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SPLINE INTERPOLATION OF BOUNDED FUNCTIONS IN THE CLASS $L_p[0, 1]$

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We shall prove the following estimates of the error for interpolation by splines with uniformly spaced nodes of order r of bounded 1-periodic functions $f \in L_p[0, 1]$: $\| f-s_{r,h}\|_{L_p[0,1]} \leq C_1(r)\tau_{r+1}(f;\ h)_{L_p[0,1]}$, and for functions $f \in W_p^{\nu}[0, 1]$: $\| f^{(i)}-s_{r,h}^{(i)}\|_{L_p[0,1]} \leq C_2(r, \nu)h^{\nu-1}\varphi_{r+1-\nu}(f^{(\nu)};\ h)_{L_p[0,1]}$ when $r \geq \nu \geq 1$ and $i=0, 1, \dots, \nu$.

The functions to be investigated are bounded, 1-periodic and belong to the class $L_p[0, 1]$, $1 \leq p \leq \infty$. For every function f we define the spline $s_{r,h}(x) = s_{r,h}(f;\ x)$ of order r with uniformly spaced nodes at the points $x_i = ih$ ($i=0, \pm 1, \pm 2, \dots;\ h=1/N$) with the properties:

1) $s_{r,h}(x_i+zh)=f(x_i+zh)$ for every integer i, where $z=0$ if r is odd or $z=1/2$ if r is even;

2) $s_{r,h}^{(r)}$ is bounded.

As far as $\sup_i \{ |\Delta_h^r f(x_i+zh)| \}$ is bounded together with $f(x)$, this spline exists and is unique [1, p. 121].

To estimate the L_p-norm of the error of this spline interpolation we use the moduli:

$$\omega_k(f;\ h)_{L_p} = \sup\{ \| \Delta_\delta^k f(.)\|_{L_p[0,1]};\ 0 \leq \delta \leq h\},$$

$$\omega_k(f,\ x;\ h) = \sup\{ |\Delta_\delta^k f(y)|;\ y,\ y+k\delta \in [x-kh/2,\ x+kh/2]\},$$

$$\omega_k(f;\ h) = \| \omega_k(f\ ,.;\ h)\|_C,$$

$$\tau_k(f;\ h)_{L_p} = \| \omega_k(f,.;\ h)\|_{L_p[0,1]}.$$

The last modulus is introduced by Bl. Sendov and P. P. Korovkin (in the case $k=1$). The main properties of this modulus and other information about it can be found in [2].

A. S. Andreev, V. A. Popov [3] proved the following estimate of the error in the case $r=2,3$: $\| s_{r,h}-f\|_{L_p} \leq C\tau_{r+1}(f;\ h)_{L_p}$.

In this paper we obtain this estimate for arbitrary r.

Theorem 1. *There exists a constant $C_1(r)$ depending only on r such that for every bounded 1-periodic function $f \in L_p[0, 1]$ the inequality $\| s_{r,h}-f\|_{L_p} \leq C_1(r)\tau_{r+1}(f;\ h)_{L_p}$ holds.*

From this theorem using the method proposed by A. Andreev [4] we obtain the following

Theorem 2. *Let* $1 \leq \nu \leq r$ *and* $f \in W_p^\nu[0, 1]$ *be bounded and 1-periodic function. Then for* $i = 0, 1, \ldots, \nu$

$$\| f^{(i)} - s_{r,h}^{(i)} \|_{L_p} \leq C_2(r, \nu) h^{\nu - i} \omega_{r+1-\nu}(f^{(\nu)}; h)_{L_p}$$

with a constant $C_2(r, \nu)$ *depending only on* r *and* ν.

In 1 we prove some lemmas and in 2 and 3 are given the proofs of Theorem 1 and Theorem 2, respectively.

Everywhere $C(.)$, $K(.)$, $L(.)$, $M(.)$, $U(.)$ are constants depending only on the arguments marked in the brackets.

