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ON A REPRESENTATION OF MIXED FINITE DIFFERENCES

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A representation of mixed differences as a combination of forward finite differences is given. Two applications of this decomposition are made.

1. Finite differences. Let X and Y be real linear spaces. For each $h \in X$ we define the translation operator $T(h): Y^x \to Y^x$ by $T(h)f(x) = f(x+h)$ for each $f: X \to Y$ and $x \in X$.

It is well-known that $\{T(h): h \in X\}$ is an abelian group with superposition as a group operation.

For $k \in \mathrm{N}$ and $h \in X$ we define the usual k-th forward difference as

$$\Delta^k(h) = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} T(ih). \tag{1}$$

Obviously $\Delta^k(h)\Delta^l(h) = \Delta^{k+l}(h)$, and the finite difference operator commutes with the translation operator. Let us also mention the formula $\Delta^k(-h) = (-1)^k T(-kh)\Delta^k(h)$.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathrm{N}^m$, $|\alpha| = \sum_{i=1}^{m} \alpha_i$ and $h = (h_1, \dots, h_m) \in X^m$. We define the α mixed finite difference with a step h by $\Delta^\alpha(h) = \Delta^{\alpha_1}(h_1)\Delta^{\alpha_2}(h_2) \dots \Delta^{\alpha_m}(h_m)$.

The aim of this section is to represent the α mixed difference as a sum of forward differences of order $|\alpha|$. Another decomposition of this type is given in [1]. The reasons forced us to consider a new representation will be discussed later.

It is easy to find that

$$\Delta^1(h_1)\Delta^1(h_2) = \Delta^2\left(\frac{h_1+h_2}{2}\right) - T(h_2)\Delta^2\left(\frac{h_1-h_2}{2}\right).$$

The above formula for a given function f says

$$f(h_1+h_2) - f(h_1) - f(h_2) + f(0)$$

$$= \left[f(h_1+h_2) - 2f\left(\frac{h_1+h_2}{2}\right) + f(0)\right] - \left[f(h_1) - 2f\left(\frac{h_1+h_2}{2}\right) + f(h_2)\right].$$

Therefore, to get our representation, we add the point $(h_1+h_2)/2$ to the points $0, h_1, h_2, h_1+h_2$. But even the difference $\Delta^1(h_1)\Delta^2(h_2)$ causes big difficulties — in the representation in [1] nine points are added to the original six points.

The following theorem is our main result.

Theorem 1. *For each $\alpha \in \mathrm{N}^m$ there are a natural number N, numbers b_ν and vectors $u^\nu, v^\nu \in \mathrm{R}^m (u^\nu = (u_1^\nu, \dots, u_m^\nu))$ $(\nu = 1, 2, \dots, N)$ such that*

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i) $0 \leq v_i^\nu,\ v_i^\nu + |\alpha| u_i^\nu \leq \alpha_i$ *for* $i=1, 2, \dots, m$ *and* $\nu = 1, 2, \dots, N$;
and for each $h \in X^m$ *we have*

ii) $\Delta^\alpha(h) = \sum_{\nu=1}^{N} b_\nu \Delta^{|\alpha|}(\sum_{i=1}^{m} u_i^\nu h_i)\, T(\sum_{i=1}^{m} v_i^\nu h_i)$.

We divide the proof of Theorem 1 to some steps.

Lemma 1. *Let* $k, M \in \mathbf{N}$, $M \geq k$, $h \in \mathbf{R}$, $h \neq 0$, $T(h)\colon \mathbf{R}^{\mathbf{R}} \to \mathbf{R}^{\mathbf{R}}$, $T_1(h) = \sum_{i=0}^{M} a_i T(ih)$ *for some real* a_i *and* $T_1(h)P \equiv 0$ *for each* $P \in H_{k-1}$ — *the set of all algebraic polynomials of one real variable of a degree at most* $k-1$. *Then there are numbres* $\{b_j\colon j=0, \dots, M-k\}$ *such that*

$$T_1(h) = \Delta^k(h)\,[\sum_{j=0}^{M-k} b_j T(jh)].$$

Proof. We get the numbers $\{b_j\}$ as a solution of the linear system

$$\sum_{r=0}^{\min(i,k)} (-1)^r \binom{k}{r} b_{i-r} = a_i \quad \text{for} \quad i=0, 1, \dots, M-k.$$

The system has unique solution because its determinant is equal to 1. To complete the proof we should show that for the numbers $\{c_i\}$ given by $c_i = \sum_{r=0}^{M-i} (-1)^{M-i-r} \binom{k}{M-i-r} b_{M-r}$ we have $c_i = a_i$ $(i = M-k+1, \dots, M)$. So we consider the operator

$$T_2(h) = T_1(h) - \Delta^k(h)[\sum_{j=0}^{M-k} b_j T(jh)] = \sum_{i=M-k+1}^{M} (a_i - c_i) T(ih).$$

From the conditions of the lemma $T_2(h)P \equiv 0$ for each $P \in H_{k-1}$. Therefore

$$\sum_{i=M-k+1}^{M} (a_i - c_i)\,(ih)^s = 0 \quad \text{for} \quad s=0, 1, \dots, k-1.$$

The determinant of this system is Vandermond's determinant and hence $a_i = c_i$. This completes the proof.

