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ON EQUIVALENT NORMS WHICH ARE UNIFORMLY CONVEX OR UNIFORMLY DIFFERENTIABLE IN EVERY DIRECTION IN SYMMETRIC SPACES

DENKA N. KUTZAROVA, STANIMIR L. TROYANSKI

In this paper necessary and sufficient conditions for existence of equivalent norms which are uniformly convex (resp. uniformly differentiable) in every direction in symmetric function spaces are obtained

1. Introduction. It has been shown in [4] that every separable Banach space admits an equivalent norm, uniformly convex in every direction. From a result of Shmulyan [11] and the construction of the norm in [4] it follows that each separable Banach space has an equivalent norm, uniformly differentiable in every direction. It has been proved in [9] that a non-separable Banach space X with a symmetric basis admits an equivalent norm which is uniformly convex (resp. uniformly differentiable) in every direction iff X is not isomorphic to $c_0(\Gamma)$ (resp. $l_1(\Gamma)$) for some uncountable set Γ. The paper [10] contains necessary and sufficient conditions for existence of such equivalent norms in Banach spaces with an unconditional basis. In [6] an equivalent lattice norm which is uniformly convex in every direction is introduced in $L_1(S, \Sigma, \mu)$ and as an application several results concerning the existence of equivalent lattice norms which are uniformly convex or uniformly differentiable in every direction in Banach lattices are given.

In the present paper we obtain generalizations of the results of [9] for symmetric function spaces.

2. Definitions and notations. The norm of a Banach space X is said to be uniformly convex in every direction if the conditions $x_n, y_n, z \in X$, $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n+y_n\| \to 2$ and $x_n-y_n=\lambda_n z$ imply $\|x_n-y_n\| \to 0$.

The norm of a Banach space X is said to be Gateaux differentiable if for any $x, y \in X$ with $\|x\|=\|y\|=1$,

$$\lim_{\tau\to 0} \tau^{-1}(\|x+\tau y\|+\|x-\tau y\|-2)=0.$$

The norm of a Banach space X is said to be uniformly differentiable in every direction if for any $x, y \in X$ with $\|y\|=1$,

$$\lim_{\tau\to 0} \tau^{-1} \sup_{\|x\|=1} (\|x+\tau y\|+\|x-\tau y\|-2)=0.$$

A Banach lattice is called σ-order complete if every order bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X has a least upper bound.

A Banach lattice X is called order continuous if for every downward directed set $\{x_\alpha\}_{\alpha\in A}$ in X with $\wedge_{\alpha\in A} x_\alpha=0$, $\lim_\alpha \|x_\alpha\|=0$.

An element $e \geq 0$ of a Banach lattice X is said to be a weak unit of X if $x \in X$, $e \wedge |x| = 0$ imply $x = 0$.

A family $\{u_\gamma\}_{\gamma \in \Gamma}$ in a Banach lattice X is said to be a generalized weak unit of X provided $u_\gamma \neq 0$, $\gamma \in \Gamma$, $|u_\gamma| \wedge |u_\beta| = 0$, $\gamma \neq \beta$ and if $x \in X$, $|x| \wedge |u_\gamma| = 0$ for all $\gamma \in \Gamma$, then $x = 0$.

Let (S, Σ, μ) be a measure space with non-negative measure. A Banach space X consisting of equivalence classes of μ-measurable real valued functions on S is called Köthe function space if X is a Banach lattice in the obvious order ($f \geq 0$ if $f(s) \geq 0$ almost everywhere) and the following conditions hold:

(i) if $|f(s)| \leq |g(s)|$ a. e. on S with f μ-measurable and $g \in X$, then $f \in X$,

(ii) if $f \in X$, then f is locally integrable, i. e. for every $A \in \Sigma$ with $\mu(A) < \infty$ there exists $\int_A f(s)d\mu$,

(iii) for every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function χ_A of A belongs to X.

A Köthe function space X on (S, Σ, μ) is said to be symmetric if there exists a constant $C > 0$ such that for any choice of finite systems $\{a_i\}_{i=1}^n$ of reals and $\{A_i\}_{i=1}^n$, $\{B_i\}_{i=1}^n \subset \Sigma$ with $\mu(A_i) \leq \mu(B_i) < \infty$, $i = 1, 2, \dots, n$, $A_i \cap A_j = \emptyset$, $B_i \cap B_j = \emptyset$, $i \neq j$, the following holds

$$\|\sum_{i=1}^n a_i \chi_{A_i}\| \leq C \|\sum_{i=1}^n a_i \chi_{B_i}\|.$$

Consider $l_\infty(\Gamma)$. An equivalent norm has been defined by Day (cf. e. g [5, p. 161])

$$D(x) = \sup\{(\sum_{i=1}^n 2^{-i} x^2(\gamma_i))^{1/2};\quad n < \infty,\ \gamma_i \in \Gamma\}.$$

For each subset $\Gamma_1 \subset \Gamma$ define a projection $P_{\Gamma_1} : l_\infty(\Gamma) \to l_\infty(\Gamma)$ by the formula $P_{\Gamma_1}x(\gamma) = x(\gamma)$ if $\gamma \in \Gamma_1$ and $P_{\Gamma_1}x(\gamma) = 0$ otherwise.

3. Equivalent norms which are uniformly convex in every direction.

Theorem 3.1. *Let X be a symmetric Köthe function space. Then the space X admits an equivalent norm, uniformly convex in every direction, if and only if X does not contain any subspace isomorphic to $c_0(\Gamma)$ for uncountable set Γ.*

Proposition 3.2 [10]. *Let X be a Banach space and $T : X \to l_\infty(\Gamma)$ be a bounded positive-homogeneous sublinear operator satisfying the following condition:*

for any $\varepsilon > 0$ there exists a partition $\{\Gamma_i^{(\varepsilon)}\}_{i=1}^\infty$ of the set Γ such that for all $x \in X$,
$$\operatorname{card}(\{\gamma \in \Gamma_i^{(\varepsilon)};\ |Tx(\gamma)| > \varepsilon \|x\|\}) \leq i.$$

