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CHARACTERISTIC CLASSES OF PARABOLIC FOLIATIONS AND SYMMETRIC FUNCTIONS

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This paper deals with homogeneous foliations, that is foliations which are defined by an explicit construction and are very convenient for the purposes of the theory of characteristic classes of foliations. The construction of homogeneous foliations is as follows. Let G be a Lie group, H its closed subgroup and π its discrete subgroup. The left cosets of H constitute a foliation on G which is invariant under the right canonical action of G on itself. There arises a foliation on G/π and this is the homogeneous foliation. We denote it by $\mathcal{F}(G, H, \pi)$. Its codimension is equal to $\dim G/H$.

The normal bundle of $\mathcal{F}(G, H, \pi)$ is trivial and is trivialized canonically if an isomorphism $\mathfrak{g}/\mathfrak{h} \simeq \mathbf{R}^N$ is fixed (here $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, and $N = \dim G/H$). Recall that the characteristic classes of codimension N foliations with trivialized normal bundles correspond to the (continuous) cohomology classes of the Lie algebra W_N of formal vector fields in $\mathbf{R}^N$; they take values in the de Rham cohomology of the total manifold of the foliation (see [1]). More algebraically, there exists a homomorphism $\alpha: H^*(W_N; \mathbf{R}) \to H^*(F\Gamma_N; \mathbf{R})$, where $F\Gamma_N$ is the homotopy fibre of the natural map from the Haefliger classifying space $B\Gamma_N$ into $BO(N)$.

A homogeneous foliation $\mathcal{F}(G, H, \pi)$ is called parabolic if G is semisimple, H is parabolic and G/π is compact. A parabolic foliation is called Grassman foliation if $G = SL(n+k, \mathbf{R})$ and H is the isotropy group of a point under the natural G-action on the Grassman manifold $G_{n,k}$ of n-planes in $\mathbf{R}^{n+k}$ (in other words H is a maximal parabolic subgroup of codimension nk).

The computation of characteristic classes of homogeneous foliations was initiated by C. Godbillon and J. Vey in their well-known paper [2], and was continued by a number of authors [3—8]. This paper is within this framework and it is a detailed exposition of the results appearing in [7, 8].

The paper consists of two parts. In the first part (Sections 1—3) we deal with parabolic foliations $\mathcal{F}(G, H, \pi)$ with non-maximal parabolic H. Our main result here is

Theorem A. *All characteristic classes (of positive degree) of parabolic foliations $\mathcal{F}(G, H, \pi)$ with non-maximal parabolic H are trivial.*

Note that this theorem contradicts the example given at the end of paper [6]. But this example is independent of the main body of [6] and we think that it could be refuted by a direct computation.

In the second part (Sections 4—7) we deal with Grassman foliations. The simplest kind of these, that is Grassman foliations with $k=1$ was investigated in [3; 4] (one could call this foliations projective because $G_{N,1} \simeq \mathbf{P}^N$). By means of projective foliations in particular it was proved that $\dim \operatorname{Im} \alpha \geq 2^{N-1}$. And

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although D. B. Fuchs [9] later proved that α is monomorphic it remains interesting to compute the dimensions of the subspace in $H^*(F\Gamma_N)$ spanned by the characteristic classes of homogeneous foliations and of the subspace in $H^*(W_N)$ spanned by the elements defining trivial characteristic classes of homogeneous foliations. D. B. Fuchs conjectured in [3] that the latter space coincides with the kernel of the characteristic homomorphism of the projective foliation but D. Baker [5] proved that this conjecture is false. We generalize the results of D. Baker in the following way.

Let N have p different decompositions into the product of two factors: $N=n_ik_i$, $i=1,\ldots,p$; $n_i>k_i$ and let $d_i=n_i-k_i$. Let A be the subspace in $H^*(W_N)$ spanned by the elements of the Vey basis $h_I c_J$ with $|J|=N$; let $B\subset A$ be the space of cohomology classes of W_N with corresponding characteristic classes being trivial for all Grassman foliations.

Theorem B. $\dim(A/B)=\Sigma 2^{d_i}$.

Corollary. *The dimension of the subspace in* $H^*(F\Gamma_N)$, *spanned by the characteristic classes of Grassman foliations, is* $\geq\Sigma 2^{d_i}$.

Remark. D. Baker [5] considered the case when N is even and $p=2$ with the decompositions $N=1\,.\,N=2\,.\,N/2$.

To state our next result fix a multi-index $I=(i_1,\ldots,i_r)$ and a number d and let $C(I,d)$ be the space spanned by the classes $h_I c_1^{N-d}\mathfrak{p}(c_1,\ldots,c_N)$, where $\mathfrak{p}$ is any polynomial of weight d.

Theorem C. *For every* I, $\dim C(I,d)/C(I,d)\cap B\leq[d/2]+1$.

Remark that $\dim C(I,d)=p(d)$ that is the number of partitions of d. Thus among characteristic classes corresponding to the elements of this space there are many which are trivial for all Grassman foliations.

The proofs of the results of the second part mostly consist in the computation of characteristic numbers of complex Grassman manifolds. Our main tool for this computation is the combinatorical technique connected with symmetric functions [10]. As a by-product of it we obtain the following result.

