Serdica
Bulgariacae mathematicae
publicationes

Сердика
Българско математическо
списание

STRICTLY G-VALUATED FIELDS

A. KONTOLATOU, J. STABAKIS

The archimedian and non-archimedian valuations of the set **Q** of rational numbers lead us directly to get the fields R of real numbers and $\mathbf{Q}_p$ of p-adic numbers, respectively. It's enough to complete **Q** by these valuations. Next, the well known extensions of the notion of valuation referred to the substitution either of **Q** by an arbitrary commutative or non-commutative field or of $\mathbf{R}^+$ by a totally ordered abelian group. The present paper is devoted to define a valuation (called valuative order), ranging over a partially ordered abelian group. The main result is: Given a commutative field $(K, +, .)$ and a partially ordered abelian group $(G, +, \leq)$ there exist an extension K_1 of K and a valuative order v of K_1 ranging onto $G\cup\{\infty\}$.

1. Preliminaries. The generalization of the notion "valuated fields" had been obtained in several ways. The non-commutative valuated fields were considered firstly by O. F. G. Schilling and independently by M. Krasner. Valuations ranging over partially ordered abelian groups have been firstly studied by L. Fuchs and a little before Zielinski had considered the particular case, where the value group was a lattice. The present paper is concerned just with the case where the value group is a partially ordered abelian group and especially in solving the following problem: *Given a field K and a partially ordered abelian group G, is it possible to extend K to a field valuated on G?*

The problem has been solved in the case of totally ordered abelian group (see [7], p. 31). Besides, when the ordered abelian group $(G, +, \leq)$ is torsion free, it is obvious that, extending the partial order to a total one, we come to the above known case. The answer is positive too, in the case where $(G, +, \leq)$ is mixed. Here it will be needed to evaluate the field K on the torsion subgroup G^* of G and extend the group $(G/G^*, +)$ to a totally ordered abelian group. So, in any case, an analogy of proofs is obtained. Through the whole work it seemed useful to give a new definition of a "valuative order" preserving the hypermetric property which, the "valuative order" ranging over a totally ordered group, had. The use of Kurepa's completion allowed us to realize it.

Let $(G, +, \leq)$ be a partially ordered abelian group, $<$ the strict order and by $x//y$ we mean that neither $x\leq y$ nor $y\leq x$. If $A\subset G$ and $x\in G$, write $A<x$ (resp. $x<A$), iff for every $y\in A$ we get $y<x$ (resp. $x<y$). Finally denote by $A\setminus B$ the complementary of B in A. Next we remind the following, referring to an order structure $(G, \leq)$. A couple (A, B), with A, B nonvoid subsets of G is said to be a cut iff it fulfils the statements:

1. $(\forall a\in A)\,(\forall\beta\in B)\;[a<\beta]$;
2. If $A<\beta$, then $\beta\in B$;
3. If $a<B$, then $a\in A$.

A (resp. B) is called lower (resp. upper) class of the cut. It A, B have no ends, the cut is a gap. On the set $G_D=G\cup L(G)$, where $L(G)$ is the set of gaps, is defined an order structure, extension of the initial one (we also denote it by $\leq$), which is a lattice. The structure $(G_D, \leq)$ is the Mac Neille completion.

Now, let L_G^- (resp. L_G^+) be the set of lower (resp. upper) classes of cuts with no ends. The Kurepa's completion (see [1]) is the order structure $(\widetilde{G}, \leq)$, where $\widetilde{G}=G\cup L_G^-\cup L_G^+$ ordered by $\leq$, extension of the initial one in the following way: Symbolize by l^- or a^- a class $A\in L_G^-$, depending on whether the cut (A, B), defined by A, is a gap l, or the upper class B has a minimum element a. Similarly every $B\in L_G^+$ is denoted by l^+ or a^+. Moreover note $a=a^0$, for any $a\in G$. Then every element $\widetilde{a}\in\widetilde{G}$ is of the form e^{ξ}, where $e\in G_D$ and ξ is one of the signs $-$, 0, $+$, called Kind of e. Putting $-<0<1/3$, define an order on G, such that:

$$e_1^{\xi_1}<e_2^{\xi_2}\Leftrightarrow e_1<e_2 \quad \text{or} \quad (e_1=e_2 \quad \text{and} \quad \xi_1<\xi_2).$$

The following is known:

(1) The completions $(G_D, \leq)$ and $(\widetilde{G}, \leq)$ of $(G, \leq)$ are complete lattices.

(2) If x, y are elements of G, and $x//y$, then $\inf_{\widetilde{G}}\{x, y\}\notin C$ (resp. $\sup_{\widetilde{G}}\{x, y\}\notin G$), where $\inf_{\widetilde{G}}A$, $A\subset E$ (resp. $\sup_{\widetilde{G}}A$, $A\subset E$) denotes the $\inf A$ (resp. $\sup A$) into $\widetilde{G}$.

Finally we recall the notion of valuative order. This is a function ν with domain a field $(K, +, .)$, ranging over a set $\hat{G}=G\cup\{\infty\}$, where $(G, +, \leq)$ is a totally ordered abelian group and ∞ is the last element of $\hat{G}$, such that for each $\gamma\in G$, $\gamma+\infty=\infty+\gamma=\infty+\infty=\infty$. The function ν fulfils the next conditions:

(i) $\nu(x)=\infty\Leftrightarrow x=0$,

(ii) $\nu(xy)=\nu(x)+\nu(y)$, for every x, y elements of K,

(iii) $\nu(x)=\nu(-x)$, for each element x of K,

(iv) $\nu(x+y)\geq\min\{\nu(x), \nu(y)\}$, for every x, y elements of K.

2. Strictly G-valuated fields. We begin defining the "valuative order" in the case where its range is a given partially ordered group.

