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STABLE LIMITS FOR SUMS OF M-DEPENDENT RANDOM VARIABLES

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The paper deals with the convergence of sums of stationary connected m-dependent random variables to a given stable limit distribution function. Especially the normal approximation of sums of m-dependent random variables with not necessary finite variances is considered. Furthermore, some bounds for the accuracy of the stable approximation are derived. The investigations are based on an estimate of the logarithm of the characteristic function of a sum of m-dependent random variables, which is proved in an earlier paper of the author. Two examples which are of interest in themselves illustrate the obtained general results.

1. Introduction. We consider a sequence $(X_k)_{k=1,2,\dots}$ of m-dependent random variables (rv's) defined on a probability space $(\Omega, \mathfrak{A}, P)$. We recall that a sequence $(X_k)_{k=1,2,\dots}$ of rv's is called m-dependent if for $1 \leqq s < t \leqq u$, $t-s > m$, the random vectors $(X_1, \dots, X_s)$ and $(X_t, \dots, X_u)$ are independent. We put $S_n = X_1 + \dots + X_n$, $F_X(x) = P(X<x)$, and $f_X(t) = Ee^{itX}$, where X stands for an arbitrary rv on $(\Omega, \mathfrak{A}, P)$. Further, $G_{\alpha\beta}(x, c)$ denotes the stable distribution function (df) whose characteristic function (cf) $g_{\alpha\beta}(t, c) = \int_{-\infty}^{\infty} e^{itx}\, dG_{\alpha\beta}(x, c)$ has the logarithm

$$\ln g_{\alpha\beta}(t, c) = -c\,|t|^{\alpha}(1 - i\beta \operatorname{sgn} t\, \omega(t, \alpha)),$$

where $0<\alpha \leqq 2$, $|\beta| \leqq 1$, $c>0$ and $\omega(t, \alpha) = \frac{2}{\pi} \ln|t|$, if $\alpha = 1$, $= \tan(\pi\alpha/2)$, else.

We briefly write $\Phi(x) = G_{20}(x, 1/2)$.

The main purpose of the present paper is to find appropriate conditions which ensure that discrepancy

$$D_n^{(\alpha,\beta)}(x, c) = F_{Z_n}(x) - G_{\alpha\beta}(x, c) \quad \text{with} \quad Z_n = (S_n = A_n)/B_n$$

uniformly tends to zero, where the centering and norming sequences A_n and B_n are suitably chosen. Furthermore, some estimates of the uniform error $\sup_x |D_n^{(\alpha,\beta)}(x, c)|$ are given. At the end of the paper some instructive examples illustrate the results in Section 2 and 4. The approximation of a given stable distributed rv by a suitably standardized sum of independent identically distributed (i. i. d.) rv's a is a well-studied problem in probability theory (see e. g. 2; 7; 8; 9]). The corresponding question for sums of weakly dependent rv's is more complicated. In [7] it is shown that in the case of a stationary α-mixing sequence (in the sense of M. Rosenblatt) the possible limit distributions of Z_n are stable and, further, if the limit df has exponent α, $0<\alpha\leqq 2$, then B_n is equal to $n^{1/\alpha}h(n)$, where $h(x)$ is a slowly varying function as $x\to\infty$, i. e. $h(xv)/h(v)\to 1$ as $v\to\infty$ for every $x>0$. The convergence of the df of a sum of φ-mixing (in the sense of I. A. Ibragimov; in particular m-dependent) rv's to an infinitely divisible df is treated in [1]. Conditions under which the df of a sum of φ-mixing Markov-dependent rv's converges to a given infinitely divisible df were found in [4]. However, it is still an unsolved problem for stationary m-dependent sequences, to find out which conditions on S_m, S_{2m} (and S_{3m}) guarantee $\sup_x |D_n^{(\alpha,\beta)}(x, c)| \to 0$ as $n \to \infty$ and enable to estimate the rate of convergence. Some results in this direction are obtained for Markov-dependent rv's in [10]. Under special dependence assumptions some stable limit theorems are proved in [3]. Estimates of the concentration function of m-dependent and Markov-dependent sums, which converge in distribution to a stable distributed rv, are given in [6]. The fundamental estimate to carry out the following investigations was derived in [5] by using a product representation of $f_{S_n}(t)$ in some neighbourhood of $t=0$.