Theorem D. *The bordism classes of complex Grassman manifolds are lineary independent over* $\mathbf{C}$.

We also obtain some new identities for symmetric functions (see Theorems E and F in Appendix).

The author expresses his deep gratitude to D. B. Fuchs for stating the problem and being a great help in the process of the work and to A. V. Zelevinsky who introduced the author to the theory of symmetric functions.

1. Topological reformulations. Here we show that the characteristic homomorphism of homogeneous foliations coincides with a cohomology homomorphism induced by a certain principal bundle map. The similar technique was used by a number of authors (cf. [3—6]).

The characteristic homomorphism of $\mathcal{F}(G,H,\pi)$ splits into a composition of two homomorphisms $H^*(W_N)\to H^*(\mathfrak{g})\to H^*(G/\pi)$. The latter does not depend on H and it is monomorphic if G is semisimple and G/π is compact. That is why we shall be concerned with the first homomorphism $H^*(W_N)\to H^*(\mathfrak{g})$ which is induced by a homomorphism $\varphi:\mathfrak{g}\to W_N$. The construction of φ is as follows. G acts canonically on G/H thus giving a homomorphism $G\to\mathrm{Diff}(G/H)$. Its infinitesimal form in a fixed formal coordinate system is φ (all this can be found in [1; 3]).

Now we replace φ with its complexification $\varphi^c : \mathfrak{g}^c \to W_N^c$ (cohomology are also taken with complex coefficients). This does not change the essence of the problem because $H^*(\mathfrak{g}^c; \mathbf{C}) \simeq H^*(\mathfrak{g}; \mathbf{R}) \otimes \mathbf{C}$ for every Lie algebra $\mathfrak{g}$.

The homomorphism φ^c corresponds (in the sense described above) to the complex foliation $\mathcal{F}(G^c, H^c)$, where G^c and H^c are the complex Lie groups of the algebras $\mathfrak{g}^c$ and $\mathfrak{h}^c$. In fact this foliation is a bundle and it is induced by the 0-dimensional foliation on G^c/H^c.

Denote the inverse image of the $2N$-base skeleton in the classifying principal $U(N)$-bundle by X_N. Recall that $H^*(W_N) \simeq H^*(X_N)$ [1].

To compute the characteristic classes of the 0-dimensional foliation on G^c/H^c we must consider the classifying bundle map from the principal tangent $GL(N, \mathbf{C})$-bundle $T(G^c/H^c) \to G^c/H^c$ into $U(N) \to X_N \to sk_{2N}BU(N)$. Then the corresponding cohomology homomorphism $H^*(W_N) \to H^*(T(G^c/H^c))$ defines the characteristic classes [11, Theorem 10. 9].

Note that the adjoint H^c-action on $\mathfrak{g}^c/\mathfrak{h}^c$ defines a representation $H^c \to GL(N, \mathbf{C})$. The following statement is easily verified.

Proposition 1.1. $T(G^c/H^c) \simeq G^c \times_{H^c} GL(N, \mathbf{C})$.

We conclude that there exists a bundle map from $G^c \to G^c/H^c$ into $T(G^c/H^c) \to G^c/H^c$. Thus we obtain the through bundle map $\mathfrak{p}$ from $G^c \to G^c/H^c$ into $X_N \to sk_{2N}BU(N)$. Let G_0 be a maximal compact subgroup of G^c with the real Lie algebra $\mathfrak{g}_0$. Using $H^*(G^c) \simeq H^*(G_0) \simeq H^*(\mathfrak{g}_0)$ and $\mathfrak{g}_0^c \simeq \mathfrak{g}^c$ we conclude from the above arguments that $\mathfrak{p}^* : H^*(X_N) \to H^*(G^c)$ is a topological analogue of $\varphi^* : H^*(W_N) \to H^*(\mathfrak{g})$.

Return to the homomorphism $\varphi : \mathfrak{g} \to W_N$ which corresponds to a parabolic foliation.

Proposition 1.2. *φ is monomorphic.*

Proof. It is easily seen that the kernel of $\mathfrak{g} \to \mathfrak{a}(G/H)$ is the intersection of all subalgebras conjugated with $\mathfrak{h}$. This intersection coincides with the intersection of all Kartan subalgebras, hence it is trivial. Our statement now follows from the fact that G-action on G/H is analytical. Q. E. D.

Recall that W_N has the subalgebra L_0 of vector fields with the trivial 0-jets. It is obvious that $\varphi(\mathfrak{h}) \subset L_0$. Recall also that $\mathfrak{gl}(N) \subset L_0$ as the subalgebra of linear vector fields. Let $\mathfrak{f} = \varphi^{-1}(\mathfrak{gl}(N))$ and $\mathfrak{f}_0$ a compact form of $\mathfrak{f}$ with a compact Lie group F_0.

Proposition 1.3. *F_0 is a maximal compact subgroup of H^c.*

Proof. Consider the homomorphism from H^c into the infinite dimensional group $\mathcal{A}$ of formal origin preserving analytical isomorphisms of $\mathbf{C}^N$ which corresponds to $\varphi^c : \mathfrak{h}^c \to L_0^c$. Our statement follows from Proposition 1.2 and the fact that $U(N)$ is a maximal compact subgroup of $\mathcal{A}$ [II, 10. 2]. Q. E. D.