Let $(K, +, .)$ be a given commutative field and $(G, +)$ an abelian group, ordered by $\leq$ (denote $(G, +, \leq)$). We also put $\hat{G}=G\cup\{\infty\}$, where ∞ is the last element of $\hat{G}$ and for each $\gamma\in G$, $\gamma+\infty=\infty+\gamma=\infty+\infty=\infty$.

Definition 1. *A function* $\nu: K\to\hat{G}$ *is called G-valuative order on K, if the following conditions are satisfied*:

(i) $\nu(x)=\infty\Leftrightarrow x=0$,

(ii) $\nu(x.y)=\nu(x)+\nu(y)$, *for every* x, y *elements of* K,

(iii) $\nu(x)=\nu(-x)$, *for every element* x of K,

(iv) $\nu(x+y)\geq\inf_{\widetilde{G}}\{\nu(x), \nu(y)\}$, *for every* x, y *elements* of K, *where G is the Kurepa's completion.*

Remark 1. By (iv) the $\nu(x+y)$ belongs to the upper class B, of the cut (A, B) of G, where A is the set of elements of G, which are smaller than $\nu(x)$ and $\nu(y)$. This last statement implies that if $\nu(x)$ and $\nu(y)$ are different and comparable, then $\nu(x+y)=\min\{\nu(x), \nu(y)\}$.

Indeed, if $\nu(x)>\nu(y)$ (1), then $\nu(x+y)\geq\inf_{\widetilde{G}}\{\nu(x), \nu(y)\}=\nu(y)$ (2).

Next $\nu(y) \geq \inf_{\widetilde{G}}\{\nu(x+y), \nu(x)\}$, that is to say, $\nu(y)$ belongs to the upper class B' of the cut (A', B'), where A' is the set of the elements which are smaller than $\nu(x+y)$ and $\nu(x)$ and because of (1) and (2), it derives that $\nu(y) = \nu(x+y)$. Besides, if $\nu(x) // \nu(y)$, the $\inf_{\widetilde{G}}\{\nu(x), \nu(y)\}$, belongs to $\widetilde{G} \setminus G$ (c. f. Section 1(2)) and this means that $\nu(x+y) > \inf_{\widetilde{G}} \nu(x), \nu(y)\}$.

According to the properties of a valuative order, the next propositions are valid:

P r o p o s i t i o n 1. (1) *If* 1 *is the unit of the field* K, *then* $\nu(1) = 0$. *Therefore, for* $x \neq 0$, $\nu(x) = -\nu(1/x)$;
(2) $\nu(x-y) \geq \inf_{\widetilde{G}}\{\nu(x), \nu(y)\}$;
(3) *If the field* K *is finite and the group* $(G, +, \leq)$ *is torsion free, then the only G-valuative order* $\nu: K \to \widehat{G}$ *which would be defined, is the trivial one.*

The proof is obvious.

P r o p o s i t i o n 2. *The function* $\nu: K \to \widehat{G}$ *is G-valuative order iff it satisfies the* (i), (ii), (iii) *of definition* 1 *and the following condition*:
(iv)′ *If* $\nu(x) > \gamma$ *and* $\nu(y) > \gamma$, *then* $\nu(x+y) > \gamma$, *for each* $\gamma \in G$ *and* x, y *arbitrary elements of* K.

P r o o f. Let ν be a function of K into G satisfying the above (i), (ii), (iii) statements. We denote by (A, B) the cut in G, where A is the set of elements which are strictly smaller than both $\nu(x)$ and $\nu(y)$.

Now, suppose that ν satisfies (iv)′. If for an element γ of G, $\nu(x) > \gamma$ and $\nu(y) > \gamma$, then $\gamma \in A$. But in this case, because of (iv)′, $\nu(x+y) > \gamma$. It means that, for each $\gamma \in A$, $\nu(x+y) > \gamma$ and consequently $\nu(x+y) \geq \inf_{\widetilde{G}}\{\nu(x), \nu(y)\}$.

Inversely if for a $\gamma \in G$, $\nu(x) > \gamma$ and $\nu(y) > \gamma$, then, by (iv), $\nu(x+y) \geq \inf_{\widetilde{G}}\{\nu(x), \nu(y)\}$. It remains to prove that $\nu(x+y) > \gamma$. But if $\nu(x) // \nu(y)$ then $\inf_{\widetilde{G}}\{\nu(x), \nu(y)\} \notin G$, so $\inf_{\widetilde{G}}\{\nu(x), \nu(y)\} > \gamma$.
Besides, if they are comparable, $\inf_{\widetilde{G}}\{\nu(x), \nu(y)\} = \min\{\nu(x), \nu(y)\}$ and $\nu(x+y) > \gamma$.

R e m a r k 2. Analogously to proposition 2, one can prove that for a function ν satisfying the (i), (ii), (iii) of definition 1, the following conditions are equivalent (suppose that x, y are arbitrary elements of K and $\gamma \in G$):

(i) The relations $\nu(x) \geq \gamma$ and $\nu(y) \geq \gamma$ imply the relation $\nu(x+y) \geq \gamma$.

(ii) If $\inf_{G_D}\{\nu(x), \nu(y)\}$ is the $\inf\{\nu(x), \nu(y)\}$ in Mac Neille's completion, then $\nu(x+y) \geq \inf_{G_D}\{\nu(x), \nu(y)\}$.

Proposition 3, referring to G-valuative orders is similar to this one, referring to valuative orders.

P r o p o s i t i o n 3. *For any G-valuative order* $\nu: K \to \widehat{G}$ *there holds*:

(1) *The set* $\nu(K - \{0\})$ *is a subgroup of* $(G, +, \leq)$ (*value group of* ν).