The next lemma is the basic step in the proof of Theorem 1.

Lemma 2. *If* $k \in \mathbf{N}$ *and* $N = k^2 - 1$, *then there are constants* $b_{ij}(i, j = 0, 1, \dots, k)$ *and* $c_{\mu\nu}$ $(\mu = 1, 2, \dots, k,\ \nu = 0, 1, \dots, N)$ *such that for each* $h_1, h_2 \in X$ *we have*

$$\Delta^k(h_1)\Delta^1(h_2) = \sum_{i=0}^{k} \sum_{j=0}^{k} b_{ij} \Delta^{k+1}(\frac{i-j}{k+1} h_1 + \frac{1}{k+1} h_2)\, T(jh_1)$$

$$+ \sum_{\mu=1}^{k} \sum_{\nu=0}^{N} c_{\mu\nu}\, \Delta^{k+1}(\frac{h_1}{k+1}) T(\frac{\nu}{k+1} h_1 + \frac{\mu}{k+1} h_2).$$

Proof. We consider the operator

$$(2) \qquad D(A, t, h) = \sum_{i=0}^{k} \sum_{j=0}^{k} a_{ij} T((i-j)th + jh),$$

where $A=\{a_{ij}: i, j=0, \ldots, k\}$, a_{ij} and t are real numbers. Using (1) and (2) we get

$$\sum_{i=0}^{k} \sum_{j=0}^{k} a_{ij}\Delta^{k+1}\left(\frac{i-j}{k+1} h_1+\frac{1}{k+1} h_2\right) T(jh_1)$$

$$=\sum_{\mu=0}^{k+1}(-1)^{k+1+\mu}\binom{k+1}{\mu} D\left(A, \frac{\mu}{k+1}, h_1\right) T\left(\frac{\mu}{k+1} h_2\right),$$

or

$$D(A, 1, h_1)T(h_2)+(-1)^{k+1}D(A, 0, h_1)=\sum_{j=0}^{k}\sum_{i=0}^{k} a_{ij}\,\Delta^{k+1}\left(\frac{i-j}{k+1} h_1\right.$$

$$\left.+\frac{1}{k+1}h_2\right)T(jh_1)+\sum_{\mu=1}^{k}(-1)^{k+\mu}\binom{k+1}{\mu}D\left(A, \frac{\mu}{k+1}, h_1\right)T\left(\frac{\mu}{k+1}h_2\right).$$

Therefore to prove Lemma 2 it is enough to show that there exists a matrix $B=\{b_{ij}: i, j=0, 1, \ldots, k\}$ with the properties:

(3) $$D(B, 1, h)=(-1)^k D(B, 0, h)=\Delta^k(h)$$

and

(4) for each $\mu=1, 2, \ldots, k$ the operator $D(B, \frac{\mu}{k+1}, h)$ is a linear combination of the operators $\{\Delta^{k+1}(\frac{h}{k+1})T(\frac{n}{k+1}h): n=0, 1, \ldots, N\}$.

In (3) and (4) we may assume that $T(h): \mathbf{R}^{\mathbf{R}}\to\mathbf{R}^{\mathbf{R}}$ because all translations are with vectors collinear with h. In view of Lemma 1 the condition

(5) $$D\left(B, \frac{\mu}{k+1}, h\right)P\equiv 0 \text{ for each } P\in H_k \text{ and each } \mu=1, 2, \ldots, k$$

provides (4). The rest part of the proof is an algorithm for construction of a matrix B satisfying (3) and (5).

For real λ and matrices $A=\{a_{ij}: i, j=0, 1, \ldots, k\}$ and $A'=\{a'_{i,j}: i, j=0, 1, \ldots, k'\}$ we get from (2)

(6) $$D(\lambda A, t, h)=\lambda D(A, t, h);$$
$$D(A+A', t, h)=D(A, t, h)+D(A', t, h) \quad \text{if} \quad k=k'.$$

We introduce an operation $*$ between two square matrices by

(7) $$A * A'=A' * A=\{a''_{ij}: i, j=0, 1, \ldots, k+k'\},$$

where $a''_{ij}=\Sigma a_{i_1j_1}a'_{i_2j_2}$ and the sum is over all indices i_1, j_1, i_2, j_2, satisfying the conditions: $i_1+i_2=i$, $j_1+j_2=j$, $0\le i_1, j_1\le k$, $0\le i_2, j_2\le k'$.

