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AN ANALYSIS OF FINITE ELEMENT SOLUTION STABILITY*

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This paper deals with the problem of finding, by the finite element method, a solution stability of a partial differential equation and for a system of ordinary differential equations. We will show that if the approximate solution converges uniformly in time and uniformly with respect to initial condition, then the stability of a partial differential equation implies that of an ordinary differential equation and vice-versa.

1. Introduction. The finite element method is a recent tool which is quite effective in finding numerical solutions of partial differential equations [1]. The method has been used extensively in structural engineering [9]. Recently some authors have reported their findings on the theoretical analysis of the finite element method. They have developed theorems about convergence, error estimates, etc. [3, 7]. But little has been done on solution stability for partial differential equations. We mentioned in this regard the work of Strang [7], Fix et al. [2, 5], and others [4, 6, 8, 10].

This paper deals with the problem of finding by the finite element method a solution stability of a partial differential equation and for a system of ordinary differential equations. We will show that if the approximate solution converges uniformly in time, then the stability of a partial differential equation implies that of an ordinary differential equation and vice-versa.

2. Definitions and Lemmas. Definition 2.1. *Let $X'=f(t, X)$ be a system of ordinary differential equations and let $Z(t)$ be any solution of it. Then if $Y(t)=X(t)-Z(t)$ we have $Y'=g(t, Y)$ with $g(t, 0)=0$.*

Definition 2.2. *For continuous functions $w(x, t)$ and vector functions $W(t)=\{w_1, w_2, \dots, w_n\}^T$, we define the norms*

$$\| W(x, t)\| = \sup_{x\in(0,1)} |w(x, t)|, \tag{2.1}$$

$$\| W(x, t)\|_n = \sup_{i=1,2,\dots,n} |w_i(t)|. \tag{2.2}$$

Let us consider the partial differential equation

$$w_t = w_{xx}+f(x, w, w_x), \tag{2.3}$$

and suppose $f(x, 0, 0)=0$, i. e. $w=0$ is a solution of Eq. (2.3) and assume that for each $q(x)$ in a suitable class, there exists a solution $w(x, t, q)$ such that $w(x, 0, q)=q(x)$.

Definition 2.3. *The solution $w=0$ of Eq. (2.3) is stable if for any $\varepsilon>0$ there exists a $\delta>0$ such that $\| q(x)\| <\delta \Rightarrow \| w(x, t, q)\| <\varepsilon$.*

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Again consider the system of ordinary differential equations

(2.4) $$\frac{dX}{dt}=P_n(X).$$

Let us suppose $P_n(0)=0$, i. e. $X=0$ is a solution of Eq. (2.4) and further assume that for all X_0 sufficiently small, there exists a solution $X(t, X_0)$ of Eq. (2.4) such that $X(0)=X_0$.

Definition 2.4. *The solution $X=0$ of Eq. (2.4) is stable if for any $\varepsilon>0$, there exists a $\delta(\varepsilon)>0$ independent of n such that $\|X_0\|_n<\delta$ $\|X(t, x_0\|_n<\varepsilon$.*

In the sequel, we denote by $u_n(x, t)$ the approximation to the exact solution $w(x, t)$ which is obtained by taking mesh size $h=1/n$ in piecewise linear finite element discretization. Then $u_n(x, t)=\Sigma_{i=1}^n u_i(t)\psi_i(x)$, where $\psi_1(x)$ are basis functions $\psi_i(x_l)=\delta_{il}$.

Let us state some lemmas which will be needed in the next section:

Lemma 2.1. *Let $\{\psi_i\}$, $i=1, 2, \ldots, n$, be the set of piecewise linear basis functions. Let $W=\{w_1, w_2, \ldots, w_n\}^T$, $w(x)=\Sigma_{i=1}^n w_i\psi_i$ then $\|W\|_n=\|w(x)\|$.*

Proof. From Eq. (2.2) we have

(2.5) $$\|w(x)\|=\sup_{x\in(0,1)}\left|\sum_{i=1}^n w_i\psi_i\right|$$

since ψ_i's are piecewise linear and the supremum occurs at a critical point, it must occur at one of the nodes x_l, where the derivative fails to exist.

