Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

THE DIRICHLET PROBLEM FOR A NONLINEAR CONVEX ELLIPTIC EQUATION

JANA D. MADJAROVA

The paper establishes $C^2(\overline{\Omega})$-a priori bounds for the classical solutions to the first boundary value problem for a nonlinear convex elliptic equation. In the case of two variables there is a $C^{2,\alpha}(\overline{\Omega})$-a priori estimate and a unique solution is proved to exist, belonging to $C^{2,\alpha}(\overline{\Omega})$.

1. Introduction. Let Ω be a bounded domain in R^n with smooth boundary ($\partial\Omega \in C^3$), $\Phi \in C^3(\mathrm{R}^n)$ and φ be the restriction of Φ on $\partial\Omega$. Consider the problem

$$(1)\qquad \left| \begin{array}{l} f(D^2u)+g(x,\ u,\ Du)=0 \ \text{ in } \Omega, \\ u|_{\partial\Omega}=\varphi, \end{array} \right.$$

where $f\in C^2(\mathrm{R}^{n^2})$, $g\in C^2(\mathrm{R}^n\times\mathrm{R}\times\mathrm{R}^n)$, $f=f(r)$, $g=g(x,\ z,\ p)$, D^2u and Du are the Hessian matrix, respectively the gradient of u. We shall suppose that the equation is uniformly elliptic, i. e. there exist constants $0<\theta\leqq\Theta<\infty$ such that

$$(2)\qquad \theta\,|\,\xi\,|^2\leqq \sum_{i,j=1}^{n} f_{ij}(r)\xi^i\xi^j\leqq\Theta\,|\,\xi\,|^2,\quad \forall r\in\mathrm{R}^{n^2},\quad \xi\in\mathrm{R}^n,\quad f(0)=0,$$

where $f_{ij}=\partial f/\partial r_{ij}=f_{ji}$.

The main assumption under which (1) will be considered is convexity of f and g with respect to the arguments r and p. According to the smoothness this is equivalent to assuming

$$(3)\qquad \sum_{i,j;\,k,l} f_{ij,kl}(r)\xi^{ij}\xi^{kl}\geqq 0,\quad \forall r\in\mathrm{R}^{n^2},\quad \xi\in\mathrm{R}^{n^2},$$

$$(4)\qquad \sum_{i,j} g_{p_ip_j}(x,\ z,\ p)\xi^i\xi^j\geqq 0,\quad \forall\xi\in\mathrm{R}^n,\quad x\in\mathrm{R}^n,\quad z\in\mathrm{R},\quad p\in\mathrm{R}^n.$$

We shall further suppose that

$$(5)\qquad g_z(x,\ z,\ p)\leqq\eta<0,$$

or

$$(5')\qquad g_z(x,\ z,\ p)\leqq 0,\quad \max_{\substack{x\in\overline{\Omega}\\ z\in\mathrm{R}}} |g_{p_i}(x,\ z,\ p)|\leqq\widetilde{G},$$

and

SERDICA Bulgaricae mathematicae publicationes. Vol. 11, 1985, p. 208—216.

(6) (i) $\max_{\substack{x\in\overline{\Omega}\\ |z|\le K}} |g_{p_i}(x, z, p)| \le G = G(K), \quad i=1, \ldots, n,$

(ii) $\max_{\substack{x\in\overline{\Omega}\\ |z|\le K}} |g_{x_k}(x, z, p)| \le G + G|p|, \quad G=G(K), \quad k=1, \ldots, n.$

In [1] Evans considers (1) under the same assumptions as above for the function f and $g=0$. Using Bernstein's method he establishes global C^2-a priori estimates for the solution, then he proves local $C^{2,\alpha}$-estimates and applying the method of continuity achieves an existence and uniqueness result for the problem.

In this paper we obtain C^2-a priori estimates for the solution of (1) and in the case $n=2$ we prove the following

Theorem 1. *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with smooth boundary and suppose* f *and* g *satisfy* (2)—(6). *Then there exists a unique function* $u\in C^{2,\alpha}(\overline{\Omega})(\alpha\in(0, 1))$ *such that*

$$\left| \begin{array}{l} f(D^2u)+g(x, u, Du)=0 \text{ in } \Omega, \\ u|_{\partial\Omega}=\varphi. \end{array} \right.$$

Acknowledgements are due to T. Genchev and N. Kutev for their helpful advice given to me during my work.