If we introduce a new norm in X by the formula

$$|||x||| = (\|x\|^2 + \sum_{i,k=1}^\infty 2^{-i-k} D^2(T_i^{(k)}x))^{1/2},\quad T_i^{(k)} = P_{\Gamma_i(1/k)}T,$$

then for any $x_n, y_n \in X$ the conditions Tx_n, $Ty_n \in c_0(\Gamma)$, $|||x_n||| \leq 1$, $|||y_n||| \leq 1$, $|||x_n + y_n||| \to 2$ imply that

$$\lim_{n\to\infty} (Tx_n(\gamma) - Ty_n(\gamma)) = 0,\ \gamma \in \Gamma.$$

L e m m a 3.3. *Let X be a Banach space such that the conditions x_n, y_n, $z\in X$, $\|x_n\|\to 1$, $\|y_n\|\to 1$, $\|x_n+y_n\|\to 2$ and $x_n-y_n=z$ imply $z=0$. Then the norm is uniformly convex in every direction.*

Let (S, Σ, μ) be a measure space and consider $L_1(S, \Sigma, \mu)$. Let $\Sigma_1\subset\Sigma$ consist of all μ-measurable sets, free of atoms. Define for any $x\in L_1(S, \Sigma, \mu)$

$$\tilde{x}(t)=\sup_{B\in\Sigma_1,\ \mu(B)\leq t}\int_B |x(s)|\,d\mu,\qquad t\in(0,\infty).$$

This function is introduced in [2].

L e m m a 3.4. *Let $x\in L_1(S, \Sigma, \mu)$ and $t, u\in(0, \mu(S)]$ with $t\leq u<\infty$. Let $B\in\Sigma_1$, $\mu(B)=u$ and $\int_B|x|\,d\mu>\tilde{x}(u)-\delta$ for some $\delta>0$. Then there exists a $Q\subset B$ with $\mu(Q)=t$ and $\int_Q|x|\,d\mu>\tilde{x}(t)-3\delta$.*

Let X be a Köthe function space on (S, Σ, μ). Denote for each $x\in X$, $k=1, 2, \ldots$

$$p_k(x)=\Big(\int_0^k \tilde{x}^2(t)dt\Big)^{1/2}.$$

Put

$$\|x\|_1=\Big(\|x\|^2+\sum_{k=1}^{\infty}2^{-k}p_k^2(x)\Big)^{1/2}.$$

It is easily seen that $\|\,.\,\|_1$ is an equivalent lattice norm in X.

L e m m a 3.5. *Let X be a Köthe function space and $x_n, y_n\in X$, $\sup_n(\|x_n\|, \|y_n\|)<\infty$, $p_k(x_n)\to 1$, $p_k(y_n)\to 1$, $p_k(x_n+y_n)\to 2$ for some integer k. Then there exists a function $v(t)$, $t\in[0, k]$ and a subsequence $\{n\}$ of indices so that $\tilde{x}_n(t)\to v(t)$, $\tilde{y}_n(t)\to v(t)$, $\widetilde{x_n+y_n}(t)\to 2v(t)$ for each $t\in[0, k]$.*

L e m m a 3.6. *Let $x_n, y_n\in L_1(S, \Sigma, \mu)$ be such that*

$$\sup_n\Big(\int_S|x_n|\,d\mu, \int_S|y_n|\,d\mu\Big)<\infty,\quad \mu(\operatorname{supp}x_n\cup\operatorname{supp}y_n)\leq k,\quad p_k(x_n)\to p,$$

$p_k(y_n)\to p$, $p_k(x_n+y_n)\to 2p$ for some integer k and

$$\lim_{\mu(B)\to 0}\ \sup_n\Big\{\Big|\int_B(|x_n|-|y_n|)d\mu\Big|\Big\}=0.$$

Denote by Ω the non-atomic part of the set $\bigcup_{n=1}^{\infty}(\operatorname{supp}x_n\cup\operatorname{supp}y_n)$. Then there exists a subsequence $\{n\}$ of indices so that $(x_n-y_n)\chi_\Omega\to 0$ in measure as $n\to\infty$.

In fact, Lemmas 3.3—3.6 are proved in [6].

L e m m a 3.7. *Let X be a Köthe function space on (S, Σ, μ) with the following property:*

$$(*)\qquad \sup_{x\in X}\mu(\{|x|>\varepsilon\|x\|\})<\infty\quad \text{for any}\quad \varepsilon>0.$$

Then the conditions $x_n, y_n, z\in X$, $x_n-y_n=z$, $\|x_n\|_1\to 1$, $\|y_n\|_1\to 1$, $\|x_n+y_n\|_1\to 2$ imply $z\chi_\Omega=0$, where Ω is the non-atomic part of $\bigcup_{n=1}^{\infty}(\operatorname{supp}x_n\cup\operatorname{supp}y_n)$.

P r o o f. It follows from $(*)$ that for each $x\in X$, the measure on $\operatorname{supp}x$ is σ-finite and therefore Ω is correctly defined.

Let x_n, y_n, $z \in X$ satisfy the above assumptions. Using the uniform convexity of l_2, the triangle inequality and the diagonal procedure, we may choose a subsequence $\{n\}$ of indices so that

(1) $$p_k(x_n) \to p_k, \quad p_k(y_n) \to p_k, \quad p_k(x_n+y_n) \to 2p_k, \quad k=1, 2, \ldots,$$

It is no loss of generality to assume that $S=\Omega$.

In order to prove that $z=0$ it suffices to show that for any $\varepsilon>0$ the equality $\mu(\{|z|>\varepsilon\})=0$ holds.

Let $\varepsilon>0$. Put $P=\{|x_n-y_n|>\varepsilon\}$, $Q_n=\{|x_n+y_n|>\varepsilon/2\}$. It follows from (*) that there exists an integer $M=M(\varepsilon)$ such that $\mu(\{|x_n|>\varepsilon/8\})\leq M$, $\mu(Q_n)\leq M$, $n=1, 2, \ldots$

Let $\delta_n>0$, $\delta_n \to 0$. Choose $S_n \subset S$, $n=1, 2, \ldots$ so that $\mu(S_n)=M$ and

(2) $$\int_{S_n} |x_n+y_n| \, d\mu > \overline{x_n+y_n}(M)-\delta_n.$$

Without affecting the generality we may assume that $Q_n \subset S_n$. Indeed, denote $A_n=Q_n \cap S_n$. We have $\mu(Q_n)\leq M=\mu(S_n)$. Since Ω is free of atoms, there exists a $B_n \subset S_n \setminus A_n$ such that $\mu(B_n)=\mu(Q_n \setminus A_n)$. Consider $S_n'=(Q_n \cup S_n)\setminus B_n$. Obviously, $\mu(S_n')=M$ and $Q_n \subset S_n'$. Moreover, by $|(x_n+y_n)\chi_{S_n \setminus Q_n}|\leq\varepsilon/2$ and $|(x_n+y_n)\chi_{Q_n}|>\varepsilon/2$ we get

$$\int_{S_n'} |x_n+y_n| \, d\mu \geq \int_{S_n} |x_n+y_n| \, d\mu > \overline{x_n+y_n}(M)-\delta_n.$$

In the sequel we shall assume that $Q_n \subset S_n$.