Lemma 1. *Let $X_1, X_2, \ldots$ be a sequence of 1-dependent rv's. If $\max_{1\le k\le n} E|e^{itX_k}-1|^2 \le 1/36$, then*

$$|\ln f_{S_n}(t) - \sum_{k=1}^{n} (f_{X_k}(t)-1) - \sum_{k=2}^{n} E(e^{itX_{k-1}}-1)(e^{itX_k}-1)$$

$$-\sum_{k=3}^{n} E(e^{itX_{k-2}}-1)(e^{itX_{k-1}}-1)(e^{itX_k}-1)| \le C_1 \max_{1\le k\le n} |f_{X_k}(t)-1| \sum_{k=1}^{n} |f_{X_k}(t)-1|.$$

Here and below $C_1, C_2, \ldots$ denote absolute positive constants (not depending on m and n). Lemma 1 is an immediate consequence of Lemma 3. 3 in [5]. That's why we omit its proof. Let $Y_1, Y_2, \ldots$ be a sequence of m-dependent rv's. Then it is obvious that the rv's $X_k = Y_{m(k-1)+1} + \ldots + Y_{mk}$, $k=1, 2, \ldots$, form a 1-dependent sequence. For the sake of convenience we assume that $N=n/m$ is an integer. Then we can reformulate Lemma 1 in the following way.

Lemma 2. *Let $Y_1, Y_2, \ldots$ be a strictly stationary sequence of m-dependent rv's. If $E|e^{itX_1}-1|^2 \le 1/36$, then*

$$\text{(1.1)} \quad |\ln f_{S_n}(t) - N(f_{X_1}(t)-1+E(e^{itX_1}-1)(e^{itX_2}-1)+E(e^{itX_1}-1)$$
$$\times(e^{itX_2}-1)(e^{itX_3}-1))| \le C_2 |f_{X_1}(t)-1| (1+N|f_{X_1}(t)-1|).$$

Here we have only used the estimates

$$|E(e^{itX_1}-1)(e^{itX_2}-1)| \le E|e^{itX_1}-1|^2 \le 2|f_{X_1}(t)-1|$$

and

$$\text{(1.2)} \quad |E(e^{itX_1}-1)(e^{itX_2}-1)(e^{itX_3}-1)| \le (E|e^{itX_1}-1|^2)^{3/2}.$$

Because the dependence between $Y_1, Y_2, \ldots$, and Y_m can be very strong we shall put the conditions on the sum $X_1 = Y_1 + Y_2 + \ldots + Y_m$. In concrete situations one has to check the conditions by utilizing the given dependence stucture between $Y_1, \ldots, Y_m$. Therefore, with exception of Section 3, we only consider 1-dependent rv's.

2. Convergence to a stable law with exponent α, $0<\alpha<2$. It is well-known that the stable df $C_{\alpha\beta}(x, c)$, $0<|\beta|\le 1$, possesses some extreme properties, especially in view of the choice of the centering sequence A_n. Therefore, to formulate the results of this section we distringuish two cases.

Theorem 1A. *Let $X_1, X_2, \ldots$ be a strictly stationary sequence of 1-dependent rv's, $0<\alpha<2$, $\alpha\neq 1$, $|\beta|\leq 1$ and $\alpha=1$, $\beta=0$, respectively. Suppose that, as $t\to 0$,*

$$f_{X_1}(t)\, e^{-iat}-1=-|t|^{\alpha}h_1(t)(1-i\beta\operatorname{sgn} t\,\omega(t,\alpha))(1+o(1))$$

and

$$f_{X_1+X_2}(t)\, e^{-2iat}-1=-|t|^{\alpha}h_2(t)(1-i\beta\operatorname{sgn} t\,\omega(t,\alpha))(1+o(1)),$$

where a is a constant and the functians $h_1(t)$, $h_2(t)$ are positive, continuous, and slowly varying (in the sense of Karamata, see e, g. [7]) as $t\to 0$. Further, assume that

$$\limsup_{t\to 0} h_1(t)/h_2(t)=b<1 \tag{2.1}$$

and put

$$A_n=an,\ h(t)=h_2(t)-h_1(t)\ \text{and}\ B_n^{-1}=\inf\{t>0: |t|^{\alpha}h(t)=\frac{1}{n}\}.$$

Then

$$\sup_x |D_n^{(\alpha,\beta)}(x, 1)| \xrightarrow[n\to\infty]{} 0$$

holds.