2. A priori estimates for u, Du, D^2u. The equation from (1) may be written in the equivalent form

$$\sum_{i,j=1}^{n} a^{ij}u_{x_ix_j} + \sum_{i=1}^{n} b^iu_{x_i} + cu + g(x, 0, 0) = 0, \tag{7}$$

where

$$\text{(i)} \quad a^{ij}(D^2u) = \int_0^1 f_{ij}(tD^2u)dt,$$

$$\text{(ii)} \quad b^i(x, u, Du) = \int_0^1 g_{p_i}(x, u, tDu)dt, \tag{8}$$

$$\text{(iii)} \quad c(x, u) = \int_0^1 g_z(x, tu, 0)dt.$$

The condition (5) yields $c<0$ and thus implies the validity of a maximum principle and hence an estimate for u of the form (if we use (5′), η is to be replaced by $\tilde{G}$)

$$\max_{\overline{\Omega}} |u| \le M = \max_{\partial\Omega} |\varphi| + C(\theta, \eta, \max_{\overline{\Omega}} |g(x, 0, 0)|).$$

The gradient of u will first be estimated near the boundary with the help of a standard barrier construction. The smoothness of $\partial\Omega$ implies the uniform exterior sphere condition. Let $x_0\in\partial\Omega$ and $B(y, R)$ be the respective exterior ball, i. e. $\overline{B}\cap\overline{\Omega}=\{x_0\}$. We shall suppose that $y=0$ (that is no loss of generality since translation preserves all the properties of the problem). We shall use the barrier $w(x)=\tau(R^{-\sigma}-r^{-\sigma})$, where $r=|x|$ and σ, τ are sufficiently large positive constants which will be chosen later. We have: $w(x)\ge 0$ for $x\in\overline{\Omega}$, $w(x_0)=0$.

We set

$$L=\sum_{i,j} a^{ij}(D^2u)\,\partial_{ij}+\sum_i b^i(x,\ u,\ Du)\partial_i$$

and apply L to w:

$$\begin{aligned} Lw&=\sum_{i,j} a^{ij}w_{x_ix_j}+\sum_i b^i w_{x_i}\\ &=\tau\sigma r^{-\sigma-4}(-(\sigma+2)\sum_{i,j} a^{ij}x_ix_j+r^2\sum_i(a^{ii}+b^ix_i))\\ &\leqq\tau\sigma r^{-\sigma-4}(-(\sigma+2)\theta\,|x|^2+r^2\sum_i(a^{ii}+b^ix_i))\\ &=\tau\sigma r^{-\sigma-2}(-(\sigma+2)\theta+\sum_i(a^{ii}+b^ix_i))<0 \end{aligned}$$

for σ large enough, as the uniform bounds on f_{ij}, g_{p_i} give us uniform bounds on a^{ij}, b^i.

Applying L to $v=\pm(u-\Phi)-w$ gives

$$Lv=\pm Lu\mp L\Phi-Lw=\mp g(x,\ u,\ 0)\mp L\Phi-Lw>0$$

for τ sufficiently large, as

$$\max_{\substack{x\in\overline{\Omega}\\ |u|\leqq M}}|g(x,\ u,\ 0)|<\infty.$$

It follows now from the ellipticity of L that v attains its maximum on the boundary; but on $\partial\Omega$ we have

$$v|_{\partial\Omega}=\pm(u|_{\partial\Omega}-\Phi|_{\partial\Omega})-w|_{\partial\Omega}=-w|_{\partial\Omega}\leqq 0$$

and consequently $v\leqq 0$ in $\overline{\Omega}$, $|u-\Phi|\leqq w$, i. e.