It follows from (2) and Lemma 3.4 that

$$\overline{x_n+y_n}(t)-3\delta_n \leq \overline{(x_n+y_n)\chi_{S_n}}(t) \leq \overline{x_n+y_n}(t), \quad t \in [0, M].$$

This and (1) give

(3) $$p_M((x_n+y_n)\chi_{S_n}) \to 2p_M.$$

Then by (3), $p_M((x_n+y_n)\chi_{S_n}) \leq p_M(x_n\chi_{S_n})+p_M(y_n\chi_{S_n})$, $p_M(x_n\chi_{S_n})\leq p_M(x_n)$, $p_M(y_n\chi_{S_n})\leq p_M(y_n)$ and (1) we get

(4) $$p_M(x_n\chi_{S_n}) \to p_M, \quad p_M(y_n\chi_{S_n}) \to p_M.$$

By (1) and Lemma 3.5 we may choose a subsequence $\{n\}$ of indices and a function $v(t)$ so that $\tilde{x}_n(t) \to v(t)$, $\tilde{y}_n(t) \to v(t)$, $\overline{x_n+y_n}(t) \to 2v(t)$, $t \in [0, M]$. In particular,

(5) $$\tilde{x}_n(M) \to v(M), \quad \tilde{y}_n(M) \to v(M), \quad \overline{x_n+y_n}(M) \to 2v(M).$$

As above, it follows from (2) and (5) that

(6) $$\int_{S_n} |x_n| \, d\mu \to v(M), \quad \int_{S_n} |y_n| \, d\mu \to v(M), \quad \int_{S_n} |x_n+y_n| \, d\mu \to 2v(M).$$

Define

$$D_n=\{|x_n|>\varepsilon/4\}, \quad E_n=\{|x_n|>\varepsilon/8\}, \quad n=1, 2, \ldots$$

We shall prove that

$$\mu(D_n \setminus S_n) \to 0. \tag{7}$$

Since μ is non-atomic and $\mu(E_n) \leq \mu(S_n)$, $D_n \subset E_n$, then there exists a $T_n \subset S_n \setminus E_n$ such that $\mu(T_n) = \mu(D_n \setminus S_n)$. Put $U_n = (S_n \cup D_n) \setminus T_n$. Since $\mu(U_n) = M$, then

$$\int_{U_n} |x_n| \, d\mu \leq \tilde{x}_n(M). \tag{8}$$

On the other hand,

$$\int_{U_n} |x_n| \, d\mu = \int_{S_n \setminus T_n} |x_n| \, d\mu + \int_{D_n \setminus S_n} |x_n| \, d\mu \geq \int_{S_n \setminus T_n} |x_n| \, d\mu + \int_{D_n \setminus S_n} \frac{\varepsilon}{4} d\mu$$

$$\geq \int_{S_n \setminus T_n} |x_n| \, d\mu + \int_{T_n} |x_n| \, d\mu + \frac{\varepsilon}{8} \mu(D_n \setminus S_n) = \int_{S_n} |x_n| \, d\mu + \frac{\varepsilon}{8} \mu(D_n \setminus S_n).$$

This and (8) give $\mu(D_n \setminus S_n) \leq 8\varepsilon^{-1} (\tilde{x}_n(M) - \int_{S_n} |x_n| \, d\mu)$. By the last inequality, (5) and (6) we obtain (7).

We shall show that

$$\mu(P \setminus S_n) \to 0. \tag{9}$$

In view of (7), it suffices to prove that $P \setminus S_n \subset D_n \subset S_n$. Indeed, let $s \in P \setminus S_n$. Since $Q_n \subset S_n$, then $s \in P \cap (S_n \setminus Q_n)$, i. e. $s \in D_n$.

Since $x_n - y_n = z$, $n = 1, 2, \ldots$ then it is easy to see that

$$\lim_{\mu(B) \to 0} \sup_n \{ | \int_B (|x_n| - |y_n|) d\mu | \} = 0.$$

Thus, according to Lemma 3.6, it follows from (3), (4), (6) and the last equality that there exists a subsequence $\{n\}$ of indices so that $(x_n - y_n)\chi_{S_n} \to 0$ in measure as $n \to \infty$. Therefore, $\mu(P \cap S_n) \to 0$. Combining this with (9) we deduce that $\mu(P) = 0$ which completes the proof.

Lemma 3.8. *Let X be a Köthe function space satisfying the condition $(*)$ from Lemma 3.7. Then X admits an equivalent lattice norm which is uniformly convex in every direction.*

Proof. Put for $k = 1, 2, \ldots$

$$\Gamma_k = \{ A \in \Sigma; \ A \text{ — atom}, \ \mu(A) \geq k^{-1}, \ \| \chi_A \| \geq k^{-1} \}.$$

It follows from $(*)$ that we may define linear operators $T_k : X \to c_0(\Gamma_k)$, $k = 1, 2, \ldots$ by the formula $T_k x(A) = k^{-1} x(a)$, $A \in \Gamma_k$, $a \in A$. Let $x \in X$. Then $|T_k x(A)| \leq |x(a)| \, \| \chi_A \| \leq \| x \|$, i. e. $\| T_k \| \leq 1$, $k = 1, 2, \ldots$

Let $\varepsilon > 0$. By $(*)$, there exists an integer $M = M(\varepsilon)$ such that $\mu(\{ |x| > \varepsilon \| x \| \}) \leq M$ for all $x \in X$, whence $\mu(\{ k^{-1} |x| > \varepsilon \| x \| \}) \leq M$, $k = 1, 2, \ldots$ Then

$$\operatorname{card}(\{ A \in \Gamma_k ; \ |T_k x(A)| > \varepsilon \| x \| \}) \leq kM, \quad x \in X, \quad k = 1, 2, \ldots \tag{10}$$

We introduce an equivalent lattice norm in X by the formula

$$||| x ||| = (\| x \|_1^2 + \sum_{k=1}^{\infty} 2^{-k} D^2(T_k(x)))^{1/2}.$$

We shall show that the norm $|||\cdot|||$ is uniformly convex in every direction. Let $x_n, y_n, z \in X$, $x_n - y_n = z$, $|||x_n|||\to 1$, $|||y_n|||\to 1$, $|||x_n+y_n|||\to 2$. It follows from (10) and Proposition 3.2 that

$$\lim_{n\to\infty} (T_k x_n(A) - T_k y_n(A)) = 0, \quad A \in \Gamma_k, \quad k=1, 2, \ldots \tag{11}$$

Then by Lemma 3.7 and (11) we obtain that $z=0$. In view of Lemma 3.3 this completes the proof.