Since there exists a natural bundle map from $G_0 \to G_0/F_0$ into $G^c \to G^c/H^c$ which induces a spectral sequences isomorphism we obtain the through bundle map g from $G_0 \to G_0/F_0$ into $X_N \to sk_{2N}BU(N)$. The above arguments show that we can identify g with the bundle map corresponding to $\varphi : (\mathfrak{g}, \mathfrak{f}) \to (W_N, \mathfrak{gl}(N))$. Thus we shall view g also as a topological analogue of φ.

Example. Let $\mathcal{F}(G, H, \pi)$ be a Grassman foliation. Then $G_0 = SU(n+k)$, $F_0 = S(U(n) \times U(K))$ and $G_c/H^c \simeq G_0/F_0 \simeq \mathbf{C}G_{n,k}$. The subalgebra $\varphi(\mathfrak{g})$ is generated by the vector fields ∂_{ij}, $\mathfrak{p}_{ij}$, g_{ij} and e_{ij}, where $e_{ij} = \Sigma_{\alpha,\beta} x_{i\beta} x_{\alpha j} \partial_{\alpha\beta}$, $\mathfrak{p}_{ij} = \Sigma_\alpha x_{i\alpha} \partial_{j\alpha}$, $g_{ij} = \Sigma_\beta x_{\beta i} \partial_{\beta j}$ with the only relation $\Sigma_i \mathfrak{p}_{ii} = \Sigma_j g_{jj}$ (here x_{ij} are coordinates in $\mathbf{R}^{nk}$, $1 \le i \le n$, $1 \le j \le k$; $\partial_{ij} = \partial/\partial x_{ij}$).

Let us summarise the results of this section. We replace the homomorphism $\varphi: \mathfrak{g} \to W_N$ with the principal bundle map from $G_0 \to G_0/F_0$ into $X_N \to sk_{2N}BU(N)$. The corresponding base map is classifying for the complex manifold G^c/H^c and the corresponding fibre homomorphism $F_0 \to U(N)$ is assosiated with $\varphi: (\mathfrak{g}, f) \to (W_N, \mathfrak{gl}(N))$.

2. Three technical results. The following result is well-known.

Proposition 2.1. *Let $\alpha: G_1 \to G_2$ be a compact Lie group homomorphism. The image of every primitive generator of $H^*(G_2)$ under α^* is a linear combination of primitive generators of $H^*(G_1)$.*

Proposition 2.2. *The ring "surviving" in the left column of the spectral sequence of $G_0 \to G_0/F_0$ is a quotient ring of $H^*(G_0)$ and it is generated (as a ring) by some linear combinations of the primitive generators of $H^*(F_0)$.*

Proof. The ring we are concerned with is the image of $H^*(G_0)$ in $H^*(F_0)$ under the inclusion $i: F_0 \to G_0$. Hence the statement follows from Proposition 2.1. Q. E. D.

Proposition 2.3. *Let G be a complex reductive Lie group and H its parabolic subgroup. Then* $\dim H^1(H) > \dim H^1(G)$.

Proof. We shall use the following formula: $H^*(G/H) = H^*(BH)/(i^*H^*(BG))$, where i is the inclusion $H \subset G$ (see [12]). Since G/H is a projective algebraic manifold $H^2(G/H) \neq 0$. So $i^*H^2(BG) \neq H^2(BH)$.

Consider the spectral sequences of the universal G and H bundles. Note that the transgressions in these spectral sequences are bijective. Since $H^2(BG)$ and $H^2(BH)$ are the $E_2^{2,0}$-terms, and $H^1(G)$ and $H^1(H)$ are the $E_2^{0,1}$-terms, $i^*H^1(G) \neq H^1(H)$.

Now consider the spectral sequence E of $H \to G \to G/H$. We have $\dim H^1(G) = \dim E_\infty^{0,1} + \dim E_\infty^{1,0}$. Since $H^1(BH) = 0$, $E_\infty^{1,0} = 0$ and so $\dim H^1(G) = \dim E_\infty^{0,1}$. But $E_\infty^{0,1} = i^*H^1(G)$, so $\dim H^1(G) < \dim H^1(H)$. Q. E. D.

Apply the above proposition to the nested parabolic subgroups $H_2 \subset H_1 \subset G$.

Corollary 2.4. $\dim H^1(H_2) - \dim H^1(G) \geqq 2$.

3. Proof of Theorem A. We return to the situation appearing in Sec. 1. Recall that g is the bundle map from $G_0 \to G_0/F_0$ into $X_N \to sk_{2N}BU(N)$ corresponding to the parabolic foliation $\mathcal{F}(G, H, \pi)$ with non-maximal H.