$$-w(x)+\Phi(x)\leqq u(x)\leqq w(x)+\Phi(x).$$

Now, if ν denotes the normal to $\partial\Omega$ in x_0 and $x-x_0=\nu$, we have

$$-(w(x)-w(x_0))+(\Phi(x)-\Phi(x_0))\leqq u(x)-u(x_0)\leqq(w(x)-w(x_0))+(\Phi(x)-\Phi(x_0))$$

and it follows immediately that $-\partial_\nu w(x_0)+\partial_\nu\Phi(x_0)\leqq\partial_\nu u(x_0)\leqq\partial_\nu w(x_0)+\partial_\nu\Phi(x_0)$ wherefrom

$$|\partial_\nu u(x_0)|\leqq|\partial_\nu\Phi(x_0)|+|\partial_\nu w(x_0)|=|\partial_\nu\Phi(x_0)|+\sigma\tau R^{-\sigma-1}.$$

The tangential derivatives of u coincide with those of Φ. Finally we reach an estimate of the form

$$\max_{\partial\Omega}|Du|\leqq C. \tag{9}$$

Further we shall use Bernstein's method in order to achieve an estimate for Du in $\overline{\Omega}$. We shall show that for N, N_1, appropriately chosen, the function $w(x)=|Du|^2+N(u+M)^2+N_1|x|^2$ can't attain its maximal value in Ω.

Suppose that w has a maximum at $x_0\in\Omega$. It follows then from the ellipticity that

$$\sum_{i,j} f_{ij}(D^2u)w_{x_ix_j}(x_0)\leqq 0 \tag{10}$$

and

$$\sum_{i,j} f_{ij}(D^2u)w_{x_ix_j}(x_0)=2\sum_{i,j,k} f_{ij}u_{x_kx_i}u_{x_kx_j}+2N_1\sum_i f_{ii}$$

$$+2\sum_{i,j,k} f_{ij}u_{x_k}u_{x_kx_ix_j}+2N\sum_{i,j} f_{ij}u_{x_ix_j}(u+M)+2N\sum_{i,j} f_{ij}u_{x_i}u_{x_j}$$

$$\geqq 2\theta nN_1+2\theta\sum_{i,k} u^2_{x_ix_k}+2\theta N|Du|^2+2\sum_{i,j,k} f_{ij}u_{x_k}u_{x_ix_jx_k}+2N(u+M)\sum_{i,j} f_{ij}u_{x_ix_j}.$$

In order to eliminate the term, containing third derivatives, we differentiate the equation from (1) with respect to x_k:

$$0=\partial/\partial_{x_k}(f(D^2u)+g(x, u, Du))$$

$$=\sum_{i,j} f_{ij}(D^2u)u_{x_ix_jx_k}+g_{x_k}(x, u, Du)+g_z(x, u, Du)u_{x_k}+\sum_i g_{p_i}(x, u, Du)u_{x_ix_k}.$$

We get

$$2\sum_{i,j,k} f_{ij}(D^2u)u_{x_k}u_{x_ix_jx_k}=-2\sum_k g_{x_k}(x, u, Du)u_{x_k}$$

$$-2(\sum_k u^2_{x_k})g_z(x, u, Du)-2\sum_{i,k} g_{p_i}(x, u, Du)u_{x_k}u_{x_ix_k}.$$

Let G be the constant from (6) corresponding to $K=M$. By (5), (6) and the obvious inequality

$$ab\leqq a^2/4\varepsilon+\varepsilon b^2, \quad \forall a,\ b\in\mathbb{R}, \quad \forall\varepsilon>0, \tag{11}$$

we have $-2|Du|^2g_z(x, u, Du)\geqq 0$,

$$2\Big|\sum_k g_{x_k}(x, u, Du)u_{x_k}\Big|\leqq 2nG|Du|+2nG|Du|^2\leqq nG+3nG|Du|^2,$$

$$2\Big|\sum_{i,k} g_{p_i}(x, u, Du)u_{x_k}u_{x_ix_k}\Big|\leqq(nG/2\varepsilon)|Du|^2+2\varepsilon G\sum_{i,k} u^2_{x_ix_k},$$

wherefrom

$$2\sum_{i,j,k} f_{ij}u_{x_k}u_{x_ix_jx_k}\geqq -nG-2\varepsilon G\sum_{i,k} u^2_{x_ix_k}-(3nG+nG/2\varepsilon)|Du|^2.$$