(2) *The set* $A_\nu = \{x \in K \mid \nu(x) \geq 0\}$ *is a ring with identity* (*valuation ring of* ν).

(3) *The set* $M_\nu = \{x \in K \mid \nu(x) > 0\}$ *is a maximal ideal of* A_ν, *while* A_ν / M_ν *is a field* (*residual field*).

T h e o r e m 1. *If* $(G, +, \leq)$ *is an ordered abelian torsion free group and* $(K, +, .)$ *a commutative field, then there is an extension* K_1 *of* K *and a G-valuative order* ν *of* K_1, *with value group* G *and residual field* K.

P r o o f. It is well known by Lorenzen-Simbireva-Everett's theorem (see [2], p. 39): Every ordered abelian group can be extended to a totally ordered one iff it is torsion free.

So, in the present case, we consider the extension of $\leq$ to a total order $\lesssim$ according to the above theorem. We can continue as in the known theorem (see [7]). We describe the basic steps of the proof.

Let $K[x]^{G^+}$ be the set of all the "generalized" polynomials with coefficients from K and exponents from $G^+=\{\gamma\in G: 0\lesssim\gamma\}$. Defining as usual the addition and the multiplication of the polynomials, we get the ring of polynomials $(K[x]^{G^+}, +, .)$. In the process we define a function $\nu: K[x]G^+\to\widehat{G}$ as following:

$$\nu(\theta)=\infty,\quad \nu(S)=\gamma_0,\quad \text{where}\quad S=a_{\gamma_0}x^{\gamma_0}+a_{\gamma_1}x^{\gamma_1}+\ldots+a_{\gamma_m}x^{\gamma_m}$$

$$(a\gamma_0\neq 0,\quad 0\lesssim\gamma_0\lesssim\gamma_1\lesssim\ldots\lesssim\gamma_m,\ \gamma_i\in G^+,\quad i\in\{1, 2, \ldots, m\}).$$

It is not difficult to verify that ν satisfies the properties of a G-valuative order. Besides, the ring $K[x]^{G^+}$ is an integral domain. Hence we can construct the field of fractions by elements of this integral domain (considering in the known way the reduced fractions).

So, $K_1=\{s\,.\,t^{-1}\mid s\in K[x]^{G^+},\ t\in K[x]^{G^+}\{0\}\}$.

Finally we extend the valuation ν, to a function ν_1 of K_1 onto $\widehat{G}$, as following:

$$\nu_1(S/t)=\nu(S)-\nu(t),\quad \text{where}\quad S/t\in K_1.$$

Now, it is easy to verify that $\nu_1: K_1\to\widehat{G}$ is a G-valuative order with value group G and residual field $A_{\nu_1}/M_{\nu_1}\cong K$.

3. The factor group G/G^* as a totally ordered group. Let $(G, +, \leq)$ be a partially ordered abelian group. We will construct a factor group of G and we'll order it by an order, induced in a natural way by $\leq$. Denote by G^* the (maximal) torsion subgroup of G and symbolize by $\overline{a}$ the class $(\mathrm{mod}\, G^*)$ of any $a\in G$.

Proposition 4. *All the elements of G^* are parallel to the neutral element of G.*

Proof. Indeed; let $a\in G^*-\{0\}$, 0 the neutral element of G. Then $[\exists n\in N)$ $[na=0]$. If $a>0$ (resp. $a<0$) then $na>0$ (resp. $na<0$), which is absurd. Therefore $a//0$.

Proposition 5. *Every class of G/G^* contains parallel elements.*

Proof. Suppose that β is an element of a given $\overline{a}$. Then $\beta=\alpha+u$, $u\in G^*$, $\beta-\alpha=u$ and $u//0$. Thus $\beta-\alpha//0$, hence $\beta//\alpha$.

Proposition 6. If α, α' *are elements of two different classes with* $\alpha'<\alpha$, *then for each* $\beta\in\overline{\alpha}$ *and* $\beta'\in\overline{\alpha'}$, *we have* $\beta'<\beta$ *or* $\beta'//\beta$.

Proof. Suppose that $\alpha+u=\beta$, where $nu=0$ and $\alpha'+u'=\beta'$, where $mu=0$. If σ is the (L. C. M.) of m, n we have $\sigma\beta-\sigma\beta'=\sigma\alpha+\sigma(u-u')-\sigma\alpha'$, thus $\sigma(\beta-\beta')=\sigma(\alpha-\alpha')>0$, $\sigma\beta>\sigma\beta'$. Hence $\beta>\beta'$ or $\beta//\beta'$.

Proposition 7. *If* $u\in G^*$ *and* $\alpha\in G\setminus G^*$ then $\alpha+u\in\overline{\alpha}$.

Proof. Let n be the order of $u(\mathrm{ord}\, u=n)$ and $\alpha+u=\beta$. If ord $(\alpha+u)$ $=m\in N$ and p is the L. C. M. of m, n then $\beta-\alpha=u$ and $p(\beta-u)=p\alpha$. Thus $p\beta-pu=0=p\alpha$ and $\alpha\in G^*$, is false. Therefore ord $(\alpha+u)=\infty$.

Definition 2. *Define on the factor group G/G^* a relation R as following:* $\overline{\alpha}R\overline{\beta}\Leftrightarrow$"$\overline{\alpha}=\overline{\beta}$" *or "there exist* $\alpha_0\in\overline{\alpha}$ *and* $\beta_0\in\overline{\beta}$ *such that* $\alpha_0<\beta_0$". *If* *the classes are noncomparable according to R, then we call them parallel* (symp. $\bar{\alpha}//\bar{\beta}$).

Proposition 8. *The R of definition 2 is an order-relation, compatible with the structure of the group* $(G/G^*, +)$.