Proof of Theorem 1A. Let T be an arbitrary positive real number. Then it suffices to prove

$$\lim_{n\to\infty} f_{Z_n}(t)=g_{\alpha,\beta}(t, 1)\quad \text{for}\quad |t|\leq T. \tag{2.2}$$

Put $X_{kn}=(X_k-a)/B_n$, $k=1,\ldots,n$. It follows from Lemma 2 and (2.2) that, for all t with $|f_{X_{1n}}(t)-1|\leq 1/72$,

$$\begin{aligned}|\ln f_{Z_n}(t)-n(f_{X_{1n}+X_{2n}}(t)-f_{X_{1n}}(t))|\\ \leq C_3|f_{X_{1n}}(t)-1|^{1/2}(1+n|f_{X_{1n}}(t)-1|).\end{aligned} \tag{2.3}$$

Since $B_n\to\infty$ relation (2.3) holds for $|t|\leq T$ if n is large enough. Using the notations and assumptions of Theorem 1A we get

$$n|f_{X_{1n}}(t)-1|\leq C_4|t|^{\alpha}\frac{h_1(t/B_n)}{h_1(1/B_n)}\frac{h_1(1/B_n)}{h(1/B_n)}\leq C_5\frac{b}{1-b}|t|^{\alpha} \tag{2.4}$$

for $|t|\leq T$ and large enough n.

By virtue of (2.1) we have

$$\left|\frac{h(vx)}{h(v)}-1\right|\leq\left(\left|\frac{h_1(vx)}{h_1(v)}-1\right|\frac{h_1(v)}{h_2(v)}+\left|\frac{h_2(vx)}{h_2(v)}-1\right|\right)/\left(1-\frac{h_1(v)}{h_2(v)}\right)\longrightarrow 0$$

as $v\to 0$ for every $x>0$, i. e. $h(x)$ is slowly varying (in the sense of Karamata) as $x\to 0$.

Therefore, our assumptions imply

$$\lim_{n\to\infty} n(f_{X_{1n}+X_{2n}}(t)-f_{X_{1n}}(t))=\ln g_{\alpha\beta}(t,\ 1)\lim_{n\to\infty}\frac{n}{B_n^\alpha}h(1/B_n)\frac{h(t/B_n)}{h(1/B_n)}$$

$$=\ln g_{\alpha\beta}(t,\ 1)$$

for $|t|\leq T$.

Thus, from (2.3) and (2.4) we obtain (2.2) and so Theorem 1A is completely proved. □

Theorem 1B. *Let $X_1, X_2, \ldots$ be a strictly stationary sequence of 1-dependent rv's. Suppose that there exist sequences A_n and $B_n>0$ and real numbers c_1, c_2 with $0<c_1<c_2$ such that*

$$f^n_{X_1}(t/B_n)\, e^{-iA_nt/B_n}\xrightarrow[n\to\infty]{} g_{1\beta}(t,\ c)$$

and

$$f^n_{X_1+X_2}(t/B_n)\, e^{-2iA_nt/B_n}\xrightarrow[n\to\infty]{} g_{1\beta}(t,\ c_2)\quad \text{for}\quad |\beta|\leq 1 \text{ and every real } t.$$

Then

$$\sup_x |D_n^{(1,\beta)}(x,\ c_2-c_1)|\xrightarrow[n\to\infty]{} 0$$

holds.

To verify Theorem 1B we put again $X_{kn}=(X_k-A_n/n)/B_n$, $k=1,\ldots,n$ and make use of the fact that

$$f_{X_{1n}}(t)-1=\ln f_{X_{1n}}(t)\,(1+o(1))\quad \text{as}\quad n\longrightarrow\infty$$

for $|t|\leq T$. Then by repeating the steps in the foregoing proof of Theorem 1A we get the assertion of Theorem 1B.

3. Convergence to the normal law. The normal approximation of sums of m-dependent rv's is a fairly well-studied problem if at least the second moment of the summands exists (see e. g. [5, 7]). Here we shall consider the normal convergence for sums of m-dependent rv's with not necessary finite variances.

Note that for arbitrary complex numbers $z_1, \ldots, z_m$ with $|z_i|\leq 1$, $i=1,\ldots,m$,

$$\Big|\prod_{i=1}^m z_i-1-\sum_{i=1}^m(z_i-1)\Big|\leq 2\sum_{1\leq i<j\leq m}|z_i-1|\,|z_j-1|$$

and

$$\Big|\prod_{i=1}^m z_i-1)-\sum_{i=1}^m(z_i-1)-\sum_{1\leq i<j\leq m}(z_i-1)(z_j-1)\Big|\leq 2\sum_{1\leq i<j<k\leq m}|z_i-1|\,|z_j-1|\,|z_k-1|.$$