Proposition 3.9. *Let X be a symmetric Köthe function space on a measure space (S, Σ, μ) with non-σ-finite measure. Then the following conditions are equivalent:*

(i) $$\lim_{t\to\infty} \inf (\|\chi_B\|;\ B\in\Sigma,\ \mu(B)\geq t) = \infty,$$

(ii) $$\sup_{x\in X} \mu(\{|x|>\varepsilon\|x\|\}) < \infty \quad \textit{for any } \varepsilon>0,$$

(iii) *X has an equivalent lattice norm which is uniformly convex in every direction,*

(iv) *X does not contain any subspace isomorphic to $c_0(\Gamma)$ for uncountable set Γ.*

Proof. (i) $\Rightarrow$ (ii). Let $\varepsilon>0$, $x\in X$ and $B=\{|x|>\varepsilon\|x\|\}$. Since $|x| > \varepsilon\|x\|\chi_B$, then $\|x\|\geq\varepsilon\|x\|\cdot\|\chi_B\|$ and therefore $\|\chi_B\|<\varepsilon^{-1}$, whence $\sup_{x\in X}\mu(\{|x|>\varepsilon\|x\|\})<\infty$, i. e. (ii) holds.

It follows from Lemma 3.8 that (ii) $\Rightarrow$ (iii).

Since $c_0(\Gamma)$ with Γ uncountable has no equivalent norm that is uniformly convex in every direction (cf. [4]), then (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). Assume that (i) is false. Since X is symmetric, then there exists a constant $c_2>0$ such that for each $B\in\Sigma$, $\mu(B)<\infty$,

$$\|\chi_B\| \leq c_2. \tag{12}$$

Since the measure is non-σ-finite, then we may construct by transfinite induction an uncountable family $\{B_\gamma\}_{\gamma\in\Gamma}$ of mutually disjoint measurable sets with finite measure so that $\|\chi_{B_\gamma}\|\geq c_1>0$ for each $\gamma\in\Gamma$. From (12) and the last inequality it follows that for any finite system $\{\gamma_i\}_{i=1}^n\subset\Gamma$, $\gamma_i\neq\gamma_j$, $i\neq j$ and any system $\{b_i\}_{i=1}^n$ of reals, $n=1, 2, \ldots$ we have

$$c_1 \max_i |b_i| \leq \max_i |b_i|\ \|\chi_{B_{\gamma_i}}\| \leq \Big\|\sum_{i=1}^n b_i\chi_{B_{\gamma_i}}\Big\|$$

$$\leq \max_i |b_i|\ \Big\|\sum_{i=1}^n \chi_{B_{\gamma_i}}\Big\| \leq \max_i |b_i|\ \|\chi_{\bigcup_{i=1}^n B_{\gamma_i}}\| \leq c_2 \max_i |b_i|,$$

i. e. $\{\chi_{B_\gamma}\}_{\gamma\in\Gamma}$ is equivalent to the natural basis of the space $c_0(\Gamma)$.

3.10. Proof of Theorem 3.1. In view of Proposition 3.9, it is enough to notice that it is shown in [6] that every Köthe function space on a σ-finite measure space has an equivalent lattice norm which is uniformly convex in every direction.

4. Equivalent norms which are uniformly differentiable in every direction.

Proposition 4.1 [10]. *Let X be a Banach space with an unconditional basis $\{u_\gamma\}_{\gamma\in\Gamma}$. Then X has an equivalent norm, uniformly differentiable in every direction, if and only if the following condition holds:*

for any $\varepsilon>0$ there exists a partition $\{\Gamma_i^{(\varepsilon)}\}_{i=1}^{\infty}$ of the set Γ such that for every choice of different elements $\{\gamma_j\}_{j=1}^{i}\subset\Gamma_i^{(\varepsilon)}$,

$$\Big\|\sum_{j=1}^{i} u_{\gamma_j}\Big\|<\varepsilon i.$$

In the present paper we generalize the above result for Banach lattices.

Theorem 4.2. *Let X be an order continuous Banach lattice with a generalized weak unit $\{u_\gamma\}_{\gamma\in\Gamma}$. Then X admits an equivalent norm, uniformly differentiable in every direction, if and only if X satisfies the condition:*

for any $\varepsilon>0$ there exists a partition $\{\Gamma_i^{(\varepsilon)}\}_{i=1}^{\infty}$ of the set Γ such that for every choice of different elements $\{\gamma_j\}_{j=1}^{i}\subset\Gamma_i^{(\varepsilon)}$,

$$(**)\qquad \Big\|\sum_{j=1}^{i} u_{\gamma_j}\Big\|<\varepsilon i.$$

Proof. Since $|u_\gamma|\wedge|u_\beta|=0$, $\gamma\neq\beta$ and X is a lattice, then $\{u_\gamma\}_{\gamma\in\Gamma}$ is an unconditional basis of $[\{u_\gamma\}_{\gamma\in\Gamma}]\subset X$. Hence, the "only if" part is an immediate consequence of Proposition 4.1.

The "if" part. Without loss of generality we may assume that $\|u_\gamma\|=1$, $\gamma\in\Gamma$. Indeed, let $\Gamma_m=\{\gamma\in\Gamma;\ \|u_\gamma\|\geq m^{-1}\}$, $m=1, 2, \ldots$

Choose an $\varepsilon>0$. By (**), $\Gamma_m=\bigcup_{k=1}^{\infty}\Gamma_{m,k}^{(\varepsilon)}$ so that for every choice of different elements $\{\gamma_j\}_{j=1}^{k}\subset\Gamma_{m,k}^{(\varepsilon)}$, $\|\Sigma_{j=1}^{k}u_{\gamma_j}\|<\frac{\varepsilon}{m}k$. Hence, $\|\Sigma_{j=1}^{k}\|u_{\gamma_j}\|^{-1}u_{\gamma_j}\|<\varepsilon k$. Moreover, $\Gamma=\bigcup_{m,k=1}^{\infty}\Gamma_{m,k}^{(\varepsilon)}$. Therefore, $\{u_\gamma/\|u_\gamma\|\}_{\gamma\in\Gamma}$ satisfies (**). Consequently, the family $\{e_\gamma\}_{\gamma\in\Gamma}$ satisfies (**), where $e_\gamma=|u_\gamma|/\|u_\gamma\|$.