Proof. (i) $(\forall\bar{\alpha})\,[\bar{\alpha}R\bar{\alpha}]$, obviously. (ii) If $\bar{\alpha}R\bar{\beta}$ and $\bar{\beta}R\bar{\alpha}$, we distinguish two cases. The first, if $\bar{\alpha}=\bar{\beta}$, the second if there exist $\alpha_0\in\bar{\alpha}$, $\beta_0\in\bar{\beta}$, $\alpha_1\in\bar{\alpha}$ and $\beta_1\in\bar{\beta}$, such that $\alpha_0<\beta_0$ and at the same time $\beta_1<\alpha_1$. The last case is false, for it contradicts Prop. 6. Thus $\bar{\alpha}=\bar{\beta}$. (iii) We prove the transitive postulate. Suppose that $\bar{\alpha}R\bar{\beta}$, $\bar{\alpha}R\bar{\gamma}$, where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are different classes one to another. (If two or three of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, coincide, proof is evident).

Then there exist $\alpha_0\in\bar{\alpha}$, β_0 and β_1 elements of $\bar{\beta}$ and $\gamma_1\in\bar{\gamma}$ such that $\alpha_0<\beta_0$ and $\beta_1<\gamma_1$. If $\beta_1=\beta_0+u$, where $u\in G^*$, then the element α_0+u belongs to $\bar{\alpha}$. Thus $\alpha_0+u<\beta_0+u$ (prop. 6), $\alpha_0+u<\beta_1<\gamma_1$. Hence $\bar{\alpha}R\bar{\gamma}$.

(iv) R is compatible with $(G/G^*, +)$. Suppose that $\bar{\alpha}R\bar{\beta}$ and $\bar{\gamma}$ are an arbitrary class. There are $\alpha_0\in\bar{\alpha}$, $\beta_0\in\bar{\beta}$, where $\alpha_0<\beta_0$. We have $\alpha_0+\gamma<\beta_0+\gamma$, thus $\overline{\alpha_0+\gamma}R\overline{\beta_0+\gamma}$. Hence $\bar{\alpha}+\bar{\gamma}R\bar{\beta}+\bar{\gamma}$.

From Proposition 8 and Lorenzen-Simbirena-Everett's theorem (G/G^* is torsion free) we conclude that

Proposition 9. The structure $(G/G^*, +, R)$ extends to the structure of total order $(G/G^*, +, \precsim)$ (where R is the relation of def. 2).

The main result of this paragraph is the next theorem:

Theorem 2. *Every partially ordered group is decomposed into a family of classes, such that, if S_1 and S_2 are two classes of the decomposition either there is no element of S_1 larger than any element of S_2 or there is no element of S_1 smaller than any element of S_2.*

Proof. We consider the totally ordered group $(G/G^*, +, \precsim)$ of Proposition 9 and define on G relation $\leq_1$ as following:

$$\alpha\leq_1\beta\Leftrightarrow\bar{\alpha}\precsim\bar{\beta},$$

where $\bar{\alpha}$, $\bar{\beta}$ belong to G/G^*. It is easy to prove that $\leq_1$ is an order, extension of the initial $\leq$ and compatible with the operation of the group. Obviously the construction or the relation R of definition 2 for the structure $(G, +, \leq_1)$ gives us the s structure $J=(G/G^*, +, \precsim)$.

The required family of subsets is the set J. Indeed if $\bar{\alpha}$, $\bar{\beta}$ are two different classes, such that $\bar{\alpha}\precsim\bar{\beta}$, the $\bar{\alpha}R\bar{\beta}$ or $\bar{\alpha}//\bar{\beta}$ (R is the relation of def. 2). Thus $(\forall x\in\bar{\alpha})(\forall y\in\bar{\beta})\,[x\leq y$ or $x//y]$ and the proof is over.

Remark 3. Evidently, if we consider the structure $(G, +, \leq_1)$ and S_1, S_2 two different classes of the above decomposition with $S_1\precsim S_2$ then $x_1\leq_1 x_2$ for each element $x_1\in S_1$ and $x_2\in S_2$.

Remark 4. If x, y are elements of C (mod G^*) and x', y' element of C' (mod G^*) then the classes of the elements $x+x'$ and $y+y'$ coincide.

4. Extension of a field to a strictly G-valuated field. We finish giving what we can characterize as the main result of this paper.

Theorem 3. *For every commutative field $(K, +, .)$ and for every partially ordered abelian group $(G, +, \leq)$, there exists an extension $\widetilde{K}$ of K and a G-valuative order ν of $\widetilde{K}$ onto $(G, +, \leq)$.*

Proof. The construction of $\widetilde{K}$ and the definition of ν will be established to two stages. The first (the torsion step of the demonstration) is comprized of the construction of a field K_1, which is an extension of K and the definition of a G-valuative order ν_1 with domain the K_1, and value group the torsion group G^*. In the second part of the proof (the torsion free step of demonstration) we extend the field K_1 to the field $\widetilde{K}$ and we extend the ν_1 to the G-valuative order ν, which has as a domain the $\widetilde{K}$ and as a value group, the group G.

I. The torsion step of the proof. Let $(G^*, +, \leq)$ be the torsion subgroup of $(G, +, \leq)$. First of all we consider G^* as an homomorphic image of a free abelian group $(F, .)$ by a function

$$g: F \to G^*. \tag{1}$$

(It is well-known that every group is an homomorphic image of a free group (see [8], p. 10). Let X be a free system of generators of the free group F (that is we consider the set X as the alphabet of the free group F). We order the set X by a total order $\prec$ (independently if X is of finite or transfinite cardinality). Suppose that x_i, $i \in I$, where I is an arbitrary index set, are the elements of X and consider the set $\mathscr{M}_1$ of all monomials of the type

$$M = x_{i_1}^{\alpha_1} . x_{i_2}^{\alpha_2} \ldots x_{i_n}^{\alpha_n}, \tag{2}$$

where the number of $x_{i_\varkappa}$ is finite, $\alpha_\varkappa$ are integers, $\varkappa \in \{1, 2, \ldots, n\}$ and $x_{i_1} \prec x_{i_2} \prec \ldots \prec x_{i_n}$.