By elementary, but rather lengthy computations we get from the last relation and Hölder's inequality that, for a stationary m-dependet sequence $Y_1, Y_2, \ldots$,

$$|f_{X_1}(t)-1+E(e^{itX_1}-1)(e^{itX_2}-1)-m(f_{Y_1}(t)-1+\sum_{j=1}^m E(e^{itY_1}-1)(e^{itY_{j+1}}-1))|$$

$$\leq 2\left(\binom{2m}{3}+\binom{m}{3}\right)E|e^{itY_1}-1|^3+\binom{m+1}{2}|f_{Y_1}(t)-1|^2.$$

$$E|e^{itX_1}-1|^2 \leq m^2 E|e^{itY_1}-1|^2,$$

and $|f_{X_1}(t)-1| \leq 2m^2|f_{Y_1}(t)-1|$. From these estimates and Lemma 2 we can deduce the following

Lemma 3. *Let $Y_1, Y_2, \ldots$ be a strictly stationary sequence of m-dependent rv's. If $|f_{Y_1}(t)-1| \leq 1/72\, m^2$, then*

$$\text{(3.1)} \qquad |\ln f_{S_n}(t) - n(f_{Y_1}(t)-1+\sum_{j=1}^{m} E(e^{itY_1}-1)(e^{it\,Y_{j+1}}-1)|$$

$$\leq C_6\, m^2 |f_{Y_1}(t)-1| + C_7\, nm^2(|f_{Y_1}(t)-1|^{3/2}+E|e^{itY_1}-1|^3).$$

It is easily seen that (3.1) holds also true if n is not a multiple of m.

Theorem 2. *Let $Y_1, Y_2, \ldots$ be a strictly stationary sequence of m-dependent rv's (m fixed) with $EY_1=0$* (i. e. $A_n=0$). *Suppose that, as $t\to 0$,*

$$\text{(3.2)} \qquad f_{Y_1}(t)-1 = -\frac{t^2}{2} h_0(t)(1+o)(1))$$

and

$$\text{(3.3)} \qquad f_{Y_1+Y_{1+j}}(t)-1 = -\frac{t^2}{2} h_j(t)\ (1+o(1)),\quad j=1,\ldots,m,$$

where the functions $h_j(t)$, $j=0, 1, \ldots, m$, are positive, continuous, and slowly varying (in the sense of Karamata) as $t\to 0$ and satisfy the condition

$$\text{(3.4)} \qquad \sum_{j=1}^{m} \liminf_{t\to 0} h_j(t)/h_0(t) > 2m-1.$$

Then

$$\sup_x |F_{S_n}(xB_n)-\Phi(x)| \xrightarrow[n\to 0]{} 0,$$

where $B_n^{-1}=\inf\{t>0:\ t^2h(t)=\frac{1}{n}\}$ *and* $h(t)=h_0(t)+\sum_{j=1}^{m}(h_j(t)-2h_0(t))$.

Proof of Theorem 2. According to Theorem 2.6.5 in [7], p. 85, condition (3.2) means that the df $F_{Y_1}(x)$ is attracted to the normal df $\Phi(x)$ and this again is equivalent to the fact (see [7], p. 83) that the function $H(z) = \int_{|x|\leq z} x^2 dF_{Y_1}(x)$ is slowly varying as $z\to\infty$.

Since $1-\cos x \geq \frac{11}{24}x^2$ for $|x|\leq 1$ and $E(1-\cos tY_1) \leq |f_{Y_1}(t)-1|$ we have in a neighbourhood of $t=0$

$$\frac{nt^2}{B_n^2} h_0(t/B_n) \geq n|f_{Y_1}(t/B_n)-1| \geq \frac{11nt^2}{24B_n^2} \int_{|x|\leq B_n/|t|} x^2 dF_{Y_1}(x),\quad t\neq 0.$$

By the definition of B_n we obtain

$$\frac{nH(B_n/|t|)}{B_n^2} \leq \frac{24h_0(t/B_n)}{11h_0(1/B_n)}(1-2m+\sum_{j=1}^{m}\frac{h_j(1/B_n)}{h_0(1/B_n)})^{-1},$$

and (3.4) leads to

$$\limsup_{n\to\infty} \frac{nH(\varepsilon B_n)}{B_n^2} < \infty \quad \text{for every} \quad \varepsilon > 0. \tag{3.5}$$

Using Karamata's representation of slowly varying functions (see [7], p. 82) one can conclude that

$$\lim_{n\to\infty} n \int_{|x|\geqq \varepsilon B_n} dF_{Y_1}(x) = 0 \quad \text{for every } \varepsilon > 0. \tag{3.6}$$

By $|e^{ix}-1| \leqq |x|$ it follows that

$$nE|e^{itY_1/B_n} - 1|^3 \leqq 8n \int_{|x|\geqq \varepsilon B_n} dF_{Y_1}(x) + \frac{\varepsilon n|t|^3}{B_n^2} H(\varepsilon B_n), \quad \varepsilon > 0.$$

The last estimate together with (3.5) and (3.6) implies

$$\lim_{n\to\infty} n \max_{|t|\leqq T} E|e^{itY_1/B_n} - 1|^3 = 0 \tag{3.7}$$

for every finite T.