Since $\{e_\gamma\}_{\gamma\in\Gamma}$ is also a generalized weak unit of X and X is order continuous, then (see e. g. [8], p. 9) X can be decomposed into a direct sum of a family of mutually disjoint ideals

$$(13)\qquad X=\sum_{\gamma\in\Gamma}\oplus X_\gamma,\quad X_\gamma=P_{e_\gamma}X,\quad \gamma\in\Gamma$$

and e_γ is a weak unit of X_γ.

Let $\varepsilon>0$ and $\{\Gamma_i^{(\varepsilon)}\}_{i=1}^{\infty}$ be a partition of Γ according to (**). We shall show that

$$(14)\qquad \max_{x^*\in X^*}\ \operatorname{card}(\{\gamma\in\Gamma_i^{(\varepsilon)};\ |x^*|(e_\gamma)>\varepsilon\|x^*\|\})<i.$$

Suppose the contrary, i. e. there exist $x^*\in X^*$ and $\{\gamma_j\}_{j=1}^{i}\subset\Gamma_i^{(\varepsilon)}$, $\gamma_j\neq\gamma_k$, $j\neq k$ so that $|x^*|(e_{\gamma_j})>\varepsilon\|x^*\|$, $j=1, 2, \ldots, i$. Then,

$$\varepsilon i\|x^*\|<|x^*|\Big(\sum_{j=1}^{i} e_{\gamma_j}\Big)\leq\|x^*\|\ \Big\|\sum_{j=1}^{i} e_{\gamma_j}\Big\|,$$

whence $\|\Sigma_{j=1}^{i}e_{\gamma_j}\|>\varepsilon i$, which is a contradiction. Consequently (14) holds.

Since for each $\gamma\in\Gamma$, X_γ is an order continuous Banach lattice with a weak unit e_γ, then (cf. e. g. [8], p. 25) there exists a probability space $(S_\gamma,$

$\Sigma_\gamma, \mu_\gamma)$, a Köthe function space $\widetilde{X}_\gamma$ on $(S_\gamma, \Sigma_\gamma, \mu_\gamma)$ and an order isometry $T_\gamma : X_\gamma \to \tilde{X}_\gamma$. Moreover, $\widetilde{X}^*_\gamma$ can be identified with the Köthe function space on $(S_\gamma, \Sigma_\gamma, \mu_\gamma)$ consisting of all μ_γ-measurable functions g for which

$$\| g \|_{\widetilde{X}^*_\gamma} = \sup \{ \int_{S_\gamma} fg d_{\mu_\gamma} ; \| f \|_{\widetilde{X}_\gamma} \leq 1 \} < \infty .$$

In the sequel, we shall briefly write (S_γ, Σ, μ).

For each $\gamma \in \Gamma$, S_γ can be represented as follows

$$(15) \qquad S_\gamma = (\bigcup_{m=1}^{\infty} A_\gamma^m) \cup \Omega_\gamma ,$$

with A_γ^m, $m=1, 2, \dots$ atoms and Ω_γ free of atoms.

Denote $f_\gamma = T_\gamma e_\gamma$, $\gamma \in \Gamma$. Since e_γ is a weak unit of X_γ and T_γ is an order isometry, then $f_\gamma(s) > 0$ a. e. in S_γ. Put

$$\Omega_\gamma^m = \{ s \in \Omega_\gamma ; \quad \frac{1}{m} \leq f_\gamma(s) < \frac{1}{m-1} \}, \quad m=1, 2, \dots$$

Consequently,

$$(16) \qquad \Omega_\gamma = \bigcup_{m=1}^{\infty} \Omega_\gamma^m , \quad \gamma \in \Gamma .$$

Let $x^* \in X^*$. Put $g_\gamma = T_\gamma^{*^{-1}}(x_\gamma^*)$, $\gamma \in \Gamma$, where x_γ^* is the restriction of x^* to X_γ. Define for $t \in [0, 1]$

$$\tilde{g}_\gamma^m(t) = \sup_{B \subset \Omega_\gamma^m,\ \mu(B) \leq t} (\int_B | g_\gamma | d\mu).$$

Since $T_\gamma^{*^{-1}}$ is an order isometry, then $T_\gamma^{*^{-1}}(| x_\gamma^* |) = | g_\gamma |$. Hence,

$$\tilde{g}_\gamma^m(1) = \int_{\Omega_\gamma^m} | g_\gamma | d\mu = m \int_{\Omega_\gamma^m} \frac{1}{m} | g_\gamma | d\mu$$

$$\leq m \int_{\Omega_\gamma^m} f_\gamma | g_\gamma | d\mu \leq m | x^* | (e_\gamma),$$

i. e.

$$(17) \qquad \tilde{g}_\gamma^m(1) \leq m | x^* | (e_\gamma), \quad \gamma \in \Gamma .$$

Define operators $V_m : X^* \to l_\infty(\Gamma)$, $m=1, 2, \dots$ by the formula

$$V_m x^*(\gamma) = \frac{1}{m} (\int_0^1 \tilde{g}_\gamma^{m^2}(t) dt)^{1/2}, \quad \gamma \in \Gamma .$$

It is clear that V_m are sublinear and positive-homogeneous. By (17), we obtain $V_m x^*(\gamma) \leq m^{-1} \tilde{g}_\gamma^m(1) \leq | x^* | (e_\gamma) \leq \| x^* \|$, $\gamma \in \Gamma$, i. e. V_m are bounded.

Next, for each $\gamma \in \Gamma$, $m=1, 2, \ldots$ we have the inequality

$$|x^*|(T_\gamma^{-1}(f_\gamma \chi_{A_\gamma^m})) \leq |x^*|(e_\gamma). \tag{18}$$

Thus, if we define oporators $W_m: X^* \to l_\infty(\Gamma)$, $m=1, 2, \ldots$ by the formula

$$W_m x^*(\gamma) = |x^*|(T_\gamma^{-1}(f_\gamma \chi_{A_\gamma^m})), \quad \gamma \in \Gamma,$$

then $W_m x^*(\gamma) \leq \|x^*\|$, i. e. W_m are bounded. Moreover, W_m are sublinear and positive-homogeneous.