1. Some lemmas. Lemma 1 [5]. *Let* $m_0 < m_1 < \ldots < m_r$; $b_i \geq 0$ $(i = 0, 1, \ldots, r)$. *We denote by* q_i *the polynomials of r-th degree, for which* $q_i(m_j) = (-1)^{i+j+1} b_j$ *for* $0 \leq j < i$ *and* $q_i(m_j) = (-1)^{i+j} b_j$ *for* $i \leq j \leq r$. *If the polynomial* p *of r-th degree is such that the inequalities* $|p(m_j)| \leq b_j$ $(j = 0, 1, \ldots, r)$ *hold, then* $|p(x)| \leq q_i(x)$ *for* $x \in [m_{i-1}, m_i]$, $i = 1, 2, \ldots, r$.

Lemma 2 (Whitney [6, 7]). *Let* f *be a bounded function on the interval* $[\alpha, \beta]$ *and let* p *be that polynomial of r-th degree which interpolates* f *at the points* $\alpha + i(\beta - \alpha)/r$. *Then*

$$|f(x) - p(x)| \leq k(r) \omega_{r+1}\left(f, \frac{\alpha+\beta}{2}; \frac{\beta-\alpha}{r+1}\right) \quad \text{for} \quad x \in [\alpha, \beta].$$

Let $\tilde{q}_i$ be the polynomials from Lemma 1 for $m_i = \alpha + i(\beta - \alpha)/r$ and $b_i = 1$. We set $L(r) = \max\{\max\{\tilde{q}_i(x);\ x \in [m_{i-1}, m_i]\};\ i = 1, 2, \ldots, r\}$. It is easy to see that $L(r) \leq \sqrt{2^r}$. Comparing Lemma 1 and Lemma 2 we get the following assertion.

Lemma 3. *For every bounded on the interval* $[\alpha, \beta]$ *function* f *for which* $|f(\alpha + i(\beta - \alpha)/r)| \leq M$, $(i = 0, 1, \ldots, r)$, *the following inequalitiy is valid*:

$$|f(x)| \leq K(r) \omega_{r+1}\left(f, \frac{\alpha+\beta}{2}; \frac{\beta-\alpha}{r+1}\right) + M \,.\, L(r) \quad \text{for } x \in [\alpha, \beta].$$

Lemma 4 [1, p. 25]. *Let the following infinite system be given*

$$\sum_{s=0}^{2\rho} b_s z_{s+m} = d_m, \quad (m = 0, \pm 1, \pm 2, \ldots), \tag{1}$$

where z_i *are unknown quantities,* $\sup_m |d_m| < \infty$ *and* $b_{2\rho} > 0$. *If all roots of the characteristic polynomial* $P_{2\rho}(z) = \sum_{s=0}^{2\rho} b_s z^s$ *are negative and different,* $p_{2\rho} = z^{2\rho} P_{2\rho}(1/z)$ *and* $P_{2\rho}(-1) \neq 0$, *then* (1) *has an unique solution* $z_m^0 = \sum_{k=-\infty}^{\infty} a_k d_{m-\rho+k}$. *Moreover*

$$\sum_{k=-\infty}^{\infty} |a_k| = |P_{2\rho}(-1)|^{-1}. \tag{2}$$

We use functions $B_r(t) = (r+1) \sum_{i=0}^{r+1} \frac{(i-t)_+^r}{\omega'(i)}$, where $\omega(x) = x(x-1) \ldots (x - r - 1)$ and $x_+^r = x^r$ if $x \geq 0$ or $x_+^r = 0$ if $x < 0$. Let function f be defined on the real axis and $\omega_r(f; h)$ be bounded (it is sufficient for the uniqueness of $s_{r,h}(f; x)$). We note

$$Spl_r(f; x) = \sum_{j=0}^{r} B_r(j+z) f(x+jh) - \sum_{j=-\infty}^{\infty} B_r\left(\frac{x_j - x}{h}\right) f(x_j - zh),$$

where $x_i = ih$ and $0 \leq z < 1$. Let s be a spline of r-th order and with nodes x_i. It is known that there exist constants γ_i such that for $y \notin [x_{k-1}, x_{k+r}]$ the equality

$$s(y) = \sum_{i=k}^{l+r} \gamma_i B_r\left(\frac{x_i - y}{y}\right)$$

holds true. From here and from the fact that $B_r\left(\frac{x_i - y}{h}\right) = 0$ for $y \in (x_i, x_{i+r+1})$ we obtain