Therefore, together with (3.1) and (3.2) we have

$$\lim_{n\to\infty} \ln f_{S_n}(t/B_n) = \lim_{n\to\infty} n\,(f_{Y_1}(t/B_n) - 1 + \sum_{j=1}^{m} E(e^{itY_1/B_n} - 1)\,(e^{itY_{j+1}/B_{n_1}}))$$

for $|t| \leqq T$.

Since (3.4) implies that $h(t)$ is slowly varying as $t\to 0$ we obtain

$$\lim_{n\to\infty} \ln f_{S_n}(t/B_n) = -\frac{t^2}{2} \lim_{n\to\infty} \frac{n}{B_n^2} h(t/B_n) = -\frac{t^2}{2}.$$

This relation is equivalent to the assertion of Theorem 2. □

R e m a r k. In general, relation (3.7) is impossible for a *df* $F_{Y_1}(x)$ which belongs to the domain of attraction of a non-normal stable *df*.

4. Rates of convergence in stable limit theorems for *m*-dependent rv's. In this section we shall extend some techniques which were developed in order to determine the accuracy of approximation to a given stable *df* in the case of i. i. d. rv's. For this purpose several kinds of so-called pseudomoments (see [2,8,9]) were introduced to express the smallness of the distance $|F_{X_1}(x) - G_{\alpha\beta}(x,c)|$ in some sense. Here we follow the concept suggested by G. C h r i s t o p h in [2] and [8]. We remark that the approach devoloped in [2] and [8] leads to best possible estimates of $D_n^{(\alpha,\beta)}(x,c)$ in the the case of i. i. d. rv's. To begin with we give a list of notations.

Let X_1, X_2, and X_3 be arbitrary rv's on $(\Omega, \mathfrak{A}, P)$. Then define

$$\Delta_1^-(x) = \Delta_1^+(x) = \Delta(x) = F_{X_1}(x) - G_{\alpha\beta}(x,c) + F_{X_1+X_2}(x) - (F_{X_1} * F_{X_2})(x)$$

$$+ F_{X_1+X_2+X_3}(x) - (F_{X_1} * F_{X_2+X_3})(x) - (F_{X_1+X_2} * F_{X_3})(x) + (F_{X_1} * F_{X_2} * F_{X_3})(x),$$

where the symbol $*$ denotes the convolution of two *df*'s,

$$\Delta_m^+(x) = \int_x^{\infty} \Delta_{m-1}^+(y)dy, \quad \Delta_m^-(x) = \int_{-\infty}^{x} \Delta_{m-1}^-(y)dy, \quad m = 2, 3, \ldots,$$

and

$$\Delta_m(x) = -\Delta_m^+(x) + (-1)^m \Delta_m^-(x), \quad \overline{\Delta_m(x)} = |\Delta_m^+(x)| + |\Delta_m^-(x)|, \quad m = 1, 2, \ldots$$

Further, define (the absolute integral difference moment, see [2; 8])

$$\varkappa_r = \begin{cases} \int\limits_{-\infty}^{\infty} |x^r| \, d\Delta(x), & 0 < r < 1, \\ r(r-1)\ldots(1+\delta) \int\limits_0^{\infty} x^{\delta} \overrightarrow{\Delta_{[r]}(x)} dx, & r \geq 1, \end{cases}$$

where $\delta = r - [r]$, ($[r]$ denotes the integer part of r, i. e. $r - 1 < [r] \leq r$) and (the integral difference moment)

$$\zeta_m = \begin{cases} 0, & m = 0, \\ m! \int\limits_0^{\infty} \Delta_m(x) \, dx, & m = 1, 2, \ldots . \end{cases}$$

Now we are in position to formulate the following analogue to Lemma 2 in [2].

Lemma 4. *Supposse that* $\varkappa_r < \infty$ *for some* $r \geq \alpha$ *and* $\zeta_0 = \ldots = \zeta_s = 0$, *where* $s = [r]$, *if* $r \neq [r]$, *and* $s = r - 1$, *else. Then, as* $t \to 0$,

$$\int\limits_{-\infty}^{\infty} e^{itx} \, d\Delta(x) = \begin{cases} o(|t|^r), & r \neq [r], \\ \frac{(it)^r}{r!} \zeta_r + o(|t|^r) & r = [r]. \end{cases} \tag{4.1}$$

The proof of Lemma 4 employs the same technique which was used to prove Lemma 2 in [2]. That's why we sketch only its main idea.