We introduce in the dual space X^* an equivalent lattice norm by the formula

$$|||x^*||| = \{\|x^*\|^2 + \sum_{m,i,k=1}^{\infty} 2^{-m-i-k}[D^2(V_{m,i}^{(k)}x^*) + D^2(W_{m,i}^{(k)}x^*)]\}^{1/2},$$

where $V_{m,i}^{(k)} = P_{\Gamma_i(1/k)} V_m$, $W_{m,i}^{(k)} = P_{\Gamma_i(1/k)} W_m$, $m, i, k = 1, 2, \ldots$

In the sequel we shall use the following assertion:

Lemma 4.3 [6]. *Let Y be an order continuous Köthe function space on a probability space (S, Σ, μ), free of atoms, and Y^* be its dual (which is also a Köthe function space). If we define in Y^* an equivalent norm*

$$|||g|||_1 = (\|g\|^2 + \int_0^1 \tilde{g}^2(t)dt)^{1/2}, \quad g \in Y^*,$$

then $|||\cdot|||_1$ is w^*-uniformly convex and w^*-lower semi-continuous.

Thus, we obtain that the norm $|||\cdot|||$ in X^* is w^*-lower semi-continuous, i. e. it is induced by an equivalent lattice norm $|||\cdot|||$ in X,

$$|||x||| = \sup\{|x^*(x)|;\ |||x^*||| \leq 1\}, \quad x \in X.$$

We shall show that the norm $|||\cdot|||$ in X is uniformly differentiable in every direction using a similar argument to that in [10]. Suppose the contrary. Then there exist x_n, $y \in X$ and numbers $\tau_n \to 0$, $a > 0$ so that $|||x_n||| = 1$, $|||y||| = 1$ and

$$|||x_n + \tau_n y||| + |||x_n - \tau_n y||| - 2 > 5a|\tau_n|. \tag{19}$$

Since X is order continuous, then by (13), (15) and (16), there exists an element $z \in X$ such that

$$|||y - z||| \leq a \tag{20}$$

and

$$\operatorname{card}(\{(\gamma, m);\ T_\gamma(z)\chi_{(A_\gamma^m \cup \Omega_\gamma^m)} \neq 0\}) < \infty. \tag{21}$$

Approximating each x_n in a similar way, we may choose x_n^*, $y_n^* \in X^*$ with the following properties:

$$|||x_n^*||| \leq 1, \quad |||y_n^*||| \leq 1; \tag{22}$$

$$|||x_n + \tau_n z||| \leq x_n^*(x_n + \tau_n z) + a|\tau_n|. \tag{23}$$

$$|||x_n-\tau_n z|||\leqq y_n^*(x_n-\tau_n z)+a|\tau_n|;$$

(24) $$\operatorname{card}(\{\gamma\,;\,x^*_{n,\gamma}\neq 0\}\cup\{\gamma\,;\,y^*_{n,\gamma}\neq 0\})<\infty,$$

where $x^*_{n,\gamma}$, $y^*_{n,\gamma}$ are the restrictions of x_n^*, y_n^* to X_γ. It is easy to see that by (19), (20), (22) and (23) we get

$$|||x_n^*+y_n^*|||-2+|\tau_n|\;|x_n^*(z)-y_n^*(z)|\geqq a|\tau_n|.$$

The last inequality and (22) imply

(25) $$|x_n^*(z)-y_n^*(z)|\geqq a,\quad n=1,\,2,\ldots$$

and

(26) $$|||x_n^*+y_n^*|||\to 2.$$

It follows from (24) that $V_m x_n^*$, $V_m y_n^*$, $W_m x_n^*$, $W_m y_n^*\in c_0(\Gamma)$, $m,\,n=1,\,2,\ldots$ Moreover, by (14), (17) and (18), we obtain that V_m and W_m, $m=1,\,2,\ldots$ satisfy the assumptions of Proposition 3.2. Consequently, (22) and (26) give

(27) $$\lim_{n\to\infty}(V_m x_n^*(\gamma)-V_m y_n^*(\gamma))=0,\quad \gamma\in\Gamma,\quad m=1,\,2,\ldots$$

and

(28) $$\lim(W_m x_n^*(\gamma)-W_m y_n^*(\gamma))=0,\quad \gamma\in\Gamma,\quad m=1,\,2,\ldots$$

Since $|||(x_n^*+y_n^*)/2|||\leqq|||y_n^*|||/3+(2/3)|||(3x_n^*+y_n^*)/4|||$, we conclude from (22) and (26) that

(29) $$|||(3x_n^*+y_n^*)/4|||\to 1.$$

Thus, by (22), (26) and (29), applying Proposition 3.2 to the sequences $\{(x_n^*+y_n^*)/2\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty$, we obtain

(30) $$\lim_{n\to\infty}\Big(V_m\Big(\frac{x_n^*+y_n^*}{2}\Big)(\gamma)-V_m x_n^*(\gamma)\Big)=0,\quad \gamma\in\Gamma,\quad m=1,\,2,\ldots$$

and

(31) $$\lim_{n\to\infty}\Big(W_m\Big(\frac{x_n^*+y_n^*}{2}\Big)(\gamma)-W_m x_n^*(\gamma)\Big)=0,\quad \gamma\in\Gamma,\quad m=1,\,2,\ldots$$

The definition of W_m, (28) and (31) give

$$\lim_{n\to\infty}(x_n^*-y_n^*)(T_\gamma^{-1}f_\gamma\chi_{A_\gamma^m})=0,\quad \gamma\in\Gamma,\quad m=1,\,2,\ldots$$

The last equality implies

(32) $$\lim_{n\to\infty}T_\gamma^{*^{-1}}(x^*_{n,\gamma}-y^*_{n,\gamma})(T_\gamma(z)\cdot\chi_{A_\gamma^m})=0,\quad \gamma\in\Gamma,\quad m=1,\,2,\ldots$$

Consider the space $\tilde{X}^m_\gamma = \{f \in \tilde{X}_\gamma;\ \operatorname{supp} f \subset \Omega^m_\gamma\}$ and its dual space $\tilde{X}^{m*}_\gamma = \{g \in \tilde{X}^*_\gamma;\ \operatorname{supp} g \subset \Omega^m_\gamma\}$. Since $\tilde{X}^m_\gamma$ is order continuous, by (27), (30) and Lemma 4.3, we obtain

$$\lim_{n\to\infty} T^{*^{-1}}_\gamma (x^*_{n,\gamma} - y^*_{n,\gamma})(T_\gamma(z)\,.\,\chi_{\Omega^m_\gamma}) = 0, \quad \gamma \in \Gamma, \quad m = 1, 2, \ldots \tag{33}$$

From (21), (32) and (33) we deduce that $x^*_n(z) - y^*_n(z) \to 0$, which contradicts (25). This concludes the proof of Theorem 4.2.