Now if $w'(x)=0$ for $x_l<x<x_{l+1}$, then $w_{l+1}-w_l=0.h$, i. e. $w_{l+1}=w_l$, which implies that $w(x)=0$ for $x_l<x<x_{l+1}$, since $\psi_i(x_l)=\delta_{il}$. Hence from Eq. (2.5) we have

$$\|w(x)\|=\sup_{x\in(0,1)}\left|\sum_{i=1}^n w_i\psi_i\right.=\sup_{i=1,2,\ldots,n}|w_i|.$$

Substituting Eq. (2.3) in the last one, we have $\|w(x)\|=\|W\|_n$. Thus the lemma is proved.

Lemma 2.2. *Let $P_n(U)$ be the function obtained by the piecewise linear finite element discretization with mesh size $h=1/n$. Let $f(x, 0, 0)=0$, then $P_n(0)=0$.*

Proof. Using the piecewise linear finite element discretization, the function $P_n(U)$ is defined by

$$h(P_n(U))_j=\frac{1}{h}(u_{j-1}-2u_j+u_{j+1})+\int_0^1 f\{x,\ \sum_{i=1}^n u_i(t)\psi_i(x),\ \sum_{i=1}^n u_i(t)\psi_i'(x)\}\psi_j(x)dx$$

or,

(2.6) $$h(P_n(U))_j=\frac{1}{h}(u_{j-1}-2u_j+u_{j+1})$$

$$+hf\{x^*,\ au_{j-1}+bu_j+cu_{j+1},\ \frac{d(u_j-u_{j-1})}{h},\ \frac{e(u_{j+1}-u_j)}{h}\},$$

where $a+b+c=1$ and $(d, e)=(1, 0)$ or $(0, 1)$.

Since $f(x, 0, 0)=0$, from Eq. (2.6) we have $P_n(0))_j=0\Rightarrow P_n(0)=0$. Thus the lemma is proved.

Lemma 2.3. *Let $P_n(U)$ be defined by Eq. (2.6). If $f(x, 0, 0)\in c(0, 1)$ and if $P_n(0)=0$ holds for all sufficiently large n, then $f(x, 0, 0)=0$.*

Proof. Suppose $f(x, 0, 0)>0$ for $x=x_0$, then there exists a neighbourhood (x_0-p, x_0+p) on which $f(x, 0, 0)>0$. But then we can choose $n>1/p$ so that $h<p$ and it is clear that there is an index $k, 1\leq k\leq n$, such that $\psi_k(x)$ is non-negative on (x_0-p, x_0+p) and zero if $x\notin(x_0-p, x_0+p)$. Then

$$h(P_n(U))_k=\int_0^1 f\{x, \sum_i^n u_i(t)\psi_i(x), \sum_{i=1}^n u_1(t)\psi_i'(x)\}\psi_k(x)dx$$

$$=hf(x^*, \quad au_{k-1}+bu_k+cu_{k+1}, \quad \frac{d(u_k-u_{k-1})}{h}, \quad \frac{e(u_{k+1}-u_k)}{h}),$$

where $a+b+c=1$ and $(d, e)=(1, 0)$ or $(0, 1)$ and $x_{k-1}<x^*<x_{k+1}$, i. e. $x^*\varepsilon(x_0-p, x_0+p)$. Since $P_n(0)=0$, then $[P_n(0)]_k=0$. But this says $hf(x^*, 0, 0)=0$, with $x^*\varepsilon(x_0-p, x_0+p)$, contradicting the fact that $f(x, 0, 0)>0$ at that interval.

Thus $f(x, 0, 0)=0$ is proved. Hence the lemma.

Finally we introduce one more definition which will be useful in the next section.