To estimate $2N(u+M)\sum_{i,j} f_{ij}u_{x_ix_j}$ we shall make use of the convexity of the functions f and g. For f it follows that

$$0=f(0)=f(D^2u)-\sum_{i,j} f_{ij}(D^2u)u_{x_ix_j}+\frac{1}{2}\sum_{i,j\,;\,k,l} f_{ij,kl}(*)u_{x_ix_j}u_{x_kx_l}$$

$$\geqq f(D^2u)-\sum_{i,j} f_{ij}(D^2u)u_{x_ix_j}=-g(x, u, Du)-\sum_{i,j} f_{ij}(D^2u)u_{x_ix_j}$$

and since $u+M\geqq 0$ in $\bar{\Omega}$, we have

$$2N(u+M)\sum_{i,j} f_{ij}u_{x_ix_j}\geqq -4MN|g(x, u, Du)|$$

(with $*$ we have denoted a point in the interval with endpoints 0 and D^2u).

The convexity of g with respect to p yields

$$g(x, u, Du)=g(x, u, 0)+\sum_i g_{p_i}(x, u, 0)u_{x_i}$$

$$+\frac{1}{2}\sum_{i,j} g_{p_ip_j}(*)u_{x_i}u_{x_j} \geqq g(x,\ u,\ 0)+\sum_i g_{p_i}(x,\ u,\ \ 0)u_{x_i},$$

$$g(x,\ u,\ 0)=g(x,\ u,\ Du)-\sum_i g_{p_i}(x,\ u,\ Du)u_{x_i}$$

$$+\frac{1}{2}\sum_{i,j} g_{p_ip_j}(*)u_{x_i}u_{x_j} \geqq g(x,\ u,\ Du)-\sum_i g_{p_i}(x,\ u,\ Du)u_{x_i},$$

wherefrom

$$g(x,\ u,\ 0)+\sum_i g_{p_i}(x,\ u,\ 0)u_{x_i} \leqq g(x,\ u,\ Du) \leqq g(x,\ u,\ 0)+\sum_i g_{p_i}(x,\ u,\ Du)u_{x_i}$$

and we get an estimate on the growth of g with respect to Du: $|g(x,\ u,\ Du)| \leqq G_1+nG|Du|$, where $G_1=\max\limits_{\substack{x\in\bar\Omega\\ |u|\leqq M}} |g(x,\ u,\ 0)|$.

Consequently

$$2N(u+M)\sum_{i,j} f_{ij}(D^2u)u_{x_ix_j} \geqq -4MNG_1-4MNnG|Du|$$

$$\geqq -4MNG_1-4MNnG(\varepsilon_1|Du|^2+1/4\varepsilon_1),\quad \varepsilon_1>0,$$

and finally

$$\sum_{i,j} f_{ij}(D^2u)w_{x_ix_j} \geqq 2(\theta-\varepsilon G)\sum_{i,k} u^2_{x_ix_k}+(2\theta N-3nG-nG/2\varepsilon-4MNnG\varepsilon_1)|Du|^2$$

$$+(2\theta nN_1-Gn-4MNG_1-MNnG|\varepsilon_1).$$

We see that for

$$\varepsilon<\theta/G,\ \varepsilon_1<\theta/(4MnG),\ N>(3nG+nG/2\varepsilon)/\theta,\ N_1>(Gn+4MNG_1+MNnG/\varepsilon_1)/(2n\theta)$$

the last expression is strictly positive — a contradiction, according to (10).

Recalling (9), finally we get $\max\limits_{\bar\Omega}|Du| \leqq C$.

To find estimates for the second derivatives we shall use ideas from [2] and [3]. It is necessary to straighten the boundary locally and consider the problem with $\Phi\equiv 0$. As we shall see later, this is no loss of generality. Let $x_0\in\partial\Omega$ and ψ be the diffeomorphism that straightens the boundary in a neighbourhood U of x_0; assume that $\psi(x_0)=0$, $\psi(U\cap\Omega)\subset\{y_n>0\}$, $\psi U(\cap\partial\Omega)\subset\{y_n=0\}$. We set $\tilde u(y)=\tilde u(\psi(x))=u\circ\psi^{-1}(y)$ and now we have

$$\tilde f(D^2\tilde u,\ D\tilde u,\ \ y)+\tilde g(D\tilde u,\ \tilde u,\ y)=0, \tag{12}$$

where

$$\tilde f_{ij}=\partial\tilde f/\partial(\tilde u_{y_iy_j})=\sum_{k,l} f_{kl}(\partial\psi_i/\partial x_k)(\partial\psi_j/\partial x_l);\ \ f_{q_i}=\partial\tilde f/\partial(\tilde u_{y_i})=\sum_{k,l} f_{kl}(\partial^2\psi_i/\partial x_k\,\partial x_l);$$

$\tilde f$ depends on y through the derivatives of ψ; hence $\tilde f_{y_k}$ has linear growth with respect to the second derivatives of the solution.

