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A CRITICAL BRANCHING PROCESS WITH DECREASING MIGRATION

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A limit theorem for a critical branching process with non-homogeneous random migration is obtained, when the probability of migration converges to zero.

Let us have on the probability space $(\Omega, \mathcal{F}, P)$ three independent collections of integer-valued random variables (r. v.), where:

1) $\{X_n(k),\ n=0,1,\dots;\ k=1,2,\dots\}$ are independent r. v. with a probability generating function (p. g. f.) $F(s)$

$$=\mathsf{E}s^{X_n(k)}=\sum_{i=0}^{\infty} f_i s^i,\ |s|\leq 1;$$

2) $\{Y_n(k),\ n=0,1,\dots;\ k=1,2,\dots\}$ are independent r. v. with p. g. f. $G(s)=\mathsf{E}s^{Y(k)}=\sum_{i=0}^{\infty} g_i s^i$;

3) $\{\xi_n,\ n=0,1,2,\dots\}$ are independent r. v. with distributions

$$(1)\qquad \begin{cases} P\{\xi_n=-1\}=p_n,\ P\{\xi_n=0\}=q_n,\ P\{\xi_n=1\}=r_n, \\ p_n+q_n+r_n=1,\ n=0,\ 1,\ 2,\dots . \end{cases}$$

Now we form the controlled functions

$$(2)\qquad \begin{cases} \varphi_n(m)=\max\{\min(m, m+\xi_n), 0\},\ \psi_n(m)=\max(0, \xi_n), \\ n, m=0, 1, 2, \dots \end{cases}$$

Then we consider a branching process with non-homogeneous random migration $\{Z_n\}$, which can be defined in the following way:

$$(3)\qquad Z_{n+1}=\sum_{k=1}^{\varphi_n(Z_n)} X_n(k)+\sum_{k=1}^{\psi_n(Z_n)} Y_n(k),\ n=0,\ 1,\ 2,\dots,$$

where as usual $\sum_{i=1}^{0}=0$.

It follows from (1)—(3) that if $q_n\equiv 1$ then $\{Z_n\}$ will be a classical Galton-Watson process characterized by the independence of particle evolutions (see [8] and [1]). In general, definition (3) describes models without this restriction, i. e. processes which admit particle interactions. If $r_n\equiv 1$ we obtain a well-known Galton-Watson process with immigration (see [1] and [8]). The critical case with $p_n\equiv 1$ is investigated by V a t u t i n [7].

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Remark that subcritical and critical processes with $p_n \equiv p$, $q_n \equiv q$, and $r_n \equiv r$ $(p+q+r=1)$ are studied in [3, 9, 10, 11].

In [4] and [12] we considered a model (3) with $F'(1) \leqq 1$ and $p_n \downarrow 0$, $q_n \uparrow q$, $r_n \uparrow r$, $p+q=1$, i. e. a process with decreasing emigration. On the other hand, in [4] and [13] we investigated a critical case $F'(1)=1$, $0 < F''(1) = 2b < \infty$ and $\lim q_n = 1$ such that $r_n \sim c/\log n$ and $p_n = o(r_n)$. In this case we obtained that

(4) $$\lim \mathsf{P}\{Z_n > 0\} = 1 - e^{-\theta}, \quad \theta = c/b > 0;$$

(5) $$\lim \mathsf{P}\{\frac{\log Z_n}{\log n} \leqq x\} = e^{-\theta(1-x)}, \quad 0 \leqq x \leqq 1;$$

(6) $$\lim \mathsf{P}\{1 - \frac{\log Z_n}{\log n} \leqq x \mid Z_n > 0\} = \frac{1-e^{-\theta x}}{1-e^{-}}, \quad 0 \leqq x \leqq 1.$$

Now we continue studying the processes (3) in the critical case when $\lim q_n = 1$ and $r_n, p_n \to 0$. Depending on the rate of this convergence we obtain another type of limit results, which are also similar to the ones of Badalbaev and Rahimov [6] for continuous time branching processes with immigration of decreasing intensity. Remark that the following results are announced in [5].

Theorem 1. *Let* $F'(1)=1$, $0 < F''(1) = 2b < \infty$, $0 < m = G'(1)$ *and* $d = G''(1) < \infty$. *Suppose* $\lim q_n = 1$ *such that* $p_n \sim C/\log n$, $C > 0$ *and* $r_n \sim L(n)/\log n$, *where* $L(n)$ *is a slowly varying function* (s. v. f.) *and* $L(n) \to \infty$, $n \to \infty$. *Then* $\lim \mathsf{P}\{Z_n > 0\} = 1$, $A_n = \mathsf{E}\, Z_n \sim mnr_n$, $B_n = \operatorname{Var} Z_n \sim mbn^2 r_n$ *and for* $x \geqq 0$,

(7) $$\lim \mathsf{P}\{L(n)(1 - \frac{\log Z_n}{\log n}) \leqq x\} = 1 - e^{-x/b}.$$