P r o o f of L e m m a 4. Put $m = [r]$. After m-fold integrating by parts we arrive at

$$\int\limits_{-\infty}^{\infty} e^{itx} \, d\Delta(x) = \sum_{k=0}^{[r[} \frac{(it)^k}{k!} \zeta_k + (it)^m (I_1 + I_2 + (-1)^m (I_3 + I_4)), \tag{4.2}$$

where

$$I_1 = \int\limits_0^{1/|t|} (e^{itx} - 1) d\Delta_{m+1}^+(x), \qquad I_2 = \int\limits_{1/|t|}^{\infty} (e^{itx} - 1) d\Delta_{m+1}^+(x),$$

$$I_3 = \int\limits_{-1/|t|}^{0} (e^{itx} - 1) d\Delta_{m+1}^-(x) \quad \text{and} \quad I_4 = \int\limits_{-\infty}^{-1/|t|} (e^{itx} - 1) d\Delta_{m+1}^-(x).$$

The inequalities $|e^{itx} - 1| \leq |tx|$ and $|e^{itx} - 1| \leq 2|tx^{\delta}|$ lead to

$$|I_1| + |I_3| \leq \begin{cases} -|t| \int\limits_0^{1/|t|} x^{1-\delta} d\left(\int\limits_x^{\infty} y^{\delta} \overline{\Delta_m(y)} \, dy\right), & r \geq 1, \\ -|t| \int\limits_0^{1/|t|} x^{1-\delta} d\left(\int\limits_{|y| > x} |y|^{\delta} |d\Delta(y)|\right), & 0 < r < 1, \end{cases}$$

and

$$|I_2|+|I_4| \leq \begin{cases} 2|t|^\delta \int\limits_{1/|t|}^{\infty} x^\delta \overline{\Delta_m(x)}\, dx, & r \geq 1, \\ 2|t|^\delta \int\limits_{|x|>1/|t|} |x|^\delta\, d\Delta(x)|, & 0<r<1. \end{cases}$$

The $\varkappa_r<\infty$ implies that $|I_1|+|I_2|+|I_3|+|I_4|=o(|t|^\delta)$ as $t\to 0$. Together with (4.2) and $\zeta_0=\ldots=\zeta_s=0$ the latter relation shows (4.1). □

Theorem 4. *Let $X_1, X_2, \ldots$ be a strictly stationary sequence of 1-dependent rv's and let $F_{X_1}(x)$ belong to the normal domain of attraction of the stable df $G_{\alpha\beta}(x, c)$ with $0<\alpha<2$, $\alpha\neq 1$, $|\beta|\leq 1$ and $\alpha=1$, $\beta=0$, respectively, i. e.*

$$f_{X_1}(t)-1=-c|t|^\alpha(1-i\beta\,\mathrm{sgn}\, t\,\omega(t,\alpha))(1+o(1)) \text{ as } t\to 0. \tag{4.3}$$

Further, suppose that in a neighbourhood of $t=0$

$$\Big|\int\limits_{-\infty}^{\infty} e^{itx}\, d\Delta(x)\Big| \leq C_8|t|^r \quad \textit{for} \quad \alpha<r\leq \min(2\alpha, 1+\alpha). \tag{4.4}$$

Then

$$\sup_x |F_{S_n}(xn^{1/\alpha})-G_{\alpha\beta}(x,c)| = o(n^{-\frac{r-\alpha}{\alpha}})$$

holds as $n\to\infty$.

Proof of Theorem 4. Take $|t|\leq \varepsilon n^{1/\alpha}$, where ε is chosen small enough, so that $|f_{X_1}(t/n^{1/\alpha})-1|$ and $|g_{\alpha\beta}(t/n^{1/\alpha}, c)-1|$ are sufficiently small. Then by Lemma 2,

$$|f_{S_n}(t/n^{1/\alpha})-g_{\alpha\beta}(t,c)| = e^{-c|t|^\alpha}|1-\exp\{\ln f_{S_n}(t/n^{1/\alpha})$$

$$-n\ln g_{\alpha\beta}(t/n^{1/\alpha}, c)\}| = e^{-c|t|^\alpha}|1-\exp n\{q_n(t/n^{1/\alpha})\}|,$$

where

$$|q_n(t/n^{1/\alpha})-\int\limits_{-\infty}^{\infty} e^{itx/n^{1/\alpha}}\, d\Delta(x)| \leq \frac{C_9}{n}|f_{X_1}(t/n^{1/\alpha})-1|$$

$$+c_{10}|f_{X_1}(t/n^{1/\alpha})-1|^2+C_{11}|g_{\alpha\beta}(t/n^{1/\alpha}, c)-1|^2.$$