Definition 2.5. *To any initial function* $q(x)$ *defined on* $(0, 1)$ *we associate the n dimensional vector* $Q_n=(q(x_1), q(x_2), \ldots, q(x_n))^T$, *then* $u(t, Q_n)$ *denotes the approximate solution for the equation*

$$(2.7)\quad \begin{cases} \text{(a)} & M\frac{dU}{dt}=P_n(U), \\ \text{(b) } \textit{with} & M_{ij}=(\psi_i, \psi_j), \\ \text{(c)} & U(0)=Q_n. \end{cases}$$

3. Stability theorems. Theorem 3.1. *Let Eq.* (2.7a) *have the solution* $U=0$ *as a stable solution for all n, sufficiently large. Let* $P_n(U)$ *be the finite element discretization of Eq.* (2.3), *then* $w(x, t)\equiv0$ *is a solution of* (2.3). *Further, if the solution* $U(t)=\{u_1(t), u_2(t), \ldots, u_n(t)\}^T$ *is such that* $u_n(x, t, Q_n)\to w(x, t, q)$ *for all functions* $q(x)$, *then* $w(x, t)\equiv0$ *is stable.*

Proof. From Lemma 2.3, $w(x, t)=0$ is a solution of Eq. (2.3). Since Eq. (2.7a) is stable for all n by hypothesis, given any $\varepsilon>0$ there exists a $\delta=\delta(n, \varepsilon)>0$ such that $\|U(t, Q_n)\|<\varepsilon/2$ for $\|Q_n\|_n<\delta$. Let us assume that the null solution of Eq. (2.3) is not stable. Then there exists an $\varepsilon>0$ such that for any $\delta>0$ there exist $\bar{t}(\delta)$, $q(x, \delta)$ for which $\|q(x)\|<\delta$, $\|w(x, \bar{t}, q)\|>\varepsilon$. Assuming the convergence, there exists $M(\varepsilon, \bar{t}, q)$, $\bar{t}>0$ and $n>M(\varepsilon, t, q)$ such that $\|u_n(x, \bar{t}, Q_n)-w(x, \bar{t}, q)\|<\varepsilon/2$. Hence from Lemma 2.1 and the last inequality, we have $\|U(\bar{t}, Q_n)\|_n>\varepsilon/2$. But $\|Q_n\|_n<\delta$, and thus the stability of the null solution of Eq. (2.7a) is contradicted. Therefore, the null solution ($w=0$) of Eq. (2.3) is stable which completes the proof of the theorem.

Theorem 3.2. *Let Eq.* (2.3) *have the null solution as a stable solution. Let* $u_n(x, t, Q_n)$ *be the solution of Eq.* (2.7 a, b). *Also let* $u_n(x, t, Q_n)$ *converge uniformly to* $w(x, t, q)$ *in time t as* $n\to\infty$, *for each* $q(x)$ *of initial functions. Then* $u=0$ *is the solution of Eq.* (2.7a) *and is stable.*

Proof. From Lemma 2.2, $U=0$ is a solution of Eq. (2.7a). Let $\varepsilon>0$, then there exists $N(\varepsilon, q)$, $n>N(\varepsilon, q)$ such that

$$(3.1)\qquad \|u_n(x, t, Q_n)-w(x, t, q)\|<\varepsilon/2.$$

Let us choose $\delta>0$ such that if $\| q(x) \| < \delta$, then $\| w(x, t, q) \| < \varepsilon/2$. Hence from Eq. (3.1) and the last inequality we have for $n > N(\varepsilon, q)$ $\| u_n(x, t, Q_n) \| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Now using Lemma 2.1. and the last equation, we have $\| U(t, Q_n) \|_n < \varepsilon$ for $\| Q_n \|_n < \delta$. Therefore, $U=0$ is a solution of Eq. (2.7a) and is stable. Hence the theorem.

4. Discussion and conclusions. It is worthwhile to note that, given a partial differential equation and the corresponding discretization obtained by piecewise linear finite element method, then the stability of the null solution is a property possessed by both or none, if the approximate solution can be shown to converge uniformly in time. This uniform convergence is not necessary for the stability of partial differential equations or for stability of the ordinary differential equation.

Finally, we conclude by remarking that it might be possible to gain information on the stability of solutions of partial differential equation from an analysis of a discretization which produces a system of ordinary differential equations.

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