1}.x^2\right]$$

$$-\frac{x(e-d)(2e-1)\prod_{i=1}^{p}(a_i)}{2e(1-e)\prod_{i=1}^{q}(b_i)}$$

$$\times\,{}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p)+\frac{1}{2},\ \frac{1}{2}(a_p)+1,\ 1+e-d,\ 1-e+d\,;\\ \frac{1}{2}(b_q)+\frac{1}{2},\ \frac{1}{2}(b_q)+1,\ \frac{3}{2},\ 2-e,\ e+1\,;\end{matrix}\ 4^{p-q-1}.x^2\right].$$

3. Special cases. Putting $p=2$, $q=1$, $a_1=2a$, $a_2=2b$ and $b_1=2c$ in (2.6), we have

$$F^{2:\,1;\,1}_{1:\,1;\,1}\left[\begin{matrix}2a,\ 2b:\ \ d-2e+1\,;\ d\,;\\ 2c:\ 2-2e;\ \ 2e\,;\end{matrix}\ x,\ -x\right] \tag{3.1}$$

$$={}_6F_5\left[\begin{matrix}a,\ b,\ \frac{2a+1}{2},\ \frac{2b+1}{2},\ \frac{1}{2}+(e-d),\ \frac{1}{2}+(d-e)\,;\\ c,\ \frac{2c+1}{2},\ \frac{1}{2},\ \frac{1+2e}{2},\ \frac{3-2e}{2}\,;\end{matrix}\ x^2\right]$$

$$-\frac{x(e-d)(2e-1)ab}{e(1-e)c}.$$

$$\times {}_6F_5\left[\begin{matrix}\frac{2a+1}{2}, \frac{2b+1}{2}, a+1, b+1, 1+e-d, 1-e+d; \\ \frac{2c+1}{2}, c+1, \frac{3}{2}, 2-e, e+1;\end{matrix} x^2\right]$$

which is the correct form of Pathan's result [10, p. 1115(3.2)].

When b, a, e and x are replaced by c, $\frac{a}{2}$, $\frac{e}{2}$ and $-x$, respectively, in (3.1) then we get a known reduction formula of Bailey [3, p. 239(4.6)]

$$\text{(3.2)} \quad F_2[a; d, d-e+1; e, 2-e; x, -x]$$

$$= {}_4F_3\left[\begin{matrix}\frac{a}{2}, \frac{a+1}{2}, (\frac{1+e}{2})-d, (\frac{1-e}{2})+d; \\ \frac{1}{2}, \frac{1+e}{2}, \frac{3-e}{2};\end{matrix} x^2\right]$$

$$+\frac{ax(e-1)(e-2d)}{e(2-e)}\times {}_4F_3\left[\begin{matrix}\frac{a+1}{2}, \frac{a+2}{2}, (\frac{2+e}{2})-d, (\frac{2-e}{2})+d; \\ \frac{3}{2}, \frac{4-e}{2}, \frac{e+2}{2};\end{matrix} x^2\right]$$

which is the correct form of Pathan [10, p. 1116(3.3)]. Here F_2 is Appell's function of second kind [1, p. 14(12)].

Putting $d=e$ in (3.1), we get the following correct form of Pathan [10, p. 1116(3.4)]

$$\text{(3.3)} \quad F^{2:\,1;\,1}_{1:\,1;\,1}\left[\begin{matrix}2a, 2b: 1-e; e; \\ 2c: 2-2e; 2e;\end{matrix} x, -x\right]$$

$$= {}_5F_4\left[\begin{matrix}a, b, \frac{2a+1}{2}, \frac{2b+1}{2}, \frac{1}{2}; \\ c, \frac{2c+1}{2}, \frac{1+2e}{2}, \frac{3-2e}{2};\end{matrix} x^2\right].$$

Replacing b, a, e and x by c, $a/2$, $e/2$ and $-x/a$, respectively, in (3.3), taking $|a|\to\infty$, using the confluence principle [6, p. 48 (§ 3.5)] and Kummer's first transformation [12, p. 125(2)], we have a known result of Preece [11, p. 375(8)]

$$\text{(3.4)} \quad {}_1F_1\left[\begin{matrix}\frac{e}{2}; \\ e;\end{matrix} x\right] {}_1F_1\left[\begin{matrix}\frac{2-e}{2}; \\ 2-e;\end{matrix} x\right] = e^x {}_1F_2\left[\begin{matrix}\frac{1}{2}; \\ \frac{1+e}{2}, \frac{3-e}{2};\end{matrix} \frac{x^2}{4}\right],$$

where ${}_1F_1$ is Kummer's confluent hypergeometric function [12, p. 123(1)].