Now (2) and (7) give

(8) $$D(A * A', t, h)=D(A, t, h)D(A', t, h).$$

We set

$$A_{1,0}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\qquad A_{1,1}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\qquad\text{and}\tag{9}$$
$$A_{k+1,r}=A_{k,r}*A_{1,0};\ A_{k+1,r+1}=A_{k,r}*A_{1,1}\quad\text{for}\quad k\geqq 1,\ 0\leqq r\leqq k.$$

The above definition is correct because the operation $*$ is commutative. Obviously

$$D(A_{1,0},\ t,\ h)=\Delta^1(h),\qquad D(A_{1,1},\ t,\ h)=\Delta^1(2th-h)T(h-th).\tag{10}$$

From (8), (9) and (10) we get

$$D(A_{k,r},\ t,\ h)=\Delta^{k-r}(h)\Delta^r(2th-h)T(rh-trh).\tag{11}$$

Now (11) gives

$$D(A_{k,r},\ t,\ h)x^s=0\quad\text{for}\quad s=0,1,\ldots,k-1;\tag{12}$$
$$D(A_{k,r},\ t,\ h)x^k=k!\ (2t-1)^r h^k$$

and

$$D(A_{k,r},\ 1,\ h)=\Delta^k(h);\tag{13}$$
$$D(A_{k,r},\ 0,\ h)=\Delta^{k-r}(h)\Delta^r(-h)T(rh)=(-1)^r\Delta^k(h).$$

Now we define B as follows:

a) if $k=2m$ then $B=\sum_{r=0}^{m}a_rA_{k,2r}$, where $\{a_r\}$ satisfy the system (with Vandermond's determinant)

$$\sum_{r=0}^{m}a_r=1;\quad \sum_{r=0}^{m}\left(\frac{2\mu}{k+1}-1\right)^{2r}a_r=0,\quad \mu=1,2,\ldots,m.\tag{14}$$

b) if $k=2m+1$ then $B=\sum_{r=0}^{m}a_rA_{k,2r+1}$, where $\{a_r\}$ satisfy the system

$$\sum_{r=0}^{m}a_r=1;\quad \sum_{r=0}^{m}\left(\frac{2\mu}{k+1}-1\right)^{2r+1}a_r=0,\quad \mu=1,2,\ldots,m.\tag{14'}$$

Now (5), (13) and (14) or (14′) give (3). From (6), (12), (14) or (14′) and the equality $\frac{2\mu}{k+1}-1=-\left(\frac{2(k+1-\mu)}{k+1}-1\right)$ we get (5). This completes the proof.

P r o o f of T h e o r e m 1. Applying α_2 times Lemma 2 with $k=\lambda_1,\lambda_1+1,\ldots,\lambda_1+\lambda_2-1$ we get Theorem 1 in the case $m=2$. Now we use some times Theorem 1 with $m=2$ to prove the part ii) for arbitrary natural m. Part i) is equivalent to the condition

$$\sum_{i=1}^{m}V_i^{\nu}h_i\in\Pi(\alpha,\ h),\quad \sum_{i=1}^{m}(v_i^{\nu}+|\alpha|u_i^{\nu})h_i\in\Pi(\alpha,\ h)\tag{iii}$$

for each ν, where $\Pi(\alpha,\ h)=\{\sum_{i=1}^{m}t_i\alpha_ih_i:\ 0\leqq t_i\leqq 1\}$. Obviously the representation in Lemma 2 satisfies i). Also iii) is conservative under the application of Lemma 2.

Remark 1. For each α the numbers b_ν and the vectors u^ν, $v^\nu(\nu=1, 2, \ldots, N)$ in Theorem 1 can be given explicitly following the proof. Obviously these numbers and vectors are not uniquely determined. Hence the following two problems can be considered (cf. Theorem 2):

Problem 1. For each $\alpha \in \mathbf{N}^m$ to find this representation of the type ii) for which $\Sigma_{\nu=1}^{N} |b_\nu|$ is minimal.

Problem 2. As Problem 1 but the representations should satisfy i).