Proposition 3.1. *The restriction of g^* on the right column of the E_∞-term of the spectral sequence of $X_N \to sk_{2N}BU(N)$ is trivial.*

Proof. Let $'E$ be the spectral sequence of $G_0 \to G_0/F_0$. Assume that $g^*(\alpha) \neq 0$ for some α from the right column of E_∞ (of the spectral sequence of $X_N \to sk_{2N}BU(N)$). Since $'E_\infty$ satisfies Poincaré duality there exists such an element $\beta \in {'E_\infty^{0,*}}$ that $\beta g^*(\alpha)$ is a non-zero multiple of the top dimensional class γ of $'E_\infty$. Note that γ is a multiple of all generators of $H^1(F_0) . \in {'E_2^{0,1}}$.

By Proposition 2.1 $g^*(\alpha)$ contains not more than one linear combination of the generators of $H^1(F_0)$. By Proposition 2.2 among the generators of $'E_\infty^{0,*}$ there are not more than $\dim H^1(G_0)$ linear combinations of the generators of $H^1(F_0)$. And finally by Corollary 2.4 $\dim H^1(G_0) = \dim H^1(G^c) < \dim H^1(H^c) - 1 = \dim H^1(F_0) - 1$. Thus $\beta g^*(\alpha)$ can not be a multiple of all generators of $H^1(F_0)$. Q. E. D.

Now we can prove Theorem A. It follows from Pittie's theorem [6] that every characteristic class of a parabolic foliation is represented by some element of the right column of the E_∞-term of the spectral sequence of

$X_N \to sk_{2N}BU(N)$. But by Proposition 3.1 every such class is trivial. Thus we have proved our first main result.

Theorem A. *All characteristic classes (of a positive degree) of parabolic foliations $\mathcal{F}(G, H, \pi)$ with non-maximal parabolic H are trivial.*

4. Further topological reformulations. Now we begin to study Grassman foliations. A complex Grassman manifold of n-planes in $\mathbf{C}^{n+k}$ will be hereafter denoted simply by $G_{n,k}$. According to the philosophy of Sec. 1 we are led to consider the bundle map g from $SU(n+k) \to G_{n,k}$ into $X_N \to sk_{2N}BU(N)$. We replace the former bundle with $U(n) \times U(k) \to U(n+k) \to G_{n,k}$; this corresponds to working with the foliation on $SL(n+k) \times \mathbf{R}/\pi$ whose leaves are of the form $L \times \mathbf{R}$, where L is a leaf in $SL(n+k)/\pi$ [5]. Then the map g is induced by the representation of $U(n) \times U(k)$ on $\mathbf{C}^{nk}$ which is the tensor product of the canonical and the conjugated representations.

Recall that the generators of $H^*(U(N))$ are denoted by $h_1, \dots, h_N$; let $c_1, \dots, c_N$ be the corresponding generators of $H^*(sk_{2N}BU(N))$. Also let $e_1, \dots, e_n$ and $d_1, \dots, d_k$ $(n \geq k)$ be the generators of $H^*(U(n))$ and $H^*(U(k))$ and let $x_1, \dots, x_n, y_1, \dots, y_k$ be the corresponding generators of $H^*(G_{n,k}) = \mathbf{C}[x_i, y_j]/(\Sigma x_\alpha y_\beta)$, $\alpha+\beta = 1, \dots, n+k$.

Proposition 4.1. *Let* $\alpha \in H^{2N}(sk_{2N}BU(N))$. 1) $g^*(h_I \alpha) \neq 0$ *if and only if* $g^*(\alpha) \neq 0$, $k \neq n$ *and* $I = (1, 2, \dots, k, j_1, \dots, j_p)$ *with* $k < j_1 < \dots < j_p \leq n$.

2) *Let* $I_1, \dots, I_q$ *be multiindexes of the type described in* 1) *and* $g^*(\alpha) \neq 0$. *Then* $g^*(a_1 h_{I_1} \alpha + \dots + a_q h_{I_q} \alpha) = 0$ if and only if $a_1 = \dots = a_q = 0$.

Proof. First of all we must note that the "if" part of the first statement was proved by D. Baker [5]. Our proof proceeds along the lines similar to those of Baker; in the "only if" part our proof relies on Proposition 2.1.

We start from the following relation: $g^*(c_i) = kx_i + (-1)^i ny_i +$ products (see [5]). Since by Proposition 2.1 $g^*(h_i)$ is product-free $g^*(h_i) = ke_i + (-1)^i nd_i$. Let E be the spectral sequence of $U(n+k) \to G_{n,k}$. As it was pointed out in the proof of Proposition 2.2 the left column of E_∞ is the image of $H^*(U(n+k))$ in $H^*(U(n) \times U(k))$ under the inclusion $j: U(n) \times U(k) \to U(n+k)$. The universal bundle map assosiated with j corresponds to direct summation of bundles. Let z_i be the ring generators of $H^*(BU(n+k))$. Then by Whitney sum formula $j^*(z_i) = \Sigma x_\alpha y_\beta$, $\alpha + \beta = i$. Thus the image of the i-th primitive generator of $H^*(U(n+k))$ under j is $e_i + d_i$ (once more there are no products by Proposition 2.1). Thus the ring $E_\infty^{0,*}$ is genereted by $e_1 + d_1, \dots, e_k + d_k, e_{k+1}, \dots, e_n$.

