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TWO REMARKS ON BOUNDED ANALYTIC FUNCTIONS

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1. Introduction. Let $f(z)=\Sigma_{k=0}^{\infty} a_k z^k$ be analytic in the unit disc $\mathsf{D}=\{z:|z|<1\}$ and bounded:

(1) $$\sup_{|z|<1} |f(z)|\leq 1.$$

The following inequalities are well-known:

(2) $$\sum_{k=0}^{\infty} |a_k|^2\leq 1,$$

(3) $$|f'(z)|\leq\frac{1-|f(z)|^2}{1-|z|^2}, \quad z\in \mathsf{D}.$$

(2) is sharp if and only if the L^2-function

(4) $$\widehat{f}(\theta)=\overline{\lim_{r\to 1}}|f(re^{i\theta})|$$

is 1 a. e. on $[0, 2\pi)$ while equality holds in (3) only for suitabie functions of the form

(5) $$f(z)=\lambda\frac{z+z_0}{1+\bar{z}_0 z}, \quad |z_0|<1, \quad |\lambda|=1.$$

In the present note we improve upon (2) under the additional assumption that the range of f is bounded away from a certain point on $\partial\mathsf{D}$ and we shall generalize (3) to higher derivatives of f.

2. Refinement of (2). Theorem 1. *Let $0<s<2$. Assume that $f(z)=\Sigma_{k=0}^{\infty} a_k z^k$ is analytic in* D *and satisfies the inequalities $|f(z)|<1$, $|f(z)+1|\geq s$ in* D. *Then*

(6) $$\sum_{k=0}^{\infty} |a_k|^2\leq 1-s^2\,\mathrm{Re}\,\frac{1-a_0}{1+a_0}.$$

(6) *is sharp for every admissible choice of s and $a_0=f(0)$.*

It is obvious how (6) must be modified if we have f bounded away from another point on $\partial\mathsf{D}$ instead of the point -1. Theorem 1 — for the cases $0<s<1$, $a_0=0$ — has first been established in [2] by a more direct approach. The method presented below was essentially suggested by J. Hersch. It rests completely on the following Lemma which is immediately verified but required considerable ingenuity to be obtained. It is due to J. Moser.

Lemma. *Let* $D_s = D \setminus \{z : |z+1| \leq s\}$. *Then*

$$v(z) = 1 - s^2 \operatorname{Re} \frac{1-z}{1+z} \tag{7}$$

is harmonic in $\bar{D}_s$ *and satisfies*

$$v(z) = |z|^2, \quad z \in \partial D_s. \tag{8}$$

Proof of Theorem 1. Let g be a conformal mapping of D onto D_s with $g(0) = f(0)$. Then the assumptions on f imply that f is subordinate to g in D and Littlewood's theorem (see [1, p. 36]) and the fact that g extends continuously to D implies that

$$\sum_{k=0}^{\infty} |a_k|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^2 d\theta, \tag{9}$$

with equality for $f \equiv g$. Now let u be the harmonic function in D with $u(e^{i\theta}) = |g(e^{i\theta})|^2$, $\theta \in [0, 2\pi)$.

The mean value property of harmonic functions implies that

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^2 d\theta. \tag{10}$$

Now let $v(z) = u(g^{-1}(z))$, $z \in \bar{D}_s$. Then v is harmonic in D_s and for $\zeta \in \partial D_s$ we get $v(\zeta) = u(g^{-1}(\zeta)) = |\zeta|^2$. Since Dirichlet's problem has a unique solution in D_s we see that v is the function (7) and by definition we deduce $v(a_0) = v(g(0)) = u(0)$. This combined with (9), (10) proves (6). It is clear that for $f \equiv g$ we have equality in (6).

3. Estimate for the *n*-th derivative of *f*. Let f be as in the introduction. In [3] we proved that for $n \in N$, $z \in D$, we have

$$|f^{(n)}(z)| \leq 2n! \frac{1-|f(z)|}{(1-|z|)^n(1+|z|)}$$

and it was conjectured that the factor 2 can be replaced by $1+|f(z)|$. We now show that this is true and that the factor of $(1-|f(z)|^2)$ is best possible although — for $n > 1$ — equality holds only in the trivial case $f \equiv \varepsilon$, $|\varepsilon| = 1$.

Theorem 2. *Let f be analytic in D, $n \geq 1$. Then*

$$|f^{(n)}(z)| \leq n! \frac{1-|f(z)|^2}{(1-|z|)^n(1+|z|)}, \quad z \in D. \tag{11}$$

For $n > 1$, $z \neq 0$, equality holds for $f \equiv \varepsilon$, $|\varepsilon| = 1$. For every $z \in D$ there exist functions f_j, $j \in N$, bounded and analytic in D such that

$$\lim_{j \to \infty} \frac{|f_j^{(n)}(z)|}{1-|f_j(z)|^2} = \frac{n!}{(1-|z|)^n(1+|z|)}.$$

Remark. Szász [4] obtained sharp estimates for $|f^{(n)}(z)|$, $z \in D$, but independent of $|f(z)|$.

The following Lemma is well-known:

Lemma. *Let $h(z) = \Sigma_{k=0}^{\infty} b_k z^k$ be analytic in D, $|h(z)| \leq 1$ in D. Then*

$$|b_k| \leq 1 - |b_0|^2, \quad k \in N. \tag{12}$$

For k fixed we have equality in (12) *only for*

$$h(z)=\varepsilon\frac{z^k+a}{1+\bar{a}z^k}, \quad |\alpha|\leq 1, \quad |\varepsilon|=1. \tag{13}$$

We note that equality in (12) *holds simultaneously for two different values of k if and only if $h\equiv\varepsilon$, $|\varepsilon|=1$.*

Proof of Theorem 2. The Lemma proves (11) for $z=0$. Now let $z\neq 0$ and define

$$h(x)=f\left(\frac{x+z}{1+\bar{z}x}\right)=\sum_{k=0}^{\infty} b_k x^k.$$

Then, by a formula of Szász [4] we obtain

$$\frac{(1-|z|^2)^n}{n!} f^{(n)}(z)=\sum_{k=1}^{n}\binom{n-1}{n-k} b_k \bar{z}^{n-k}$$

and $b_0=h(0)=f(z)$. Using (12) we get

$$\frac{(1-|z|^2)^n}{n!}|f^{(n)}(z)|\leq(1-|b_0|^2)\sum_{k=1}^{n}\binom{n-1}{n-k}|z|^{n-k}=(1-|f(z)|^2)(1+|z|)^{n-1}, \tag{14}$$

which is equivalent to (11). We also note that equality holds in (14) only if we have simultaneous equality in (12) for $k=1,\ldots,n$ which is possible only for $h\equiv\varepsilon$, $|\varepsilon|=1$, and therefore for $f\equiv\tilde{\varepsilon}$, $|\tilde{\varepsilon}|=1$.

Now fix $z\in D$, $z\neq 0$, and choose a sequence $a_j\in D$ with $a_j\to z/|z|$. Then for the bounded analytic functions

$$f_j(z)=\frac{z-a_j}{1-\bar{a}_j z}$$

we find

$$\frac{|f_j^{(n)}(z)|}{1-|f_j(z)|^2}=\frac{n!\,|a_j|^{n-1}}{|1-\bar{a}_j z|^{n-1}(1-|z|^2)}\to\frac{n!}{(1-|z|)^{n-1}(1-|z|^2)}$$

for $j\to\infty$. This proves the second claim of Theorem 2.

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