We consider these monomials in a "reduced form", that is, if an exponent α_i is zero, the factor $x_{i_\varkappa}^{\alpha_\varkappa}$ is missed, except in the case where all the exponents $\alpha_\varkappa$ are zero, in which case we write 1 in the place of the monomial.

By the known way we define the multiplication of two monomials and also the inverse of a given monomial, where, as the reduced type of a product $M_1 M_2$ we mean the form, which is produced when one does all the commutations and the monomial gets the form (2). (By this way we have written all the elements of $\mathscr{M}_1$ in a fixed form and so the structures $(\mathscr{M}_1, .)$ and $(F, .)$ coincide.)

Next, we introduce in the set $\mathscr{M}_1$ an order $\prec$, extension of the already defined order $\prec$ (that's the reason of the same notation) in the following manner:

put $M = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_n}^{\alpha_n} \succ 1$, if $\alpha_1 > 0$ and $M = x_{i_1}^{\alpha_1} \ldots x_{i_n}^{\alpha_n} \prec 1$, if $\alpha_1 < 0$. We also define that if M_1, M_2 belong to $\mathscr{M}_1$ then

$$M_2 \prec M_1, \quad \text{iff} \quad M_1 . M_2^{-1} \succ 1. \tag{3}$$

It is evident that the relation $\prec$ is compatible with the structure $(F, .)$. So we can consider F as a totally ordered free group $(F, ., \prec)$.

Now, we construct the ring $\mathscr{A}$ of the generalized polynomials of the form

$$A_i = u_1 M_{i_1} + u_2 M_{i_2} + \ldots + u_n M_{i_n}, \tag{4}$$

where $u_i \in K$ and the monomials M_{i_j}, $j \in \{1, \ldots, n\}$ have been put according to increasing order, that is $M_{i_1} \prec M_{i_2} \prec \ldots \prec M_{i_n}$. On the set $\mathscr{A} \times \mathscr{A}^*$, where

$\mathscr{A}^* = \mathscr{A} - \{0\}$ and 0 the neutral element of addition, we define the known equivalence $(A, B) \sim (A', B') \Leftrightarrow AB' = A'B$, that gives us the field of fractions $K_1 = \mathscr{A} \times \mathscr{A} / \sim$.

Next, describe the definition of the above mentioned G-valuative order ν_1 onto $G^* \cup \{\infty\}$ (the symbol ∞ has the well-known properties).

We begin regarding that if $A = \dfrac{u_1 M_{i_1} + \ldots + u_n M_{i_n}}{U_1 M_{j_1} + \ldots + U_m M_{j_m}}$, $B = \dfrac{S_1 M_{K_1} + \ldots + S_\nu M_{K\nu}}{t_1 M_{\lambda_1} + \ldots + t_p M_{\lambda_p}}$

and $A \sim B$, then $M_{i_1} \cdot M_{j_1}^{-1} = M_{K_1} \cdot M_{\lambda_1}^{-1}$. So, if F is the above mentioned free group and $\bar{A}$ the image of the canonical projection of the above A into $\mathscr{A} \times \mathscr{A}^* / \sim = K_1$, then the map

$$\partial : K_1^* \to F \text{ with value } \partial(\bar{A}) = M_{i_1} M_{j_1}^{-1} \text{ is well defined.}$$

Consequently, if g is the homomorphism (1) we define the function $\nu_1^* : K_1^* \to G^*$, with $\nu_1^*(\bar{A}) = g o h(\bar{A})$. The construction of ν_1 finishes putting $\nu_1(0) = \infty$. To simplify we'll use the symbolism $\nu_1(A)$ for $A \in K_1$ instead of the right $\nu_1(\bar{A})$.

It is a simple consequence that ν_1 is an epimorphism of K_1 onto G^*. Besides, we have that $\nu_1(0) = \infty$, $\nu_1(A \cdot B) = \nu_1(A) + V_1(B)$, $\nu_1(A) = \nu_1(-A)$.

Finally we'll prove the hypermetric property, at first for polynomials and next for fractions of polynomials. Let $S = u_1 M_{i_1} + \ldots + u_n M_{i_n}$, $t = U_1 M_{j_1} + \ldots + U_m M_{j_m}$.

We have the cases:

(i) $M_{i_1} \neq M_{j_1}$ and $\nu_1(S) = \nu_1(t)$. Then $\nu_1(S+t) = g[\min\{M_{i_1}, M_{j_1}\}] = \nu_1(S) = \nu_1(t)$.

(ii) $M_{i_1} \neq M_{j_1}$ and $V_1(S) \neq \nu_1(t)$, hence $\nu_1(S) // \nu_1(t)$, $\nu_1(S+t) \in G^*$ and therefore is larger than $\inf_{\tilde{G}^*}\{G^*\}$ that is the minimum element of the complement.

(iii) $M_{i_1} = M_{j_1}$ and $\nu_1(S) = \nu_1(t)$. Then $S + t = (u_1 + U_1) M_{i_1} + \ldots$, hence

$$\nu_1(S+t) = M_{i_1} \geq \inf_{\tilde{G}^*}\{\nu_1(S), \nu_1(t)\} = \nu_1(S).$$

Let now S, t be as above and $S' = u_1' M_{i_1}' + \ldots + u_\varkappa' M_{i_\varkappa}'$, $t' = U_1' M_{j_1}' + \ldots + U_\lambda' M_{j_\lambda}'$.