If $g^*(h_I \alpha) \neq 0$ then there exists such an element $\beta \in E_\infty^{0,*}$ that $\beta g^*(h_I \alpha)$ is a non-zero multiple of the top dimensional class of E_∞. Since $g^*(h_I \alpha) = (ke_{i_1} \pm nd_{i_1}) \dots (ke_{i_r} \pm nd_{i_r}) g^*(\alpha)$ this can take place only if $n > k$ and I is of the form required.

To prove the second statement of the proposition let $\eta = a_1 h_{I_1} \alpha + \dots + a_q h_{I_q} \alpha$ and $J_i = I_i - (1, 2, \dots, k)$. Note that $g^*(h_{J_i})$ is a product of several e_j with $k < j \leq n$. If $\eta \neq 0$ there exists such a class $\beta \in \Lambda^*(e_{k+1}, \dots, e_n) \subset E_\infty^{0,*}$ that $\beta g^*(a_1 h_{J_1} + \dots + a_q h_{J_q}) = e_{k+1} \dots e_n$. Set $\gamma = \beta(e_1 + d_1) \dots (e_k + d_k)$. Then $\gamma g^*(\eta)$ is a non-zero multiple of the top dimensional class of E_∞. Q. E. D.

The point of the above proposition is that instead of computing the characteristic classes represented by the right column of E_∞-term of the spectral sequence of $X_N \to sk_{2N}BU(N)$ we can compute the corresponding characteristic numbers of Grassman manifolds.

5. Characteristic classes of Grassman manifolds. From now on we shall use the descriptton of $H^*(G_{n,k})$ as a quotient ring of the ring Λ of symmetric functions. Recall some basic facts about Λ (the reference for this is [10]).

Given a partition $\lambda=(l_1, \ldots, l_r)$ let S_λ be the corresponding Schur function. These functions constitute a basis of Λ. Let p_i be the power sum of degree i and for any partition $\mu=(m_1, \ldots, m_r)$ write $p_\mu=p_{m_1}\ldots p_{m_r}$. The functions p_μ also constitute a basis. Let h_i be the complete symmetric function of degree i and e_i the i-th elementary symmetric function. Define h_μ and e_μ by analogy with p_μ; h_μ and e_μ constitute two other bases in Λ.

Define an involution t in Λ by $t(S_\lambda)=S_{\lambda'}$, where λ' is the partition conjugated to λ. Then $t(p_i)=(-1)^{i-1}p_i$. Define an inner product in Λ considering S_λ to be an orthonormal basis. The basis p_μ is orthogonal and t is an isometry.

We identify Λ with the ring $H^*(G_{\infty,\infty})$; S_λ corresponds to Schubert cocycles under this isomorphism. $H^*(G_{n,k})$ becomes a quotient ring of Λ which is spanned (as a space) by S_λ with $\lambda=(l_1, \ldots, l_k)$, $l_i\leq n$. Fix the notation π for the partition $(n, \ldots, n)$ of nk. The function S_π corresponds to the orientation cocycle of $G_{n,k}$ and the projection $\Lambda\to H^{2N}(G_{n,k})$ is described by the formula $x\longmapsto <x, S_\pi>S_\pi$.

Let A be the ring $\Lambda[n, k]$ and define an involution τ: $n\longmapsto -n$, $k\longmapsto -k$, $x\longmapsto t(x)$ for $x\in\Lambda$.

P r o p o s i t i o n 5.1. *There exist such elements $ch_i\in A$ that*

1) *ch_i corresponds to the i-th component of Chern character of the tangent bundle $TG_{n,k}$.*

2) $ch_i=-(n+k)p_i/i!$ *for odd i.*

3) $\tau(ch_i)=(-1)^i ch_i$.

4) *The degree of ch_i in the variables n and k is not greater than 1.*

Proof. Let γ_n and γ_k be the canonical bundles of dimensions n and k over $G_{n,k}$. Then $TG_{n,k}=\gamma_n^*\otimes\gamma_k$ and $\mathrm{ch}\,\gamma_n=n+\Sigma p_i/i!$. Since $\gamma_n+\gamma_k$ is trivial $ch\,\gamma_k=k-\Sigma p_i/i!$. Hence $ch\,TG_{n,k}=(n+\Sigma(-1)^i p_i/i!)(k-\Sigma p_i/i!)$. From this our statement follows. Q. E. D.

6. Proofs of Theorems B and C. Assume that the dimension N permits q decompositions $N=n_i k_i$, $i=1, \ldots, q$ with $n_i\geq k_i$ and $n_1>n_2>\cdots$. Write $\pi_i=(n_i, \ldots, n_i)$ (k_i times).

P r o p o s i t i o n 6.1. *There exist such partitions $\mu_1, \ldots, \mu_q$ of N with odd parts that*

1) $\langle p_{\mu_i}, S_{\pi_i}\rangle\neq 0$,

2) $\langle p_{\mu_i}, S_{\pi_j}\rangle=0$ *for $i<j$ and $N>4$.*

The proof will be given in the next section.

Let g_i be the classifying map from G_{n_i,k_i}, into $sk_{2N}BU(N)$ and set $C=\cap_1^q\mathrm{Ker}\,g_i^*\cap H^{2N}(sk_{2N}BU(N))$. One can easily derive the following statement from Propositions 5.1 and 6.1.