Proof. Let $H_n(s) = \mathsf{E} s^{Z_n}$, $|s| \leqq 1$, $n \geqq 0$, where without any restriction we can suppose that $Z_0 = 0$ a. s., i. e. $H_0(s) \equiv 1$. Then it follows from (1)—(3) that

(8) $$H_{n+1}(s) = \mathsf{E}\{\mathsf{E}(s^{Z_{n+1}} \mid Z_n)\}$$
$$= a_n(s) H_n(F(s)) + p_n H_n(0)(1 - F^{-1}(s)),$$

where

(9) $$a_n(s) = p_n F^{-1}(s) + q_n + r_n G(s).$$

Repeated application of relation (8) gives

(10) $$H_{n+1}(s) = U_n(n, s) + \sum_{k=0}^{n} p_{n-k} H_{n-k}(0)(1 - F_{k+1}^{-1}(s)) U_{k-1}(n, s),$$

where

(11) $$U_k(n, s) = \prod_{i=0}^{k} a_{n-i}(F_i(s)), \quad U_{-1}(n, s) \equiv 1,$$

and $F_i(s)$ denotes its functional iterate of $F(s)$, i. e. $F_0(s) = S$, $F_{i+1}(s) = F(F_i(s))$.

Later on we will use the following well-known results for critical Galton-Watson processes (see [1] or [8]):

(12) $0<F_n(0)\leqq F_n(s)\leqq 1, \quad F_n(s)\uparrow 1$, uniformly for $0\leqq s\leqq 1$;

(13) $1-F_n(0)\sim (bn)^{-1}, \quad n\to\infty$;

(14) $1-F_n(s)=(1-s)\,(1+\varepsilon_n(s))\,(1+bn\,(1-s))^{-1}$,

where $\lim \varepsilon_n(s)=0$ uniformly for $0\leqq s\leqq 1$.

Then from (9) and (12) it follows that

$$(15)\qquad 1-a_{n-1}(F_i(s))=\frac{r_{n-i}\,(1-F_{i+1}(s))}{F_{i+1}(s)}\Big[F_{i+1}(s)\frac{1-G(F_i(s))}{1-F_{i+1}(s)}-\frac{p_{n-1}}{r_{n-i}}\Big]\to 0$$

as $n\to\infty$ uniformly by $i\leqq n$ and $0\leqq s\leqq 1$.

On the other hand, there exists N, such that for $n\geqq N$, $0\leqq i\leqq n-N$ and $0\leqq s\leqq 1$, $\frac{1-G(F_i(s))}{1-F_{i+1}(s)}F_{i+1}(s)\geqq\frac{p_{n-i}}{r_{n-i}}$ because of $\frac{1-G(F_i(s))}{1-F_{i+1}(s)}\to m$ as $F_i(s)\to 1$, and conditions of the theorem.

Hence $0\leqq a_{n-i}(F_i(s))\leqq 1$ and

$$(16)\qquad U_n(n,s)=\prod_{i=0}^{n-N} a_{n-i}(F_i(s))\prod_{i=n-N+1}^{n} a_{n-i}(F_i(s))=\Pi_1\,.\,\Pi_2,$$

where

$$(17)\qquad \Pi_2=\prod_{j=0}^{N-1} a_j(F_{n-j}(s))\to 1,\ n\to\infty,$$

and, using (15),

$$(18)\qquad \log\Pi_1=\sum_{i=0}^{n-N}\log\{1-(1-a_{n-i}(F_i(s)))\}$$

$$\sim-\sum_{i=0}^{n-N}(1-a_{n-i}(F_i(s)))=-V_n(s)+W_n(s),$$

as $n\to\infty$ uniformly for $0\leqq s\leqq 1$.

It is not difficult to obtain that for $0\leqq s\leqq 1$

$$(19)\qquad 0\leqq W_n(s)=\sum_{i=0}^{n-N}\frac{1-F_{i+1}(s)}{F_{i+1}(s)}p_{n-i}$$

$$\leqq\frac{1}{f_0}\sum_{i=0}^{n}(1-F_{i+1}(0))\,p_{n-i}=O(1).$$

Indeed, using (12), (13) and conditions of the theorem one can see that

$$(20)\qquad \sum_{i\leqq n/2}(1-F_i(0))\,p_{n-i}\leqq\frac{C+\varepsilon}{\log(n/2)}\sum_{i\leqq n/2}(1-F_i(0))=O(1);$$

$$(21)\qquad \sum_{n/2<i\leqq n}(1-F_i(0))\,p_{n-i}\leqq(1-F_{[n/2]}(0))\sum_{j<n/2}p_j=O(1/\log n).$$

On the other hand, for $0 \leqq s \leqq \rho < 1$ and each $0 < \delta < 1$ we have

$$(22) \qquad V_n(s) = \sum_{i=0}^{n-N} (1 - F_{i+1}(s))\, r_{n-i}$$

$$\geqq \sum_{i \leqq \delta(n-N)} (1 - F_{i+1}(s)) \frac{L(n-i)(1-\varepsilon)}{\log(n-i)}$$

$$\geqq (1-\varepsilon)\big(\min_{j \geqq n(1-\delta)+\delta N} L(j)\big) \sum_{i \leqq \delta(n-N)} (1 - F_{i+1}(s))/\log(n-i) \to \infty$$

as $n \to \infty$ uniformly by $0 \leqq s \leqq \rho < 1$.