In the case $\alpha=(1, 1)$ the optimal representation for Problem 2 is given in the beginning of the paper and for it $\Sigma|b_\nu|=2$. For Problem 1 it is

$$\Delta^1(h_1)\,\Delta^1(h_2)=\frac{1}{2}(\Delta^2(h_1)+\Delta^2(h_2)-T(2h_2)\Delta^2(h_1-h_2)).$$

Remark 2. In Theorem 1 the vectors h_i, $i=1, 2, \ldots, m$ can be linearly dependent. In particular they can be collinear to a given vector.

Remark 3. The reason we introduce the representation in Theorem 1 is the following: the representation of the type ii) in [1] does not satisfy i). In other words the vectors in the right-hand side of the decomposition do not belong to the convex hull $\Pi(\alpha, h)$ of the vectors in the left-hand side. This would make difficult such types of applications of the representation as in the next two points because it need proper extensions of functions outside the domains they are defined. Also the existence of such extensions sets additional restrictions on the domain.

2. A relation between two moduli. In this and the next point "domain" means a closure of an open connected set. Also $\overline{S}$ denotes the closure of the set S.

Let D be a domain in $\mathbf{R}^m$. We set

$$D(k, h)=\{x \in D:\ x+kht \in D,\ 0\leq t\leq 1\} \text{ for } k \in \mathbf{N},\ h \in \mathbf{R}^m \text{ and}$$

$$D(\alpha, h)=\{x \in D:\ x+\sum_{i=1}^{m}\alpha_i t_i h_i \in D,\quad 0\leq t_i\leq 1\} \text{ for } \alpha \in \mathbf{N}^m,\ h \in \mathbf{R}^m.$$

For $f \in L_p(D)$ $(1\leq p\leq\infty)$ we define the following moduli:

$$\omega_k(f;\ \delta)_p=\sup\{\|\Delta^k(h)f\|_{L_p(D(k,\ h))}:\ h \in \mathbf{R}^m,\ \sum_{i=1}^{m}|h_i|\leq\delta\} \tag{15}$$

for $k \in \mathbf{N}$ and $\delta>0$, and

$$\omega_\alpha(f;\ \varepsilon)_p=\sup\{\|\Delta^\alpha(h)f\|_{L_p(D(\alpha,\ h))}: h \in \mathbf{R}^m,\ |h_i|\leq\varepsilon_i,\ i=1, \ldots, m\}$$

for $\alpha \in \mathbf{N}^m$ and $\varepsilon \in \mathbf{R}^m$, $\varepsilon_i>0, i=1, 2, \ldots, m$.

It is easy to get by Theorem 1 the following relation between these moduli

Theorem 2. *For $f\in L_p(D)$, $\alpha \in \mathbf{N}^m$, $\varepsilon \in \mathbf{R}^m$, $\varepsilon_i>0$, $i=1, 2, \ldots, m$ we have*

$$\omega_\alpha(f;\ \varepsilon)_p\leq(\sum_{\nu=1}^{N}|b_\nu|)\,\omega_{|\alpha|}(f;\ |\alpha|^{-1}\sum_{i=1}^{m}\alpha_1\varepsilon_1)_p,$$

where b_ν are given in Theorem 1.

In this application of Theorem 1 the condition i) provides

$$x+\sum_{i=1}^{m} v_i^{\nu} h_i \in D(|\alpha|, \sum_{i=1}^{m} u_i^{\nu} h_i) \quad \text{if} \quad x \in D(\alpha, h) \text{ (cf. Remark 3).}$$

For functions periodic in $\mathbb{R}^m$ and properly defined moduli an analog of Theorem 2 holds true. We can derive this analog from Theorem 1, or from the representation in [1]. But Theorem 2 as it is stated cannot be derived from the representation in [1].

3. A generalization of Whitney's theorem. We use the notation $\omega_k(f; D)_p = \omega_k(f; k^{-1}.\operatorname{diam} D)_p$, $\omega_k(f; D) = \omega_k(f; D)_\infty$. In [2] Whitney proves

Theorem A. *For each $n \in \mathbb{N}$ there is a constant K_n such that for each function f continuous in $[a, b]$ we have*

$$\|f-P\|_{C[a,b]} \leqq K_n \omega_n(f; [a, b]),$$

where $P \in H_{n-1}$ and $P(a+i(b-a)/(n-1)) = f(a+i(b-a)/(n-1))$ for $i=0, 1, \ldots, n-1$.

There are many generalizations of this theorem at the present moment (see e. g. [3]). We shall consider only the following one given in [1].