We'll prove that $\nu_1\left(\frac{S}{t} + \frac{S'}{t'}\right) \geq \inf_{\tilde{G}^*}\left\{\nu_1\left(\frac{S}{t}\right), \nu_1\left(\frac{S'}{t'}\right)\right\}$; $\nu_1\left(\frac{St' + S't}{tt'}\right) = \nu_1(St' + S't)$

$-\nu_1(tt') \geq \inf_{\tilde{G}^*}\{\nu_1(St'), \nu_1(S't)\} - \nu_1(tt') = \inf_{\tilde{G}^*}\{\nu_1(u_1 M_{i_1} U_1' M_{j_1}' + \ldots), \nu_1(u_1' M_{i_1}' U_1 M_{j_1}$

$+ \ldots)\} - \nu_1(U_1 M_{j_1} U_1' M_{j_1}' + \ldots) = \inf_{\tilde{G}^*}\{g(M_{i_1} M_{j_1}'), g(M_{i_1}' M_{j_1})\} - g(M_{j_1} M_{j_1}')$

$$= \inf_{\tilde{G}^*}\{g(M_{i_1}) + g(M_{j_1}'), g(M_{i_1}') + g(M_{j_1})\} + g(M_{j_1} - g(M_{j_1}').$$

But $\inf_{\tilde{G}^*}\{\alpha, \beta\} - \gamma = \inf_{\tilde{G}^*}\{\alpha - \gamma, \beta - \gamma\}$. Indeed, if $\inf_{\tilde{G}^*}\{\alpha, \beta\} = \delta$ then $\delta \leq \alpha$, $\delta \leq \beta$. Therefore $\delta - \gamma \leq \inf_{\tilde{G}^*}\{\alpha - \gamma, \beta - \gamma\}$. If $\delta_1 = \inf_{\tilde{G}^*}\{\alpha - \gamma, \beta - \gamma\}$, then $\delta_1 \leq \alpha - \gamma$, $\delta_1 \leq \beta - \gamma$, hence $\delta_1 - \gamma \leq \alpha$, $\delta_1 - \gamma \leq \beta$, so $\delta_1 - \gamma \leq \inf_{\tilde{G}^*}\{\alpha, \beta\} = \delta$.

Thus

$$\inf_{\tilde{G}^*}\{g(M_{i_1}) + g(M_{j_1}'), g(M_{i_1}') + g(M_{j_1})\} - g(M_{j_1}) - g(M_{j_1}')$$

$$= \inf_{\tilde{G}^*}\{g(M_{i_1}) + g(M_{j_1}') - g(M_{j_1}) - g(M_{j_1}'), \quad g(M_{i_1}') + g(M_{j_1}) - g(M_{j_1}) - g(M_{j_1}')\}$$

$$=\inf_{\widetilde{G^*}}\{g(M_{i_1}M_{j_1}^{-1}),\quad g(M'_{i_1}M_{j_1}^{-1})\}=\inf_{\widetilde{G^*}}\{\nu_1(\frac{S}{t}),\quad \nu_1(\frac{S'}{t'})\}.$$

II. The torsion free step of the proof. We consider again the maximal torsion subgroup G^* of G, the extension K_1 of the field K as well as the valuative order ν_1, of K_1 of the previous step. We also consider the structure $(G/G^*, +, <1)$ of Proposition 8, where the factor group G/G^* is totally ordered by 1, induced by $<$ and being an extension of this $<$.

Finally, let $\Gamma=(\gamma_i)_{i\in I}$ be a system of representatives of $(G/G^*, +, <1)$, where the representative of G^* is always 0 and $\Gamma^+=\{\gamma_i\in\Gamma : 0<_1\gamma_i\}$. As we have described in Theorem 3, if $\gamma_i<\gamma_j$, then $\gamma_i<_1\gamma_j$, too. Construct now the ring $K_1[x]^{\Gamma^+}$ of the reduced polynomials $\alpha_{\gamma_1}x^{\gamma_1}+\alpha_{\gamma_2}x^{\gamma_2}+\ldots+\alpha_{\gamma_n}x^{\gamma_n}$, where the indexes (and exponents) have been given in increasing order $0<_1\gamma_1<_1\gamma_2<_1\ldots<_1\gamma_n$) and the components α_{ν_i} are elements of K_1. To each element $\varkappa$ of Γ we correspond the class $C_\varkappa(\text{mod } G^*)$.

On the ring $K_1[x]^{\Gamma^+}$ we define the multiplication of monomials: $x^{i_1}.x^{j_1}=x^\varkappa$, where $\varkappa\in\Gamma$ and $x^{i_1-j_1}\in C_\varkappa$. Analogously, the multiplication of polynomials is defined in the known way. Likewise $x^{i_1}.x^{j_1-1}=x^\varkappa$, where $\varkappa\in\Gamma$ and $x^{i_1+j_1}\in C_\varkappa$.

Remark that in the multiplication

$$(\alpha_{\gamma_1}x^{\gamma_1}+\alpha_{\gamma_2}x^{\gamma_2}+\ \ldots\ \alpha_{\gamma_n}x^{\gamma_n}).(\alpha_{\beta_1}x^{\beta_1}+\alpha_{\beta_2}x^{\beta_2}+\ \ldots+\alpha_{\beta_m}x^{\beta_m}),$$

putting $x^{\gamma_1}x^{\beta_1}=x^{\varkappa_1^1}$ and $x^{\gamma_i}x^{\beta_j}=x, \varkappa_i^j$ will be $\varkappa_1^1\leq\varkappa_i^j$.

Next we construct the field of fractions $\widetilde{L}$ of the polymomials $K_1[x]^{\Gamma^+}$ and put $\widehat{G}=G\cup\{\infty\}$. We can suppose that this ∞ and that one, we had put in the torsion step of the proof, coincide.