Obviously the equation (12) is elliptic and $\tilde f_{ij}$, $\tilde f_{q_i}$ are uniformly bounded. We set $\tilde M_2=\max|D^2\tilde u|$, $M_2=\max|D^2u|$.

Consider the function $w(y)=2\tilde M_2^{1/2}y_n-\tilde M_2^{3/4}y_n^{3/2}+b|y'|^2$ in the cylinder $Q=\{|y'|<\delta,\ 0<y_n<\tilde M_2^{-1/2}\}$. For $b>1/\delta^2$ we have $w(y)\geqq 0$ on the boundary of the cylinder, $w(0)=0$. In Q:

$$\sum_{i,j}\tilde{f}_{ij}w_{y_iy_j}=2b\sum_{k=1}^{n-1}\tilde{f}_{kk}-(3/4)\tilde{M}_2^{3/4}y_n^{-1/2}\tilde{f}_{nn}\leqq C_1-C_1\tilde{M}_2.$$

Further we shall use the auxiliary function $z=\pm\tilde{u}_{y_k}-\tau w$, $k=1,\ldots,n-1$.

Having in mind the properties of the derivatives of $\tilde{f}$ and differentiating (12) with respect to y_k, we obtain

$$\Big|\sum_{i,j}\tilde{f}_{ij}(\tilde{u}_{y_k})_{y_iy_j}\Big|\leqq C_2+C_2\tilde{M}_2.$$

If z attains its maximum at an interior point y_0, then

$$0\geqq\sum_{i,j}\tilde{f}_{ij}z_{y_iy_j}(y_0)\geqq-(C_2+C_2\tilde{M}_2)-\tau(C_1-C_1\tilde{M}_2)=\tau(C_1-C_2)\tilde{M}_2-(\tau C_1+C_2).$$

For $\tau>C_2/C_1$ we get $\tilde{M}_2\leqq(\tau C_1+C_2)/(\tau C_1-C_2)$, i. e. an estimate for $D^2\tilde{u}$.

Let z attain its maximum on the boundary of Q. For $y_n=\tilde{M}_2^{-1/2}$ or $|y'|=\delta$ we have $z\leqq\pm\tilde{u}_{y_k}-\tau<0$ for τ large enough; on the other hand, $z\leqq0$ for $y_n=0$ since $\tilde{u}|_{y_n=0}=0$ and $k\leqq n-1$; consequently $-\tau w\leqq\tilde{u}_{y_k}\leqq\tau w$, $k=1,\ldots,n-1$, in $\bar{Q}$.

Recalling $w(0)=\tilde{u}(0)=0$, we get $-2\tau\tilde{M}_2^{1/2}\leqq\tilde{u}_{y_ky_n}(0)\leqq2\tau\tilde{M}_2^{1/2}$, $k=1,\ldots,n-1$ in a way similar to the one used for obtaining (9) and from the equation $\tilde{u}_{y_ny_n}(0)\leqq C'+C'M_2^{1/2}$.

Finally

$$\max_{\partial\Omega}|D^2u|\leqq C+C(\max_{\bar{\Omega}}|D^2u|)^{1/2}.$$

For non-zero boundary conditions we set $v=u-\Phi$; then $v|_{\partial\Omega}=0$ and v is a solution of the equation $f(D^2v+D^2\Phi)+g(x,\ v+\Phi,\ Dv+D\Phi)=0$, which after straightening the boundary acquires the form (12).

Further we once more apply Bernstein's method with the auxiliary function $w(x)=(u_{\xi\xi}^-)^2+N|Du|^2+N_1|x|^2$, where $\xi\in\mathrm{R}^n$, $|\xi|=1$ and $u_{\xi\xi}^-=\min(0,\ u_{\xi\xi})$. We shall use an idea of Evans [1]: it is sufficient to establish one-sided bounds on $u_{\xi\xi}$ for arbitrary ξ; canonizing the equation we can get two-sided estimates for any second derivative.