We shall apply Theorem 4.2 to the symmetric Köthe function spaces.

Theorem 4.4. *Let X be a symmetric Köthe function space. Then the space X admits an equivalent norm, uniformly differentiable in every direction, if and only if X does not contain any subspace isomorphic to $l_1(\Gamma)$ for uncountable set Γ.*

Lemma 4.5. *Let X be a Köthe function space. If there exists an element $x \in X$ such that the measure on* $\operatorname{supp} x$ *is not σ-finite, then X contains a subspace isomorphic to l_∞. Moreover, the unit vectors of l_∞ correspond, under this isomorphism, to mutually disjoint elements of X.*

Proof. Let $x \in X$ satisfy the assumptions of Lemma 4.5. It is no loss of generality to assume that $x \geq 0$. Then there exists an $\eta > 0$ such that the measure on the set $B = \{x \geq \eta\}$ is not σ-finite. Since X is a Köthe function space, the inequality $\eta\chi_B \leq x$ implies $\chi_B \in X$. Since the measure on B is not σ-finite, we may construct by transfinite induction a family $\{B_\gamma\}_{\gamma\in\Gamma}$, card $(\Gamma) = \aleph_1$, of mutually disjoint measurable sets with $0 < \mu(B_\gamma) < \infty$, $B_\gamma \subset B$, $\gamma \in \Gamma$. In particular, we may choose a sequence of different indices $\{\gamma_i\}_{i=1}^\infty \subset \Gamma$ so that $0 < c_1 \leq \|\chi_{B_{\gamma_i}}\|$, $i = 1, 2, \ldots$ Moreover, $\|\chi_{B_{\gamma_i}}\| \leq \|\chi_B\| < \infty$, $i = 1, 2, \ldots$ Since $B_{\gamma_i} \cap B_{\gamma_j} = \emptyset$, $i \neq j$ and X is a Köthe function space, it follows from the above inequalities that for any bounded sequence $\{b_i\}_{i=1}^\infty$ of reals,

$$c_1 \sup_i |b_i| \leq \Big\| \sum_{i=1}^\infty b_i \chi_{B_{\gamma_i}} \Big\| \leq \|\chi_B\| \sup_i |b_i|,$$

which completes the proof.

Proposition 4.6. *Let X be a symmetric Köthe function space on a measure space (S, Σ, μ) with non σ-finite measure. Then X admits an equivalent norm, uniformly differentiable in every direction, if and only if X is order continuous and satisfies the following condition:*

$$\inf\{\|\chi_B\|/\mu(B);\ B \in \Sigma,\ t \leq \mu(B) < \infty\} = 0 \quad \textit{for any} \quad t > 0. \tag{***}$$

Proof. Let X be order continuous. Then, by a theorem of Kakutani (cf. e. g. [8, p. 9]), the space X can be decomposed into a direct sum

$$X = \sum_{\gamma\in\Gamma} \oplus X_\gamma, \tag{34}$$

where for each $\gamma \in \Gamma$, X_γ is an ideal with a weak unit e_γ and $e_\gamma \wedge e_\beta = 0$, $\gamma \neq \beta$.

Since l_∞ is not order continuous, we get from Lemma 4.5 the measure on $\operatorname{supp} x$ is σ-finite for each $x \in X$, i. e.

$$\operatorname{supp} e_\gamma = \bigcup_{i=1}^{\infty} B_\gamma^i, \ \gamma \in \Gamma \quad \text{with} \quad B_\gamma^j \cap B_\gamma^i = \emptyset, \quad i \neq j, \quad \mu(B_\gamma^i) < \infty.$$

Considering another index set Γ, with no loss of generality we may assume that

$$(35) \qquad e_\gamma = \chi_{B_\gamma}, \quad 0 < \mu(B_\gamma) < \infty, \ \mu(B_\gamma \cap B_\beta) = 0 \quad \gamma \neq \beta.$$

The family $\{e_\gamma\}_{\gamma\in\Gamma}$ is obviously a generalized weak unit of X. We shall show that (***) implies (**). Put $\Gamma_m = \{\gamma \in \Gamma;\ m^{-1} \leq \mu(B_\gamma) \leq m\}$, $m = 1, 2, \ldots$. By (34), $\Gamma = \bigcup_{m=1}^{\infty} \Gamma_m$.

Let $\varepsilon > 0$. Fix an integer m and denote $\delta = \varepsilon/(2mC)$ where C is the symmetric constant of X. By (***), there exists a measurable set A such that

$$(36) \qquad m \leq \mu(A) < \infty, \quad \| \chi_A \| < \delta\mu(A).$$

Put $i = [\mu(A)/m]$. Thus, (36) implies $i > 0$. Let $\{\gamma_j\}_{j=1}^{i} \subset \Gamma_m$, $\gamma_j \neq \gamma_k$, $j \neq k$ and denote $B = \bigcup_{j=1}^{i} B_{\gamma_j}$. Since $\mu(B) \leq im \leq \mu(A)$, then $\| \chi_B \| \leq C \| \chi_A \|$. Hence, (36) gives

$$\| \sum_{j=1}^{i} \chi_{B_{\gamma_j}} \| < C \frac{\varepsilon}{2mC} \mu(A) \leq \varepsilon i.$$

Renumbering Γ_m, we thus obtain the condition (**). Therefore, it follows from Theorem 4.2 that X has an equivalent lattice norm which is uniformly differentiable in every direction.

The "only if" part. Since l_∞ has no equivalent Gateaux differentiable norm (cf. [3]), then by Lemma 4.5 the measure on $\operatorname{supp} x$ is σ-finite for each $x \in X$. Hence (cf. e. g. [8, p. 29]), X is σ-order complete. Thus, since X does not contain a subspace isomorphic to l_∞, it follows from a theorem of Lozanovskii (see e. g. [8, p. 7]) that X is order continuous. Consequently, (34) and (35) hold.