Just as in the first part of the proof we consider the known equivalence $\sim$ and after that, putting $\widetilde{K}=\widetilde{L}/\sim$ define the function ν of $\widetilde{K}$ as following: $\nu(0)=\infty$, $\nu(\sum_{i=1}^{n}\alpha_{\gamma_i}x^{\gamma_i})=\nu_1(\alpha_{\gamma_1})+\gamma_1$, where ν_1 is the valuative order of K_1 onto $G^*\cup\{\infty\}$ (as exposed in the previous part; hence $\nu_1(\alpha_{\gamma_1})\in G^*$).

$$\nu\left(\frac{\sum_{i=1}^{n}\alpha_{\gamma_i}x^{\gamma_i}}{\sum_{i=1}^{m}\alpha_{\beta_i}x^{\beta_i}}\right)=\nu\left(\sum_{i=1}^{n}\alpha_{\gamma_i}x^{\gamma_i}\right)-\nu\left(\sum_{i=1}^{m}\alpha_{\beta_i}x^{\beta_i}\right).$$

Firstly we remark that if $A=\dfrac{\sum_{i=1}^{n}\alpha_{\gamma_i}x^{\gamma_i}}{\sum_{i=1}^{m}\alpha_{\beta_i}x^{\beta_i}}$, $B=\dfrac{\sum_{i=1}^{r}\alpha_{\delta_i}x^{\delta_i}}{\sum_{i=1}^{s}\alpha_{\varepsilon_i}x^{\varepsilon_i}}$ are equivalent (mod $\sim$)

then $\alpha_{\gamma_1}.\alpha_{\varepsilon_1}=\alpha_{\beta_1}.\alpha_{\delta_1}$ and $\gamma_1+\varepsilon_1=\beta_1+\delta_1$; then $\nu_1(\alpha_{\gamma_1})+\nu_1(\alpha_{\varepsilon_1})+\gamma_1+\varepsilon_1=\nu_1(\alpha_{\beta_1})+\nu_1(\alpha_{\delta_1})+\beta_1+\delta_1$, whence $\nu_1(\alpha_{\gamma_1})+\gamma_1-\nu_1(\alpha_{\beta_1})-\beta_1=\nu_1(\alpha_{\delta_1})+\delta_1-\nu_1(\alpha_{\varepsilon_1})-\varepsilon_1$. Therefore $\nu(A)=\nu(B)$, hence if $\bar{A}$ is the canonical projection of A onto $\widetilde{K}$, we can put $\nu(\bar{A})=\nu(A)$.

ν is a valuative order of $\widetilde{K}$.
(We use the above symbolisms of A and B and by $\bar{A}, \bar{B}$ we mean the corresponding classes (mod $\sim$) of A and B.) Indeed

(i) $\nu(0)=\infty$ by definition,

(ii) $\nu(A \,.\, B)=\nu(A)+\nu(B)$ because

$$\nu(A \,.\, B)=\nu_1(\alpha_{\gamma_1})+\nu_1(\alpha_{\delta_1})+\gamma_1+\delta_1-\nu_1(\alpha_{\beta_1})-\nu_1(\alpha_{\varepsilon_1})-\beta_1-\varepsilon_1$$

and $\nu(A)+\nu(B)=[\nu_1(\alpha_{\gamma_1})+\gamma_1-\nu_1(\alpha_{\beta_1})-\beta_1]+[\nu(\alpha_{\delta_1})+\delta_1-\nu_1(\alpha_{\varepsilon_1})-\varepsilon_1]$. Finally, we'll prove that if A and B belong to $\widetilde{L}$, then the relation $\nu(A+B)\geq \inf_{\widetilde{G}}\{\nu(A), \nu(B)\}$ holds.

We'll proceed as in the previous step, working firstly with polynomials and next with fractions.

Let $S, t \in K_1[x]^{\Gamma^+}$

$$S=\alpha_{\gamma_1}x^{\gamma_1}+\ldots+\alpha_{\gamma_n}x^{\gamma_n}, \quad t=K_{\beta_1}x^{\beta_1}+\ldots+K_{\beta_m}x^{\beta_m}.$$

Distinguish the cases:

(i) $\nu(S)=\nu(t)$ or equivalently $\gamma_1=\beta_1$ and $\nu_1(\alpha_{\gamma_1})=\nu_1(K_{\beta_1})$ because

$$\nu_1(\alpha_{\gamma_1})+\gamma_1'=\nu_1(K_{\beta_1})+\beta_1',$$

$$S+t=(\alpha_{\gamma_1}x^{\gamma_1}+\ldots+\alpha_{\gamma_n}x^{\gamma_n})+(K_{\beta_1}x^{\beta_1}+\ldots K_{\beta_m}x^{\beta_m})=(\alpha_{\gamma_1}+K_{\beta_1})x^{\gamma_1}+\ldots$$

$$\nu(S+t)=\nu_1(\alpha_{\gamma_1}+K_{\beta_1})+\gamma_1\geq \inf_{\widetilde{G}^*}\{\nu_1(\alpha_{\gamma_1}), \quad \nu_1(K_{\beta_1})\}+\gamma_1=\inf_{\widetilde{G}}\{\nu(S), \nu(t)\}=\nu(S).$$

(ii) $\nu(S)\neq\nu(t)$ and $\gamma_1\neq\beta_1$; thus they belong to different classes, whence they are comparable by <1, let be $\nu(S)<_1\nu(t)$. So

$$\nu(S+t)=\nu[\alpha_{\gamma_1}x^{\gamma_1}+\ldots+K_{\beta_1}x^{\beta_1}+\ldots]=\nu_1(\alpha_{\gamma_1})+\gamma_1=\min\{\nu(S), \nu(t)\}.$$

(iii) $\nu(S)\neq\nu(t)$ and $\gamma_1=\beta_1$; then $\nu_1(\alpha_{\gamma_1})\neq\nu_1(K_{\beta_1})$, $\nu(S)//\nu(t)$ and $\nu(S+t)$ $=\nu[(\alpha_{\gamma_1}+K_{\beta_1})x^{\gamma_1}+\ldots]=\nu_1(\alpha_{\gamma_1}+K_{\beta_1})+\gamma_1$.
Besides, $\nu(S)$, $\nu(t)$ and $\nu(S+t)$ belong to the same class (mod G^*) not coinciding and therefore $\nu(S+t)>\inf_{\widetilde{G}}\{\nu(S), \nu(t)\}$.