Suppose that W attains its maximum at a point x_0, where $u_{\xi\xi}(x_0)<0$, i. e. $u_{\xi\xi}<0$ and $u_{\xi\xi}^-=u_{\xi\xi}$ in a whole neighbourhood of x_0. At x_0 we have

$$0\geqq\sum_{i,j}f_{ij}(D^2u)w_{x_ix_j}\geqq2\theta nN_1+2\sum_{i,j}f_{ij}u_{\xi\xi_{x_i}}u_{\xi\xi_{x_j}}+2u_{\xi\xi}\sum_{i,j}f_{ij}u_{\xi\xi_{x_ix_j}}$$

$$+2N\sum_{i,j,k}f_{ij}u_{x_ix_k}u_{x_jx_k}+2N\sum_{i,j,k}f_{ij}u_{x_k}u_{x_ix_jx_k}\geqq2\theta nN_1+2\theta\sum_i u^2_{\xi\xi_{x_i}}+2N\theta\sum_{i,j}u^2_{x_ix_j}$$

$$-NGn-N(3nG+2G+nG/2\varepsilon)|Du|^2-2\varepsilon NG\sum_{i,j}u^2_{x_ix_j}+2u_{\xi\xi}\sum_{i,j}f_{ij}u_{\xi\xi_{x_ix_j}}.$$

Again we made use of the ellipticity and the inequality (11). Let $\xi=\Sigma\alpha_ix_i$, $\|\alpha\|=1$. Differentiating the equation twice with respect to ξ gives

$$\sum_{i,j;\,k,l}f_{ij,kl}(D^2u)u_{x_ix_j\xi}u_{x_kx_l\xi}+\sum_{i,j}f_{ij}u_{x_ix_j\xi\xi}+\Sigma\beta_{ij}g_{x_ix_j}(x,\ u,\ Du)+2\Sigma\alpha_ig_{x_iz}(x,\ u,\ Du)u_\xi$$

$$+2\sum_{i,k}\alpha_i g_{x_i p_k}(x,\ u,\ Du)u_{x_k\xi}+g_{zz}u_\xi^2+2\sum_i g_{zp_i}u_\xi u_{x_i\xi}+g_z u_{\xi\xi}+\sum_i g_{p_i}u_{x_i\xi\xi}$$

$$+\sum_{i,j} g_{p_ip_j}(x,\ u,\ Du)u_{x_i\xi}u_{x_j\xi}=0.$$

The convexity of f and of g with respect to p yields

$$\sum_{i,j\,;\,k,l} f_{ij,kl}\,u_{x_ix_j\xi}u_{x_kx_l\xi}\geqq 0,\quad \sum_{i,j} g_{p_ip_j}u_{x_i\xi}u_{x_j\xi}\geqq 0.$$

Since $u_{\xi\xi}(x_0)<0$, we have

$$2u_{\xi\xi}\sum_{i,j} f_{ij}u_{\xi\xi x_ix_j}\geqq -2\sum_{i,j}\beta_{ij}g_{x_ix_j}u_{\xi\xi}-4\sum_i\alpha_i g_{x_iz}u_\xi u_{\xi\xi}-4\sum_i\alpha_i g_{x_ip_k}u_{x_k\xi}u_{\xi\xi}$$

$$-2g_{zz}u_\xi^2u_{\xi\xi}-4\sum_i g_{zp_i}u_\xi u_{x_i\xi}u_{\xi\xi}-g_z u_{\xi\xi}^2-\sum_i g_{p_i}u_{x_i\xi\xi}u_{\xi\xi}.$$

To the last term we apply (11). For N, N_1 large enough we obtain the contradictory inequality $0\geqq\sum_{i,j} f_{ij}w_{x_ix_j}>0$, which shows that w attains its maximum on the boundary, i. e. $w(x)\leqq(C+CM_2^{1/2})^2+\bar{C}$ and consequently