P r o p o s i t i o n 6.2. $\dim H^{2N}(sk_{2N}BU(N))/C=q$.

Recall that A is the subspace in $H^*(W_N)$ spanned by $h_I\alpha$ with $\alpha\in H^{2N}(sk_{2N}BU(N))$, $B\subset A$ is the space of the classes which are trivial as characteristic classes of all Grassman foliations and $d_i=n_i-k_i$ for $n_i>k_i$. Now we can prove our second main result.

T h e o r e m B. $\dim A/B=\Sigma 2^{d_i}$.

Proof. Let F_i be the set of multi-indexes $I=(1, 2, \ldots, k_i, \alpha, \ldots, \beta)$ with $\beta \leq n_i$ and set $G_i=F_i-F_{i+1}$. Then $|F_i|=2^{d_i}$, $|G_i|=2^{d_i}-2^{d_{i+1}}$. Let $A_i \subset A$ be the space spanned by the classes $h_I \alpha$ with $I \in G_i$ and set $B_i=B \cap A_i$. Then $A=\bigoplus A_i$ and $B=\bigoplus B_i$. By Propositions 5.1 and 6.2 $\dim A_i/B_i=i\,|G_i|$. So $\dim A/B=\Sigma i\,(2^{d_i}-2^{d_{i+1}})=\Sigma 2^{d_i}$. Q. E. D.

Let us now proceed to the proof of Theorem C. Given a partition $\mu=(m_1, \ldots, m_r)$ write $|\mu|=\Sigma m_i$, $l(\mu)=r$ and $r(\mu)=|\mu|-l(\mu)$. Define the function in the variables n and k by $\mathfrak{p}_\mu=\langle p_\mu p_1^{N-|\mu|}, S_\pi\rangle/\langle p_1^N, S_\pi\rangle$.

Proposition 6.3. *$\mathfrak{p}_\mu$ is a polynomial in the variable $n-k$ of degree $\leq r(\mu)$.*

The proof will also be given in the next section.

Recall that at the end of the previous section the classes ch_i were defined. Given a partition $\mu=(m_1, \ldots, m_r)$ write $ch_\mu=ch_{m_1}\ldots ch_{m_r}$. Define the function g_μ by $g_\mu=\langle ch_\mu p_1^{N-|\mu|}, S_\pi\rangle/\langle p_1^N, S_\pi\rangle$. It is obvious that g_μ is symmetric in n and k. Set $S=n+k$. It follows that g_μ is a polynomial in S with coefficients depending on N. By Proposition 6.3 and 5.1 (4) the degree of g_μ is not greater than $|\mu|$ and by Proposition 5.1 (3) g_μ is either odd or even. So the polynomial $\langle ch_\mu ch_1^{N-|\mu|}, S_\pi\rangle/\langle p_1^N, S_\pi\rangle=S^{N-|\mu|}g_\mu$ belongs to the space spanned by the monomials $S^N, S^{N-2}, \ldots, S^{N-2[|\mu|/2]}$. Its dimension is equal to $[|\mu|/2]+1$.

Recall that $C(I, d)$ is the subspace in $H^*(W_N)$ spanned by the classes $h_I C_1^{N-d}\mathfrak{p}(c_1, \ldots, c_N)$, where $\mathfrak{p}$ is a polynomial of weight d. We obtain the following result.

Theorem C. $\dim C(I, d)/C(I, d) \cap B \leq [d/2]+1$.

7. Computations in the ring of symmetric functions. Recall the algorithm for computation of inner products $\langle p_\mu, S_\lambda\rangle$ — the so-called Murnagan-Nakayama rule ([10]). A skew-hook of a Young diagram is a connected part of its boundary with its complement being also a Young diagram. Let $\mu=(m_1, \ldots, m_r)$. Then $\langle p_\mu, S_\lambda\rangle=\Sigma_\nu(-1)^{i-1}\langle p_{\mu_1}, s_\nu\rangle$, where $\mu_1=(m_2, \ldots, m_r)$, $\lambda-\nu$ is a skew-hook consisting of m_1 points and i is its number of rows.

Example. $\langle p_{31}, S_{22}\rangle=-\langle p_1, s_1\rangle=-1$; $\langle p_{31}, S_{31}\rangle=0$.

Now we shall prove Proposition 6.1. Recall that $N=n_ik_i$, $i=1, \ldots, q$; $n_i \geq k_i$; $n_1>\cdots>n_q$ and $\pi_i=(n_i, \ldots, n_i)$ (k_i times). Write: $S_i=n_i+k_i$. Then $S_1>\cdots>S_q$. Proceeding inductively assume the proposition to be proved for every number $<N$.

1) If S_i is even then set $\mu_i=(S_i-1, \nu_i)$, where ν_i is the partition for which $\langle p_{\nu_i}, S(n_{i-1}, \ldots, n_{i-1})\rangle \neq 0$. By Murnagan-Nakayama rule $\langle p_{\mu_i}, S_{\pi_i}\rangle \neq 0$ and $\langle p_{\pi_i}, S_{\pi_j}\rangle=0$ when $i<j$ because $s_j<s_i$.