$$\sum_{j=0}^{r} B_r(j+z)s(x+jh) = \sum_{j=0}^{r} B_r(j+z) \sum_{l=k+j}^{k+j+r} \gamma_l B_r\left(\frac{x_l - x - jh}{h}\right)$$

$$= \sum_{j=0}^{r} \sum_{i=k}^{k+r} B_r(i+z)\gamma_{i+j} B_r\left(\frac{x_i - x}{h}\right) = \sum_{i=k}^{k+r} B_r\left(\frac{x_i - x}{h}\right) \sum_{j=0}^{r} \gamma_{i+j} B_r\left(\frac{jh+zh}{h}\right)$$

$$= \sum_{i=k}^{k+r} B_r\left(\frac{x_i - x}{h}\right) \sum_{l=i}^{i+r+1} \gamma_l B_r\left(\frac{x_l - x_i + zh}{h}\right) = \sum_{i=-\infty}^{\infty} B_r\left(\frac{x_i - x}{h}\right) s(x_i - zh).$$

From that equation it follows in particular that $Spl(s_{r,h}(f;\, x);\, x) = 0$. Denoting $g(x) = f(x) - s_{r,h}(f;\, x)$ we receive

$$Spl_r(f;x) = Spl_r(g;\, x) = \sum_{j=0}^{r} B_r(j+z)g(x+jh).$$

Let $x = x_i + y$, where $y \in [0,\, h]$, and let $z_i = g(x_i + y)$. From the last equation follows the infinite system

$$\sum_{j=0}^{r} B_r(j+z)z_{i+j} = Spl_r(f;\, x_i + y). \tag{3}$$

In [1, p. 123—126] is proved that the characteristic polynomial with coefficients ($z = 0$ if r is odd; $z = 1/2$ if r is even)

$$b_s = \sum_{j=0}^{r+1} (-1)^{j+r+1} \binom{r+1}{j} (j-s+z)_+^r = \sum_{j=0}^{r+1} \frac{(r+1)!}{\omega'(j)} (j-s+z)_+^r = r!\; B_r(s+z)$$

satisfies the conditions of Lemma 4. On the other hand,

$$\sup_x |Spl_r(f;\, x)| = \sup_x |Spl_r(g;\, x)| \leq \{\sum_{j=0}^{r} |B_r(j+z)|\} K(r-1)\omega_r(g;\, h)$$

$$\leq C(r)\{\omega_r(f;\, h) + h^r \| s_{r,h}^{(r)} \|_C\} < \infty$$

and therefore we can apply Lemma 4 to the system (3), which gives

$$g(x_i + y) = \sum_{k=-\infty}^{\infty} a_k Spl(f;\, x_{i-\rho+k} + y),$$

where $\rho = [r/2]$. Changing $x = x_i + y$ we receive

Lemma 5. *For every function f, for which $\omega_r(f;\, h) < \infty$ the equation*

$$f(x) - s_{r,h}(f;\, x) = \sum_{k=-\infty}^{\infty} a_k Spl_r(f;\, x + (k-\rho)h)$$

holds

2. Proof of Theorem 1. We use $Spl_r(f; x)$ for $x=x_i+lh/r(l=0, 1, \dots, r-1)$. Because of $B_r(x)=0$ for $x\notin(0, r+1)$ we have

$$Spl_r(f; x_i+lh/r)=\sum_{j=0}^{r} B_r(j+z)f(x_i+(l+rj)h/r)-\sum_{j=i+1}^{i+r+1} B_r(j-i-l/r)f(x_j-zh),$$

i. e. $Spl_r(f; x_i+lh/r)$ is a linear combination of $f(x_i+mh/r)$ for $m=0, 1, \dots, r^2+r-1$. On the other hand, it is zero for polynomials of r-th degree and therefore there exist constants $u_{l,m}$ such that

$$Spl_r(f; x_i+lh/r)=\sum_{m=0}^{r^2-2} u_{l,m}\Delta_{h/r}^{r+1} f(x_i+mh/r).$$

Let us note $U_m(r)=\max\{|u_{l,m}|; l=0, 1, \dots, r-1\}$. From Lemma 5 and (2) we obtain