Since μ is not σ-finite, then Γ is uncountable. Hence, there exists a constant $c_1 > 0$ and a subset $\Gamma_0 \subset \Gamma$ with Γ_0 uncountable so that

$$(37) \qquad \| e_\gamma \| \geq c_1, \quad \mu(B_\gamma) \geq c_1, \quad \gamma \in \Gamma_0.$$

Fix an arbitrary $t > 0$. Let $\varepsilon > 0$. Put

$$(38) \qquad \delta = \min(\varepsilon c_1,\ c_1^2/t).$$

Then, by Theorem 4.2, there exists a corresponding partition $\{\Gamma_i^{(\delta)}\}_{i=1}^{\infty}$ of the set Γ. Since Γ_0 is uncountable, we may find an integer k such that $\operatorname{card}(\Gamma_k^{(\delta)} \cap \Gamma_0) \geq k$. Let $\{\gamma_j\}_{j=1}^{k} \subset \Gamma_k^{(\delta)} \cap \Gamma_0$, $\gamma_j \neq \gamma_q$, $j \neq q$. Combining this with (37), we have

$$(39) \qquad c_1 \leq \| \sum_{j=1}^{k} e_{\gamma_j} \| < k\delta.$$

From (38) and (39) follows immediately

$$(40) \qquad t < c_1 k.$$

Denote $B=\cup_{j=1}^{k}B_{\gamma_j}$. By (37) and (40), we obtain

$$\mu(B)\geq c_1k>t. \tag{41}$$

Moreover, (38), (39) and (41) give

$$\|\chi_B\|=\|\sum_{j=1}^{k}e_{\gamma_j}\|<\varepsilon c_1k\leq\varepsilon\mu(B).$$

Then, (41) and the above inequality imply that

$$\inf\{\|\chi_B\|/\mu(B);\quad t\leq\mu(B)<\infty\}=0,$$

which concludes the proof.

L e m m a 4.7 [6]. *Let X be a Köthe function space on a σ-finite measure space. Then X admits an equivalent lattice norm, uniformly differentiable in every direction, if and only if X contains no subspace, isomorphic to l_∞.*

L e m m a 4.8 (cf. e. g. [7, p. 120]). *Let $\{x_n\}_{n=1}^{\infty}$ be a symmetric basis of a Banach space X such that $\limsup_n\|x_1+x_2+\ldots+x_n\|/n>0$. Then $\{x_n\}_{n=1}^{\infty}$ is equivalent to the natural basis of l_1.*

4.9. P r o o f o f T h e o r e m 4.4. The "only if" part is an immediate consequence of the fact that $l_1(\Gamma)$ with Γ uncountable has no equivalent Gateaux differentiable norm (see [3]).

The "if" part. Since l_∞ contains a subspace isomorphic to $l_1(\Gamma)$ with $\operatorname{card}(\Gamma)>\aleph_0$ (cf. e. g. [1, p. 254]), then X contains no subspace isomorphic to l_∞. If the measure μ is σ-finite the assertion thus results in view of Lemma 4.7.

Let now the measure be non-σ-finite. Since X contains no subspace isomorphic to l_∞, then by Lemma 4.5, X is σ-order complete and by a theorem of Lozanovskii (cf. e. g. [8, p. 7]), X is order continuous. Hence, in view of Proposition 4.6, it remains to show that (***) holds. Suppose the contrary, i. e. there exist t_1, $c_1>0$ so that

$$\|\chi_B\|\geq c_1\mu(B)\quad\text{for any}\quad B\in\Sigma\quad\text{with}\quad t\leq\mu(B)<\infty. \tag{42}$$

Since μ is non-σ-finite, we may construct by transfinite induction an uncountable family $\{B_\gamma\}_{\gamma\in\Gamma}$ of mutually disjoint measurable subsets with finite measure so that

$$\inf_{\gamma\in\Gamma}\mu(B_\gamma)\geq t,\quad \sup_{\gamma\in\Gamma}\|\chi_{B_\gamma}\|\leq c_2. \tag{43}$$

Then, for any finite system $\{\gamma_i\}_{i=1}^{n}\subset\Gamma$, by (42) and (43), we obtain

$$\|\sum_{i=1}^{n}\chi_{B_{\gamma_i}}\|=\|\chi_{\bigcup_{i=1}^{n}B_{\gamma_i}}\|\geq c_1\mu(\bigcup_{i=1}^{n}B_{\gamma_i})\geq c_1tn. \tag{44}$$

It is easy to verify that $\{\chi_{B_\gamma}\}_{\gamma\in\Gamma}$ is a symmetric basic sequence. Therefore by (44) and Lemma 4.8, $\{\chi_{B_\gamma}\}_{\gamma\in\Gamma}$ is equivalent to the natural basis of the space $l_1(\Gamma)$. The contradiction shows that X admits an equivalent lattice norm which is uniformly differentiable in every direction.

REFERENCES

1. C. Bessaga, A. Pelczynski. Selected topics in infinite-dimensional topology. Warszawa, 1975.
2. W. Davis, N. Ghoussoub, J. Lindenstrauss. A lattice renorming theorem and applications to vector-valued processes. *Trans. Amer. Math. Soc.*, **263**, 1981, 531-540.
3. M. Day. Strict convexity and smoothness. *Trans. Amer. Math. Soc.*, **78**, 1955, 516-528.
4. M. Day, R. James, S. Swaminathan. Normed linear spaces that are uniformly convex in every direction. *Canad. J. Math.*, **23**, 1971, 1051-1059.
5. M. Day. Normed linear spaces. Berlin — Heidelberg — New York, 1973.
6. D. N. Kutzarova, S. L. Troyanski. On equivalent lattice norms which are uniformly convex or uniformly differentiable in every direction in Banach lattices with a weak unit. *Serdica*, **9**, 1983, 249-262.
7. J. Lindenstrauss, L. Tzafriri. Classical Banach spaces I. Sequence spaces. Berlin—Heidelberg — New York, 1977.
8. J. Lindenstrauss, L. Tzafriri. Classical Banach spaces II. Function spaces. Berlin — Heidelberg — New York, 1978.
9. S. L. Troyanski. On non-separable Banach spaces with a symmetric basis. *Studia Math.*, **53**, 1975, 153-163.
10. С. Л. Троянски. О равномерной выпуклости и гладкости в каждом направлении в несепарабельных пространствах Банаха с безусловным базисом. *Доклады БАН*, **30**, 1977, 1243—1246.
11. В. Л. Шмульян. О дифференцируемости нормы в пространстве Банаха. *ДАН СССР*, 27, 1940, 643—648.

Centre for Mathematics and Mechanics
Sofia 1090 P. O. Box 373

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