2) If s_i is odd and $k_i \neq 1$ then set $\mu_i=(S_i-2, S_i-2, \nu_i)$, where ν_i is the partition for which $\langle p_{\nu_i}, S(n_{i-2}, \ldots, n_{i-2})\rangle \neq 0$. As in 1) μ_i has the required properties.

3) If n_i is even and $k_i=1$ (in this case $i=1$) then set $\mu_1=(N-1, 1)$. Then $\langle p_{\mu_1}, S_{\pi_1}\rangle \neq 0$ and $\langle p_{\mu_1}, S_{\pi_j}\rangle=0$ when $j>1$ and $N>4$. Proposition 6.1 is proved.

It follows from Proposition 5.1 (2) that the characteristic numbers matrix of the manifolds G_{n_i, k_i} is non-degenerate. We obtain the following result.

Theorem D. *The bordism classes of complex Grassman manifolds are lineary independent over* $\mathbb{C}$.

Now we shall prove Proposition 6.3. Given a partition $\lambda=(l_1, \ldots, l_r)$ write $m_i=l_i-i+r$. We say that a symmetric function is of rank q if it is a linear combination of p_μ with $r(\mu)\leq q$ and some p_μ with $r(\mu)=q$ are involved. Given a partition λ of N set $\mathfrak{p}_\mu=\langle p_\mu p_1^{N-|\mu|}, S_\lambda\rangle/\langle p_1^N, S_\lambda\rangle$.

Proposition 7.1. *$\mathfrak{p}_\mu$ is a symmetric polynomial of rank $\leq r(\mu)$ in the variables m_i.*

Proof. Let $D(x_1,\ldots,x_n)$ be Van der Mond determinant in variables x_i. We need some further results concerning symmetric functions [10].

1) $\langle p_1^{|\lambda|}, S_\lambda\rangle=|\lambda|!\, D(m_1,\ldots,m_r)/\Pi\, m_i!$.

2) Let δ_r be the operator in Λ dual to multiplication by p_r. Then in the basis of complete symmetric functions $\delta_r=\Sigma h_i\,\partial/\partial h_{i+r}$.

3) $S_\lambda=\det(h_{l_i+j-i})$.

To simplify the notations denote $D(m_1,\ldots,m_r)$ by D and $D(m_1,\ldots, m_i-q,\ldots,m_r)$ by $D_i(q)$. Proceeding inductively assume $p_\mu=p_q p_\nu$. Then by properties 1)—3) above $\mathfrak{p}_\mu=\Sigma m_i(m_i-1)\ldots(m_i-q+1)D_i(q)\mathfrak{p}_\nu(m_1,\ldots,m_i-q \ldots,m_r)/D$ and $rk(\mathfrak{p}_\nu)\leq r(\nu)$. Let $\alpha=(\alpha_1,\alpha_2,\ldots)$ and consider the sum $\Sigma\, m_i(m_i-1)\ldots(m_i-q+1)D_i(q)\,p_\alpha(m_1,\ldots,m_i-q,\ldots,m_r)/D$. It equals to $\Sigma_i m_i(m_i-1)\ldots(m_i-q+1)D_i(q)\,\Pi_j\,(p_{\alpha_j}(m)+(m_{i-q})^{\alpha_j}-m^{\alpha}{}_i)/D$, which is a linear combination of $p_\gamma(m)\Sigma D_i(q)m_i^k/D$ with $r(\gamma)+k\leq r(\alpha)+q$.

It is easily seen that $rk(\Sigma D_i(q)\,m_i^k/D)\leq k-1$. So $rk(\mathfrak{p}_\mu)\leq r(\alpha)+q-1\leq r(\mu)$. Q. E. D

Now let λ be a Young diagram with $|\lambda|=d$. Every point of λ corresponds to a pair of non-negative integers (i, i). The content C_{ij} of the point (i, j) is the number $j-i$.

Proposition 7.2.

$$\frac{\langle s_\lambda p_1^{N-d}, S_\pi\rangle}{\langle p_1^N, S_\pi\rangle}=\frac{(N-d)!}{N!\,d!}\langle S_\lambda, p_1^d\rangle \prod_{(i,j)\in\lambda}(n-c_{ij})(k+c_{ij}).$$

This statement directly follows from the formula 1) of the proof of Proposition 7.1.

By the above proposition $\mathfrak{p}_\mu$ is a polynomial in $n-k$. To complete the proof of Proposition 6.3 it remains to demonstrate that $\mathfrak{p}_\mu$ contains no monomials $n^i k^j$ with $i-j>r(\mu)$. But this follows from Proposition 7.1 and the statement given below.

Proposition 7.3. *Let the numbers m_i correspond to the Young diagram π (i. e. $m_i=n+k-i$). Then $m_1^r+\cdots+m_k^r$ contains no monomials $n^i k^j$ with $i-j\geq r$.*

Proof. Set $N_r(l)=1^r+\cdots+l^r$. It is well known that $N_r(l)$ is a polynomial in l of degree $r+1$. It is also clear that $m_1^r+\cdots+m_k^r=N_r(n+k-1)-N_r(n-1)$. The latter difference is a polynomial in n and k of a degree not greater than $r+1$ which is divisible by k. Q. E. D.