$$|f(x_i+lh/r)-s_{r,h}(x_i+lh/r)|=|\sum_{k=-\infty}^{\infty} a_k Spl_r(f; x_i+lh/r+(k-\rho)h)|$$

$$\leqq \sum_{k=-\infty}^{\infty} |a_k| \sum_{m=0}^{r^2-2} U_m(r)|\Delta_{h/r}^{r+1} f(x_{i+k-\rho}+mh/r)|$$

$$\leqq \sum_{k=-\infty}^{\infty} |a_k| \sum_{m=0}^{r^2-2} U_m(r)\omega_{r+1}(f, x_{i+k-\rho}+\frac{mh}{r}+\frac{h}{2}; \frac{h}{r}).$$

Denote the last expression by M and apply Lemma 3 in the interval $[x_i, x_{i+1}]$ for the function $g(x)=f(x)-s_{r,h}(x)$:

$$|f(x)-s_{r,h}(x)|\leqq K(r)\omega_{r+1}(f, x_i+\frac{h}{2}; \frac{h}{r+1})+$$

$$L(r)\sum_{k=-\infty}^{\infty} |a_k| \sum_{m=0}^{r^2-2} U_m(r)\omega_{r+1}(f; x_i+(k-\rho+\frac{m}{r}+\frac{1}{2})h; \frac{h}{r})\leqq K(r)\omega_{r+1}(f, x; \frac{2h}{r+1})$$

$$+L(r)\sum_{k=-\infty}^{\infty} |a_k| \sum_{m=0}^{r^2-2} U_m(r)\omega_{r+1}(f; x+(k-\rho+\frac{m}{r})h; \frac{h}{r}+\frac{h}{r+1}).$$

Obviously the upper inequation is valid for $x\in[0, 1]$. After receiving the L_p-norm on both sides and applying the triangular inequality we obtain

$$\|f-s_{r,h}\|_{L_p[0,1]}\leqq K(r)\tau_{r+1}(f; \frac{2h}{r+1})_{L_p}+L(r)\sum_{k=-\infty}^{\infty} |a_k| \sum_{m=0}^{r^2-2} U_m(r)\tau_{r+1}(f; \frac{h}{r}+\frac{h}{r+1})_{L_p}$$

$$\leqq \{K(r)+L(r)\sum_{k=-\infty}^{\infty} |a_k| \sum_{m=0}^{r^2-2} U_m(r)\}\,\tau_{r+1}(f; \frac{2h}{r})_{L_p}.$$

3. Proof of Theorem 2. Lemma 6 [4, Lemma 3]. *Let* $0\leqq\nu\leqq r+1$ $f^{(\nu)}\in L_p[0, 1+(r+1)t]$ *and let* P *be a polynomial of order* r. *Then*

$$\|f^{(\nu)}-P^{(\nu)}\|_{L_p[0,1]}\leqq\omega_{r+1}(f^{(\nu)}; t)_{L_p[0,1+(r+1)t]}+C(\nu, r)t^{-\nu}\|f-P\|_{L_p[0,1+t(r+1)]}.$$

Lemma 6′ [4, Lemma 3′]. *Let* $0\leqq\nu\leqq r+1$, $f^{(\nu)}\in L_p[-(r+1)t, 1]$ *and let* P *be a polynomial of order* r. *Then*

$$\|f^{(\nu)}-P^{(\nu)}\|_{L_p[0,1]}\leqq\omega_{r+1}(f^{(\nu)}; t)_{L_p[-(r+1)t,1]}+C(\nu, r)t^{-\nu}\|f-P\|_{L_p[-(r+1)t,1]}.$$

Let $\alpha_0<\alpha_1<\ldots<\alpha_n$ and $\breve{\alpha}=\min\{\alpha_i-\alpha_{i-1};\ i=1,2,\ldots,n\}$. The following assertion is proved analogously to Lemma 4 in [4].