Thus, from (4.3) and (4.4),

$$n|q_n(t/n^{1/\alpha})| \leq C_{12}(|t|^r n^{-\frac{r-\alpha}{\alpha}}+\frac{|t|^\alpha}{n}+\frac{|t|^{2\alpha}}{n}),$$

so that, by $|e^z-1|\leq |z|e^{|z|}$ and $r\leq 2\alpha$,

$$|f_{S_n}(t/n^{1/\alpha})-g_{\alpha\beta}(t,c)| \leq C_{13} n^{-\frac{r-\alpha}{\alpha}}(|t|^r+|t|^\alpha+|t|^{2\alpha})\, e^{-\frac{c}{2}|t|^\alpha}, \tag{4.5}$$

where n is taken large enough and ε sufficiently small. Now, we apply Esseen's well-known estimate (see [7]):

$$\sup_x |F_{S_n}(xn^{1/\alpha})-G_{\alpha,\beta}(x, c)| \leqq \frac{1}{\pi} \int_{|t|\leqq \varepsilon n^{1/\alpha}} |f_{S_n}(t/n^{1/\alpha})-g_{\alpha\beta}(t/n^{1/\alpha}, c)| \frac{dt}{|t|}$$

$$+\frac{24}{\pi\varepsilon n^{1/\alpha}} \sup_x \frac{d}{dx} G_{\alpha\beta}(x, c).$$

Taking into account that $r\leqq 1+\alpha$ the latter estimate and (4.5) show the validity of Theorem 4. □

Remark. Obviously, if $\varkappa_r<\infty$ for $r\geqq\alpha$ and $\zeta_0=\ldots=\zeta_s=0$, then, by Lemma 4, condition (4.4) is satisfied. In some cases it is more convenient to check (4.4) directly (e. g. if structure or estimates of $E(e^{itX_1}-1)(e^{itX_2}-1)$ and $E(e^{itX_1}-1)(e^{itX_2}-1)(e^{itX_3}-1)$ are known) than to verify the finiteness of $\varkappa_r$. This will be demonstrated in Section 5.

5. Examples. Lemma 5. *Let $\xi_1, \xi_2, \ldots$ be i. i. d. rv's having the common df $G_{\alpha 0}(x, 1)$, $0<\alpha<2$. Then*

$$\lim_{n\to 0} P((n \ln n)^{-1/\alpha} \sum_{k=1}^{n} \xi_k\xi_{k+1}<x)=G_{\alpha 0}(x, 1) \tag{5.1}$$

holds fos every real x.

Proof of Lemma 5. Remembering the fact that the cf of the product of two independent rv's X and Y is given by

$$f_{XY}(t)=\int_{-\infty}^{\infty} f_X(tx)\, dF_Y(x) = \int_{-\infty}^{\infty} f_Y(tx)\, dF_X(x) \tag{5.2}$$

we can write

$$1-f_{\xi_1\xi_2}(t)=2\int_0^{\infty}(e^{-(|t|x)^\alpha}-1)d(1-G_{\alpha 0}(x, 1))$$

$$=2\alpha|t|^\alpha \int_0^{\infty}(1-G_{\alpha 0}(x, 1))x^{\alpha-1}e^{-(|t|x)^\alpha}\, dx.$$

Since $1-G_{\alpha 0}(x, 1)=(\frac{1}{2}+a(x))\,x^{-\alpha}$, $x>0$, with $a(x)\to 0$ as $x\to\infty$ (see [7] p. 93), we have

$$\left|\int_0^{\infty}(1-G_{\alpha 0}(x, 1))\, x^{\alpha-1}e^{-(|t|x)^\alpha}\, dx-\frac{1}{2}\int_1^{1/|t|}\frac{dx}{x}\right|$$

$$\leqq \int_0^1 x^{\alpha-1}\, dx+\int_1^{1/|t|}\frac{a(x)}{x}e^{-(|t|x)^\alpha}\, dx+\frac{1}{2}\int_1^{1/|t|}\frac{1-e^{-(|t|x)^\alpha}}{x}\, dx$$

$$+\int_{1/|t|}^{\infty}\frac{(1/2+a(x))}{x}\, e^{-(|t|x)^\alpha}\, dx \quad \text{for} \quad 0<|t|<1.$$

It is easily seen that

$$\int_1^{1/|t|} \frac{a(x)}{x}\, dx = o(\ln \frac{1}{|t|}) \quad \text{as} \quad t \to 0$$

and the other terms on the right side of the latter inequality are bounded.