Appendix. Some new identities for symmetric functions are represented here. These identities are not directly connected with the main subject of the paper but in view of the fact that the great part of their proof is already made we find it possible to give them here.

The starting-point for us is Proposition 7.1. Let $\mathfrak{p}\in\Lambda$, $\deg\mathfrak{p}=d$ and set $\varphi(\mathfrak{p})=\langle \mathfrak{p}p_1^{N-d}, S_\lambda\rangle/\langle p_1^N, S_\lambda\rangle$, where $|\lambda|=N$. We view $\varphi(\mathfrak{p})$ as a symmetric function in $m_i(m_i=l_i-i+N)$, where $\lambda=(l_1,\ldots,l_N)$, with coefficients from the field $k(N)$. By Proposition 7.1 $rk\varphi(\mathfrak{p})\leq rk\,\mathfrak{p}$. Set $A=k(N)\otimes\Lambda/(p_1-1)=k(N)$ [1,

$p_2, p_3, \dots]$ and $B = k(N) \otimes \Lambda/(p_1 - C^2_{N+1}) = k(N)[1, p_2, p_3, \dots]$. A and B are filtered by giving p_μ filtration equal to $r(\mu)$. Since $\varphi(\mathfrak{p}) = \varphi(\mathfrak{p}p_1)$ for every $\mathfrak{p}$ and $p_1(m_1, \dots, m_N) = C^2_{N+1}$, φ is a morphism of filtered spaces.

Proposition A.1. *φ is an isomorphism of the filtered spaces A and B.*

Proof. Since $A \simeq B$ it is sufficient to demonstrate that φ is injective. Let $\varphi(\Sigma c_\mu p_\mu) = 0$, where no partition μ contains 1. Then for every λ satisfying $|\lambda| = N$, $\Sigma c_\mu \langle p_\mu p_1^{N-|\mu|}, S_\lambda\rangle = 0$. So $\Sigma c_\mu p_\mu p_1^{N-|\mu|} = 0$. Since no partition μ contains 1, $c_\mu = 0$ for every μ. Q. E. D.

Since for every $\mathfrak{p}, g \in \Lambda$, $rk(\mathfrak{p}g) \leq rk\,\mathfrak{p} + rk\,g$ we obtain the following result.

Corollary A.2. *Let α and β be partitions of a number N. There exist such coefficients c_γ that for every partition λ of the same N the following relation holds*

$$\frac{\langle p_\alpha, S_\lambda\rangle \langle p_\beta, S_\lambda\rangle}{\langle p_1^N, S_\lambda\rangle^2} = \sum_\gamma c_\gamma \frac{\langle p_\gamma, S_\lambda\rangle}{\langle p_1^N, S_\lambda\rangle},$$

where the summation is taken over partitions γ of N satisfying $r(\gamma) \leqq r(\alpha) + r(\beta)$.

Theorem E. *Let $\alpha_1, \dots, \alpha_k$ be partitions of N. If $r(\alpha_1) > r(\alpha_2) + \dots + r(\alpha_k)$ then $\Sigma_\lambda (\langle p_1^N, S_\lambda\rangle)^{2-k} \langle p_{\alpha_1}, S_\lambda\rangle \cdots \langle p_{\alpha_k}, S_\lambda\rangle = 0$.*

Proof. By the previous corollary the sum above is a linear combination of the sums $\Sigma_\lambda \langle p_{\alpha_1}, S_\lambda\rangle \langle p_\beta, S_\lambda\rangle$, where $r(\beta) \leqq r(\alpha_2) + \dots + r(\alpha_k)$. The latter sum is equal to $\langle p_{\alpha_1}, p_\beta\rangle$ which is zero because $\{p_\mu\}$ is an orthogonal basis and $p_{\alpha_1} \neq p_\beta$ since $r(\beta) < r(\alpha_1)$. Q. E. D.

Denote now by m_λ the symmetrisations of the monomials corresponding to the partitions λ and by $\mathfrak{p}_\lambda$ the "forgotten" symmetric functions corresponding to λ. Then the basis $\{m_\gamma\}$ is dual to $\{h_\mu\}$ and $\{\mathfrak{p}_\lambda\}$ is dual to $\{e_\mu\}$ [10].

Theorem F. *Let $\alpha_1, \dots, \alpha_k$ be partitions of N. If $r(\alpha_1) > r(\alpha_2) + \dots + r(\alpha_k)$ then $\Sigma_\lambda (\langle p_1^N, h_\lambda\rangle)^{2-k} \langle p_{\alpha_1}, m_\lambda\rangle \langle p_{\alpha_2}, h_\lambda\rangle \cdots \langle p_{\alpha_k}, h_\lambda\rangle = \Sigma_\lambda (\langle p_1^N, e_\lambda\rangle)^{2-k} \langle p_{\alpha_1}, \mathfrak{p}_\lambda\rangle \langle p_{\alpha_2} e_\lambda\rangle \cdots \langle p_{\alpha_k}, e_\lambda\rangle = 0$.*

The proof is the same except that we use symmetric function in variables l_i instead of functions in m_i.

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