(13) $$u_{\xi\xi}\geqq-(C+CM_2^{1/2}).$$

Canonizing the equation at a fixed point gives us $|u_{y_iy_i}|\leqq(n-1)C(M_2)+C(\theta,\ \Theta)\max|g|$, choosing $\xi=y_i+y_j$, $\eta=y_i-y_j$ and making use of the one-side bounds on $u_{\xi\xi}$ and $u_{\eta\eta}$ we finally get $\max_{\bar\Omega}|D^2u|^2\leqq C+(C+CM_2^{1/2})^2$, i. e. $\max_{\bar\Omega}|D^2u|\leqq C$. Thus we proved the following

Theorem. *Let u be a smooth solution of* (1) *under the assumptions* (2)—(6). *There exists a constant* $C=C(\Omega,\ \theta,\ \Theta,\ n,\ G)$ *such that*

$$\max_{\bar\Omega}|u|\leqq C,\quad \max_{\bar\Omega}|Du|\leqq C,\quad \max_{\bar\Omega}|D^2u|\leqq C.$$

3. The case $n=2$. In the case $n=2$ we can obtain $C^{2,\alpha}$-a priori estimates for u. Let us differentiate the equation from (1) with respect to x_k and set $v=u_{x_k}$:

(14) $$\sum_{i,j=1}^{2} f_{ij}(D^2u)v_{x_ix_j}+\sum_{i=1}^{2} g_{p_i}(x,\ u,\ Du)v_{x_i}+g_zv+g_{x_k}=0.$$

Equation (14) is uniformly elliptic as well. We can apply Theorem 11.4 ([4], p. 247) and thus establish an interior $C^{1,\alpha}$-estimate for v: $|v|_{1,\alpha;\,\Omega'}\leqq C$, where $\Omega'\subset\subset\Omega$. As a result $|u|_{2,\alpha;\,\Omega'}\leqq C$, $\Omega'\subset\subset\Omega$, where C depends on θ, Θ, Φ, Ω', Ω, $|u|_{2;\,\Omega}$.

To obtain estimates near the boundary it is necessary to straighten it; we shall suppose, that in a neighbourhood U of $x_0\in\partial\Omega$ the boundary is given by the equation $x_2=0$ and $\Omega\cap U\subset\{x_2>0\}$. Again we consider (14) with $v=u_{x_1}$, v is a solution of the problem with boundary condition $v|_\Gamma=u_{x_1}|_\Gamma=\varphi_{x_1}$, $\Gamma=U\cap\partial\Omega$.

A $C^{1,\alpha}$-estimate for v in $u\cup\Gamma$ results from [4, p. 248] (i. e. an estimate for the tangential derivative of u). What remains is to establish a bound for the

Hölder norm of $u_{x_2x_2}$. We shall use the implicit function theorem: the equation is of the form $F(x, u, u_{x_1}, u_{x_2}, u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2})=0$.

The function F is well-defined and smooth in a neighbourhood of $(0, u(0), u_{x_1}(0), u_{x_2}(0), u_{x_1x_2}(0), u_{x_2x_2}(0))$, and $F_{u_{x_2x_2}} \neq 0$ (by the uniform ellipticity). That makes it possible to solve the equation, i. e. locally $u_{x_2x_2}$ is a smooth function of the remaining variables, whose C^α-norms have already been estimated. Finally

(15) $$|u|_{2,\alpha;\bar\Omega} \leqq C.$$

Now we can apply the method of continuity. Consider the problem

(16) $$\begin{cases} \theta(1-\lambda)\Delta u+\lambda F[u]=0 \text{ in } \Omega \\ u|_{\partial\Omega}=\varphi, \end{cases}$$

where $\lambda \in [0, 1]$, $F[u]=f(D^2u)+g(x, u, Du)=F(x, u, Du, D^2u)$. Since the function $\lambda f(r)+\theta(1-\lambda)(r_{11}+r_{22})$ is convex, this is a problem of the same type as (1) and the a priori estimate (15) is valid for its solutions. Let Λ be the set of all λ, for which (16) is solvable. We know that $0 \in \Lambda$. We shall show that Λ is closed and relatively open in $[0, 1]$, and thus $\Lambda=[0, 1]$. Let u_λ be the solution of (16) for $\lambda \in [0, 1]$. Without loss of generality $\Phi \equiv 0$. Let us denote $B=\{u \in C^{2,\alpha}(\bar\Omega) \mid u|_{\partial\Omega}=0\}$; obviously B is a closed subspace of $C^{2,\alpha}(\bar\Omega)$. The problem (16) is equivalent to