Therefore, as $t \to 0$,

$$1 - f_{X_1}(t) = \alpha\, |t|^\alpha \ln \frac{1}{|t|} (1 + o(1)),$$

i. e. $X_1 = \xi_1\xi_2$ belongs to the non-normal domain of attraction of the stable df $G_{\alpha 0}(x, 1)$.

Because $X_1 + X_2 = \xi_2(\xi_1 + \xi_3)$ is a product of two independent rv's it follows from (5.2) that

$$1 - f_{X_1+X_2}(t) = 2 \int_0^\infty (e^{-2(|t|x)^\alpha} - 1)\, d(1 - G_{\alpha 0}(x, 1)).$$

By repeating the above procedure we obtain

$$1 - f_{X_1+X_2}(t) = 2\alpha\, |t|^\alpha \ln \frac{1}{|t|} (1 + o(1)) \quad \text{as} \quad t \to 0.$$

We see that the assumptions of Theorem 1A with $a = 0$, $b = 1/2$, and $B_n = (n \ln n)^{1/\alpha}(1 + o(1))$ as $n \to \infty$ are fulfilled. Therefore, by Theorem 1A, the desired relation (5.1) is proved. □

Lemma 6. *Let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically normally distributed rv's with $E\xi_1 = 0$ and $E\xi_1^2 = 1$. Then*

$$\sup_x \left| P\left(\frac{1}{n} \sum_{k=1}^n \frac{\xi_{k+1}}{\xi_k} < x\right) - G_{10}(x, 1) \right| \le \frac{C_{14}}{n} \tag{5.3}$$

holds.

Proof of Lemma 6. It is well-known that the rv's $X_k = \xi_{k+1}/\xi_k$, $k = 1, \ldots, n$, are Cauchy distributed, i. e, $Ee^{it\xi_{k+1}/\xi_k} = e^{-|t|}$. Since $\frac{d}{dz} F_{1/\xi_1}(z) = \frac{1}{\sqrt{2\pi z^2}} e^{-1/2z^2}$ and $\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2}\, dx = \pi$, it follows that

$$1 - f_{X_1+X_2}(t) = 1 - e^{-|t|} + \int_0^\infty e^{-\frac{t^2}{2x^2} - \frac{x^2}{2}} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos txz}{z^2} e^{-\frac{1}{2z^2}}\, dz\right) dx$$

and

$$\left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos txz}{z^2} e^{-\frac{1}{2z^2}}\, dz - |t|\,|x| \right| \le \frac{t^2 x^2}{\pi} + \frac{1}{2\pi} \int \frac{t^2 x^2}{z^2}\, dz = \frac{3t^2 x^2}{2\pi}.$$

Therefore, as $t \to 0$,

$$1 - f_{X_1+X_2}(t) = |t| + |t| \int_0^\infty e^{-\frac{t^2}{2x^2} - \frac{x^2}{2}} x\, dx + o(t^2)$$

$$=2|t|+|t|\int_0^{|t|}(e^{-\frac{t^2}{2x^2}}-1)e^{-\frac{x^2}{2}}xdx+|t|\int_{|t|}^{\infty}(e^{-\frac{t^2}{2x^2}}-1)e^{-\frac{x^2}{2}}xdx+o(t^2)=2|t|+o(t^2).$$

In like fashion we get

$$1-f_{X_1+X_2+X_3}(t)=1-f_{X_1+X_2}(t)+\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{t^2}{2x^2}-\frac{x^2}{2}-\frac{y^2}{2}+\frac{itx}{y}}$$

$$\times\int_{-\infty}^{\infty}\frac{1-\cos tyz}{z^2}e^{-\frac{1}{2z^2}}dzdydx=2|t|+\frac{|t|}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{t^2}{2x^2}-\frac{x^2}{2}-\frac{y^2}{2}+\frac{itx}{y}}$$

$$\cdot|y|dydx+o(t^2)=2|t|+|t|e^{-|t|}+o(t^2)=3|t|+o(t^2).$$

Combining all these estimates we have $\int_{-\infty}^{\infty}e^{itx}d\Delta(x)=o(t^2)$ as $t\to 0$.

Finally, a straightforward application of Theorem 4 with $r=2$ and $\alpha=1$ implies (5.3). □

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