Relations (16)—(22) show that for $0 \leqq s \leqq \rho < 1$

$$(23) \qquad \lim U_n(n, s) = 0.$$

Since from (10) $0 \leqq H_{n+1}(0) = \mathsf{P}\{Z_n = 0\} \leqq U_n(n, 0)$ then by (23) we obtain the first statement of the theorem, i. e. $\lim \mathsf{P}\{Z_n > 0\} = 1$.

On the other hand, for $0 \leqq s \leqq 1$

$$(24) \qquad \Big|\sum_{k=0}^{n} p_{n-k} H_{n-k}(0)(1 - F_{k+1}^{-1}(s))\, U_{k-1}(n, s)\Big|$$

$$\leqq f_0^{-(N+1)} \sum_{k=0}^{n} p_{n-k} H_{n-k}(0)(1 - F_{k+1}(0))$$

$$= o\big(\sum_{k=0}^{n} p_{n-k}(1 - F_k(0))\big) = o(1), \quad n \to \infty,$$

because of (20) and (21).

Denote $y_n = \exp\{-\lambda x^{-b/r_n} n^{-1}\}$ for $\lambda > 0$ and $0 < x < 1$. Since r_n is a s. v. f. there is (as it is noted in [6]) a function $\alpha_r(n) \to \infty$, $n \to \infty$, such that for each function $\alpha(n)$, $1 \leqq \alpha(n) \leqq \alpha_r(n)$ we have $\lim (r_{[n/\alpha(n)]})/r_n = 1$. Then

$$(25) \qquad V_n(y_n) = \sum_{k=N}^{n} r_k (1 - F_{n-k}(y_n))$$

$$= \sum_{N \leqq k \leqq n/\alpha_r(n)} + \sum_{n/\alpha_r(n) < k \leqq n(1-x^{b/r_n})} + \sum_{n(1-x^{b/r_n}) < k \leqq n}$$

$$= S_1(n) + S_2(n) + S_3(n),$$

where

$$0 \leqq S_1(n) \leqq (1 - F_{n-[n/\alpha_r(n)]}(0)) \sum_{j \leqq n} r_j = o(1), \quad n \to \infty,$$

$$0 \leqq S_3(n) \leqq \big(\max_{n(1-x^{b/r_n}) \leqq k \leqq n} r_k\big)(1 - y_n)\, n x^{b/r_n} = o(1), \quad n \to \infty,$$

and using (14)

$$S_2(n) \sim \sum_{n/\alpha_r(n) < k \leqq n(1-x^{b/r_n})} \frac{r_k(1-y_n)}{1 + b(n-k)(1-y_n)}$$

$$\sim r_n \Big[\sum_{n/\alpha_r(n) < k \leqq n(1-x^{b/r_n})} \frac{1-y_n}{1 + b(n-k)(1-y_n)}\Big].$$

$$\text{Since } \int_l^{m+1}(1+Cx)^{-1}\,dx \leq \sum_{k=l}^{m}\frac{1}{1+Ck} \leq \int_{l-1}^{m}(1+Cx)^{-1}\,dx,$$

then similarly to ([6], p. 280), one can find that $\lim S_2(n)=-\log x$, and from (25) $\lim V_n(y_n)=-\log x$.

On the other hand, similarly (25), it is not difficult to see that from (19)

$$W_n(y_n) \leq \frac{1}{f_0}\sum_{k=N}^{n} p_k(1-F_{n-k}(y_n)) = \frac{1}{f_0}(T_1(n)+T_2(n)+T_3(n)),$$

where $T_1(n), T_3(n)\to 0$ and $0 \leq T_2(n) \leq \max\limits_{n/\alpha_r(n)\leq k \leq n(1-x^{b/r_n})} L(k)^{-1} S_2(n) \to 0$.

Hence, from (16)—(18) it follows that $\lim U_n(n, y_n)=x$, $0<x<1$, and by (10) and (24) $\lim H_n(y_n)=x$. The limit is a Laplace transform of a distribution with mass $1-x$ at infinity and x at the origin and by continuity theorem (see [2], p. 408) we obtain $\lim \mathsf{P}\{Z_n/nx^{b/r_n} \leq u\}=x$, $u>0$, which is equivalent to (7).

The moments of Z_n can be obtained by differentiating (8) or (10) and putting $s \uparrow 1$:

$$A_n = H_n'(1) = \sum_{k=0}^{n-1}\{r_k m - p_k(1-H_k(0))\},\ A_0=0,$$

$$B_n = H_n''(1) = \sum_{k=0}^{n-1}\{2A_k(b+r_k m-p_k)+r_k d+2(1-b)p_k(1-H_k(0))\}.$$

Now asymptotic behaviour of A_n and B_n follows from Theorem 1 ([2], Ch. 8, § 9).

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