Lemma 7. *If $f\in L_p[\alpha_0,\alpha_n]$ and $0<t\leqq\breve{\alpha}/(2r+2)$, then*

$$\sum_{i=1}^{n}\{\omega_{r+1}(f;t)_{L_p[\alpha_{i-1},\alpha_i]}\}^p\leqq\{C(r)\omega_{r+1}(f;t)_{L_p[\alpha_0,\alpha_n]}\}^p.$$

Lemma 8. *Let $0\leqq\nu\leqq r+1$, $f\in W_p^\nu[\alpha_0,\alpha_n]$, $0\leqq t\leqq\breve{\alpha}/(2r+2)$ and s be a spline of order r with nodes $\alpha_0,\alpha_1,\ldots,\alpha_n$. Then*

$$\|f^{(\nu)}-s^{(\nu)}\|_{L_p[\alpha_0,\alpha_n]}\leqq C(r,\nu)\{\omega_{r+1}(f^{(\nu)};t)_{L_p[\alpha_0,\alpha_n]}+t^{-\nu}\|f-s\|_{L_p[\alpha_0,\alpha_n]}\}.$$

Proof. We apply Lemma 6 and Lemma 6′, respectively, in the intervals $[\alpha_{i-1},(\alpha_{i-1}+\alpha_i)/2]$ and $[(\alpha_{i-1}+\alpha_i)/2,\alpha_i]$ for $t\leqq\breve{\alpha}/(2r+2)$ and obtain

$$\|f^{(\nu)}-s^{(\nu)}\|_{L_p[\alpha_{i-1},\alpha_i]}\leqq\|f^{(\nu)}-s^{(\nu)}\|_{L_p[\alpha_{i-1},(\alpha_{i-1}+\alpha_i)/2]}+\|f^{(\nu)}-s^{(\nu)}\|_{L_p[(\alpha_{i-1}+\alpha_i)/2,\alpha_i]}$$
$$\leqq 2\{\omega_{r+1}(f^{(\nu)};t)_{L_p[\alpha_{i-1},\alpha_i]}+C(\nu,r)t^{-\nu}\|f-s\|_{L_p[\alpha_{i-1},\alpha_i]}\}.$$

This inequality gives the proposition of the lemma in the case $p=\infty$. For $1\leqq p<\infty$ we have

$$\|f^{(\nu)}-s^{(\nu)}\|^p_{L_p[\alpha_{i-1},\alpha_i]}\leqq 2^p\,.\,2^{p-1}\{(\omega_{r+1}(f^{(\nu)};t)_{L_p[\alpha_{i-1},\alpha_i]})^p$$
$$+(C(\nu,r)t^{-\nu})^p\|f-s\|^p_{L_p[\alpha_{i-1},\alpha_i]}\}.$$

Adding the above inequalities for $i=1,2,\ldots,n$ and applying Lemma 7 we obtain

$$\|f^{(\nu)}-s^{(\nu)}\|^p_{L_p[\alpha_0,\alpha_n]}=\sum_{i=1}^{n}\|f^{(\nu)}-s^{(\nu)}\|^p_{L_p[\alpha_{i-1},\alpha_i]}$$
$$\leqq 4^p\{\sum_{i=1}^{n}(\omega_{r+1}(f^{(\nu)};t)_{L_p[\alpha_{i-1},\alpha_i]})^p+C(\nu,r)t^{-\nu})^p\sum_{i=1}^{n}\|f-s\|^p_{L_p[\alpha_{i-1},\alpha_i]}\}$$
$$\leqq 4^p\{(C(r)\omega_{r+1}(f^{(\nu)};t)_{L_p[\alpha_0,\alpha_n]})^p+(C(\nu,r)t^{-\nu})^p\|f-s\|^p_{L_p[\alpha_0,\alpha_n]}\}$$

and therefore

$$\|f^{(\nu)}-s^{(\nu)}\|_{L_p[\alpha_0,\alpha_n]}\leqq 4\{C(r)\omega_{r+1}(f^{(\nu)};t)_{L_p[\alpha_0,\alpha_n]}+C(\nu,r)\,t^{-\nu}\|f-s\|_{L_p[\alpha_0,\alpha_n]}\}.$$

The lemma is proved.

From the properties of the moduli see [2]): $\tau_k(f;\delta)_{L_p}\leqq C(k)\,\delta\omega_{k-1}(f';\delta)_{L_p}$, $\omega_k(f;\delta)_{L_p}\leqq\delta\omega_{k-1}(f;\delta)_{L_p}$ and after applying Theorem 1 and Lemma 8 we obtain the proposition of Theorem 2.

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