(17) $$\theta u=\Delta^{-1}(\theta\lambda\Delta u-\lambda F[u]), \quad u \in B.$$

It is easily seen that $|\theta\Delta u-F[u]|_{0,\alpha;\bar\Omega} \leqq C|u|_{2,\alpha;\bar\Omega}$; since $\Delta^{-1}: C^\alpha(\bar\Omega) \to B$ is linear and bounded, for λ small enough (17) defines a contraction mapping, i. e. (16) is solvable. Hence, Λ contains an interval of the form $[0, \lambda_0]$.

Let us show that Λ is closed in $[0, 1]$. Let $\lambda_i \in \Lambda$, $\lambda_i \to \lambda'$. By the uniform a priori bound

(18) $$|u_{\lambda_i}|_{2,\alpha;\bar\Omega} \leqq C$$

follows that we can choose a subsequence of $\{u_{\lambda_i}\}$, converging in $C^{2,\beta}(\bar\Omega)$ for some $0<\beta<\alpha$; by (18) $u_{\lambda'}=\lim\limits_i u_{\lambda_i} \in C^{2,\alpha}(\bar\Omega)$ and $|u_{\lambda'}|_{2,\alpha;\bar\Omega} \leqq C$. Continuity implies that $u_{\lambda'}$ solves (16) with parameter λ'.

To prove that Λ is relatively open in $[0, 1]$ we shall use the implicit function theorem in Banach spaces [5]. Let $\bar\lambda \geqq \lambda_0$, $\bar\lambda \in \Lambda$. We denote $\psi(x, \lambda, u) =\theta(1-\lambda)\Delta u+\lambda F[u]$.

By supposition $\bar\lambda \in \Lambda$, i. e. there exists $u_{\bar\lambda} \in B$ such that $\psi(x, \bar\lambda, u_{\bar\lambda})=0$.

The Frechet derivative of ψ with respect to u_λ is

$$(D_{u_\lambda}\psi)h=\theta(1-\lambda)\Delta h+\lambda\sum_{i,j} f_{ij}(D^2u_\lambda)h_{x_ix_j}+\sum_i g_{p_i}h_{x_i}+g_zh.$$

The operator $D_{u_2}\psi$ is linear, elliptic and according to (5) it is an isomorphism from B onto $C^\alpha(\bar\Omega)$. Consequently we can apply the implicit function theorem and conclude that for $\lambda \in (\bar\lambda-\delta, \bar\lambda+\delta)$ there exists $u=u(\lambda) \in B$ such that $\psi(x, \lambda, u(\lambda))=0$, i. e. $(\bar\lambda-\delta, \bar\lambda+\delta) \subset \Lambda$.

This means that $\Lambda=[0, 1]$ and (1) is also solvable in $C^{2,\alpha}(\bar\Omega)$. Thus Theorem 1 is proved.

Remark. The solution u belongs to $C^{2,\beta}(\Omega)$ for arbitrary $\beta\in(0, 1)$. The coefficients of (14) belong to $C^{\alpha}(\Omega)$, consequently $v\in C^{2,\alpha}(\Omega)$, i. e. $u\in C^{3,\alpha}(\Omega)$ and standard imbedding theorems imply that $u\in C^{2,\beta}(\Omega)$ for each $0<\beta<1$.

REFERENCES

1. L. C. Evans. Classical solutions of fully nonlinear, convex, second-order elliptic equations. *Comm. Pure Appl. Math.*, **25**, 1982, 333-363.
2. L. C. Evans, P. L. Lions. Résolution des équations de Hamilton-Jacobi-Bellman. *C. R. Acad. Sci. Paris*, June 1980.
3. P. L. Lions. Résolution analytique des problèmes de Bellman-Dirichlet. *Acta. Math.*, **146**, 1981, 151-166.
4. D. Gilbarg, N. S. Trudinger. Elliptic partial differential equations of second order. New York, 1977.
5. J. T. Schwartz. Nonlinear functional analysis. New York, 1969.

Centre for Mathematics and Mechanics
Sofia 1090 *P. O. Box 373*

Received 5. 8. 1983