In (2.6) putting $e=1/2$, we have

$$\text{(3.5)} \quad F^{p:\,1;\,1}_{q:\,1;\,1}\left[\begin{matrix}(a_p): d; d; \\ (b_q): 1; 1;\end{matrix} x, -x\right]$$

$$= {}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p),\ \frac{1}{2}(a_p)+\frac{1}{2},\ d,\ 1-d\,; \\ \frac{1}{2},\ 1,\ 1,\ \frac{1}{2}(b_q),\ \frac{1}{2}(b_q)+\frac{1}{2}\,;\end{matrix}\ 4^{p-q-1}\cdot x^2\right].$$

In (3.2) replacing x by x/a, taking $|a|\to\infty$, using confluence principle, we have another known result of Preece [11, p. 378(11), see also 8, p. 395, Ex 110(ii)]

$$(3.6)\qquad {}_1F_1\left[\begin{matrix}d\,; \\ e\,;\end{matrix}\ x\right]\ {}_1F_1\left[\begin{matrix}d-e+1\,; \\ 2-e\,;\end{matrix}\ -x\right]$$

$$= {}_2F_3\left[\begin{matrix}(\frac{1+e}{2})-d,\ (\frac{1-e}{2})+d\,; \\ \frac{1}{2},\ \frac{3-e}{2},\ \frac{e+1}{2}\,;\end{matrix}\ \frac{x^2}{4}\right] - \frac{(1-e)(e-2d)x}{e(2-e)}$$

$$\times {}_2F_3\left[\begin{matrix}\frac{e}{2}+(1-d),\ (1+d)-\frac{e}{2}\,; \\ \frac{3}{2},\ \frac{4-e}{2},\ \frac{2+e}{2}\,;\end{matrix}\ \frac{x^2}{4}\right].$$

The result (3.6) may also be obtained from (2.6) by suitable adjustment of parameters and variables, with $p=q=0$ and using the fact that the empty product is interpreted as unity. Similarly in (3.6), replacing x by x/d and $|d|\to\infty$, we have a known result of Bailey [2, p. 245(2.05)]

$$(3.7)\qquad {}_0F_1\left[\begin{matrix}-\,; \\ e\,;\end{matrix}\ x\right]\ {}_0F_1\left[\begin{matrix}-\,; \\ 2-e\,;\end{matrix}\ -x\right]$$

$$= {}_0F_3\left[\begin{matrix}\overline{\qquad\qquad}\,; \\ \frac{1}{2},\ \frac{3-e}{2},\ \frac{e+1}{2}\,;\end{matrix}\ \frac{-x^2}{4}\right] + \frac{2x(1-e)}{e(2-e)}\times {}_0F_3\left[\begin{matrix}\overline{\qquad\qquad}\,; \\ \frac{3}{2},\ \frac{4-e}{2},\ \frac{2+e}{2}\,;\end{matrix}\ \frac{-x^2}{4}\right],$$

where ${}_0F_1$ is Bessel function [12, p. 74(3)].

When $d=0$ or $d=2e-1$, (2.6) reduces to a result in the form

$$(3.8)\qquad {}_{p+1}F_{q+1}\left[\begin{matrix}(a_p),\ 1-2e\,; \\ (b_q),\ 2-2e\,;\end{matrix}\ x\right]$$

$$= {}_{2p+1}F_{2q+2}\left[\begin{matrix}\frac{1}{2}(a_p),\ \frac{1}{2}(a_p)+\frac{1}{2},\ \frac{1-2e}{2}\,; \\ \frac{1}{2},\ \frac{1}{2}(b_q),\ \frac{1}{2}(b_q)+\frac{1}{2},\ \frac{3-2e}{2}\,;\end{matrix}\ 4^{p-q-1}\cdot x^2\right]$$

$$+\frac{x(1-2e)\prod_{i=1}^{p}(a_i)}{2(1-e)\prod_{i=1}^{q}(b_i)}$$

$$\times {}_{2p+1}F_{2q+2}\left[\begin{matrix}\frac{1}{2}(a_p)+\frac{1}{2},\ \frac{1}{2}(a_p)+1,\ 1-e\,; \\ \frac{1}{2}(b_q)+\frac{1}{2},\ \frac{1}{2}(b_q)+1,\ \frac{3}{2},\ 2-e\,;\end{matrix}\ 4^{p-q-1}\cdot x^2\right].$$

It is noteworthy that results similar to (3.8) were also obtained by Barr [4, p. 591(1)], Carlson [5, 234(10)], Manocha [9, p. 43(3)], Sharma [13, p. 145(2)] and Srivastava [16, p. 191(3)]. When $e=2d$, (3.2) reduces in the form

$$(3.9)\qquad F_2[a\,;\ d,\ 1-d\,;\ 2d,\ 2-2d\,;\ x,\ -x]={}_3F_2\left[\begin{matrix}\frac{a}{2},\ \frac{a+1}{2},\ \frac{1}{2}\,; \\ \frac{1+2d}{2},\ \frac{3-2d}{2}\,;\end{matrix}\ x^2\right],$$

which is the special case of Bailey [3, p. 239(4.4)].

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