Now, let S and t be as above and $S'=\alpha'_{\gamma_1}x^{\gamma_1'}+\ldots+a'_{\gamma_K}x^{\gamma_K'}$, $t'=K_{\beta_1'}x^{\beta_1'}+\ldots$ $+K_{\beta_\lambda'}x^{\beta_\lambda'}$. We'll prove that $\nu(\frac{S}{t}+\frac{S'}{t'})\geq\inf_{\widetilde{G}}\{\nu(\frac{S}{t}), \nu(\frac{S'}{t'})\}$:

$$\nu(\frac{S}{t}+\frac{S'}{t'})=\nu(\frac{St'+S't)}{tt'})=\nu(St'+S't)-\nu(tt')\geq\inf_{\widetilde{G}}\{\nu(St'), \nu(S't)\}-\nu(tt')$$

$$=\inf_{\widetilde{G}}\{\nu(\alpha_{\gamma_1}x^{\gamma_1}+\ldots)(K'_{\beta_1}x^{\beta_1'}+\,.\,)], \nu[(\alpha'_{\gamma_1}x^{\gamma_1'}+\ldots).(K_{\beta_1}x^{\beta_1}+\ldots)]\}-\nu(t)-\nu(t')$$

$$=\inf_{\widetilde{G}}\{\nu_1(\alpha_{\gamma_1}.K'_{\beta_1})+p, \quad \nu_1(\alpha'_{\gamma_1}K_{\beta_1})+\mu\}-\nu_1(K_{\beta_1})-\beta_1-\nu_1(K'_{\beta_1})-\beta_1',$$

where $\gamma_1+\beta_1'\in C_p$, $\gamma_1'+\beta_1\in C_\mu$ and p, μ belong to Γ.

According to remark 4 (p. 10), the classes (mod G^*) of $p-\beta_1'$, γ_1 coincide; likewise the classes of the elements $\mu-\beta_1$ and γ_1' coincide, too.

Also as we proved in (iii) of Theorem (p. 13, 15) it holds

$$\inf_{\widetilde{G}}\{\nu_1(\alpha_{\gamma_1}K_{\beta_1'})+p,\ \nu_1(\alpha_{\gamma_1}'K_{\beta_1})+\mu\}-\nu_1(K_{\beta_1})-\beta_1-\nu_1(K_{\beta_1'})-\beta_1'$$

$$=\inf_{\widetilde{G}}\{\nu_1(\alpha_{\gamma_1})+\nu_1(K_{\beta_1'})+p-\nu_1(K_{\beta_1})-\beta_1-\nu_1(K_{\beta_1'})-\beta_1',$$

$$\nu_1(\alpha_{\gamma_1}')+\nu_1(K_{\beta_1})+\mu-\nu_1(K_{\beta_1})-\beta_1-\nu_1(K_{\beta_1'})-\beta_1'\}$$

$$=\inf_{\widetilde{G}}\{\nu_1(\alpha_{\gamma_1})+\gamma_1-\nu_1(K_{\beta_1})-\beta_1,\ \ \nu_1(\alpha_{\gamma_1}')+\gamma_1-\nu_1(K_{\beta_1}')-\beta_1'\}$$

$$=\inf_{\widetilde{G}}\{\nu(\frac{S}{t}),\quad \nu(\frac{S'}{t'})\}.$$

The proof is over with the remark that ν is on $\widehat{G}$.

Indeed, if we get the polynomials αx^{γ}, where α runs through K_1, then $\nu(\alpha)\in G^*$ and covers the whole group G^*. Hence $\nu(\alpha)+\gamma$, for all α will give us the corresponding class C_γ. We proceed in the same way for all γ into the system Γ of representatives and cover the group G.

REFERENCES

1. L. Docas. Complétés de Dedekind et de Kurepa des ensembles partiellement ordonnés. *C. R. Ac. Sc. Paris*, **256**, 1963, 2504—6.
2. L. Fuchs. Partially ordered algebraic systems. Oxford, 1963.
3. M. Gennoun. Thèse. Paris, 1978.
4. M. Krasner. Cours d'Algèbre Supérieure. Introduction à la théorie des valuations. Université de Clermont Fd., 1960.
5. W. Krull. Allgemeine Bewertungs theorie. *J. reine u. angew Math.*, **167**, 1932, 160—196.
6. D. Kupera. Ensembles ordonnés et ramifiés. Thèse, Paris, 1935.
7. P. Ribenboim. Théories des valuations. Montréal, 1968.
8. Scott. Group Theory. New Jersey, 1964.
9. O. Schilling. The theory of valuations. Math. Surveys 1950, A. M. S. N. York.
10. N. Varouchakis. Groupes ultramétroides valués. *Symposium Karatheodory*. Athens, 1973, 635—640.
11. N. Varouchakis. Anneaux ultramétroides valués. *Math. Balk.*, 1975, 279—293.

Department of Mathematics
University of Patras
Patras, Greece

Received 8. 8. 1983