Theorem B. *Let D be bounded Lipschitz-graph domain in $\mathbb{R}^m$ and $1 \leqq p \leqq \infty$. Then for each n there is a constant K, depending on n and D, such that for each $f \in L_p(D)$ there is $P \in H_{n-1}^m$ such that*

$$\|f-P\|_{L_p(D)} \leqq K\omega_n(f; n^{-1} \operatorname{diam} D)_p,$$

where $$H_k^m = \{P(x) = \Sigma a_\alpha x^\alpha : \alpha \in \mathbb{Z}_+^m, |\alpha| \leqq k, x \in \mathbb{R}^m, a_\alpha \in \mathbb{R}\}.$$

Here Lipschitz-graph domain means that for each $x \in \partial D$ in a neighbourhood of x the boundary can be given as the graph of a function satisfying the Lipschitz condition.

In Theorem B the restriction that D is a Lipschitz-graph domain comes from the need to extend properly f outside D (cf. Remark 3). Now we shall generalize Theorem B in Theorem 3 for $p=\infty$ and Theorem 4 for $1 \leqq p < \infty$. These theorems cannot be improved with respect to D in some sense as it is shown in Example 1. We can simply get Theorem 3 from Theorem B and Lemma 3 as we proceed in the proof of Theorem 4 but we wish to show that Theorem 3 can be obtained as a consequence of Theorem 1 and Theorem A avoiding the rather technical Theorem B.

Let $\operatorname{Ext} D$, $\operatorname{Int} D$ and $\Gamma = \mathbb{R}^m \setminus (\operatorname{Ext} D \cup \operatorname{Int} D)$ stand for the exterior, the interior and the boundary of the domain D, respectively. Then we have $D = \Gamma \cup \operatorname{Int} D = \overline{\operatorname{Int} D}$. We set $B(\varepsilon) = \{y \in \mathbb{R}^m : \sum_{i=1}^m y_i^2 < \varepsilon^2\}$ for $\varepsilon > 0$ and $h^\perp = \{y \in \mathbb{R}^m : \sum_{i=1}^m y_i h_i = 0\}$ for $h \in \mathbb{R}^m$.

The domain $D \subset \mathbb{R}^m$ is said to be a segment-graph domain if D is bounded and for each $x \in \Gamma$ there exists $h \in \mathbb{R}^m$, real numbers $t' < 0 < t''$ and an open set $V \subset \mathbb{R}^m$, $V \ni 0$, such that for each $z \in V \cap h^\perp$ the set $\Gamma(x, h; z) = \{y \in \Gamma : y = x+z+th, t' < t < t''\}$ is closed and connected, i. e. $\Gamma(x, h; z)$ is a closed interval or a point.

For example each Lipschitz-graph domain is a segment-graph domain because then $\Gamma(x, h; z)$ is always aone-point set and moreover there exists

$\varphi: V\cap h^{\perp}\to \mathsf{R}$ satisfying the Lipschitz condition such that $\Gamma(x, h; z)=\{x+z+h\varphi(z)\}$ for each $z\in V\cap h^{\perp}$.

Theorem 3. *Let D be a segment graph domain in R^m. Then for each n there exists $c=c(n, D)$ such that for each $F\in C(D)$ there exists $P\in H^m_{n-1}$ such that $\| F-P\|_{C(D)}\leqq c\cdot\omega_n(F; D)$.*

To prove Theorem 3 we need some lemmas.

Lemma 3. *Let $0<a<b$ and for $g\in C[0, b]$ we have $\| g\|_{C[0,a]}\leqq K\omega_n(g; [0, a])$. Then $\| g\|_{C[0,b]}\leqq(2^{rn}K+2^{(r-1)n})\omega_n(f; [0, b])$, where $r\in \mathsf{N}$, $r-1<(\ln b-\ln a)/(\ln n-\ln(n-1))\leqq r$.*

Proof. Let $b\leqq na/(n-1)$. Then for $x\in(a, b]$ we have

$$(16)\qquad |g(x)|\leqq|\Delta^n(\tfrac{x}{n})\, g(0)|+\sum_{i=0}^{n-1}\binom{n}{i}|g(\tfrac{i}{n}x)|\leqq((2^n-1)K+1)\omega_n(g; [0, b]).$$

If $b>na/(n-1)$, we set $a_j=(\frac{n}{n-1})^j a$, $j=0, 1,\dots, r$ (r is given above) and apply r times (16) with $a=a_{j-1}$, $b=a_j$, $j=1, 2,\dots, r-1$ and $a=a_{r-1}$, $b=b$.

Lemma 4. *Let $g\in C[0, a]$ and $|g(air^{-1})|\leqq M$, $i=0, 1,\dots, r$. Then $\| g\|_{C[0,a]}\leqq C_1(r)\omega_{r+1}(g; [0, a])+C_2(r)M$.*

Proof. Theorem A provides us $P\in H_r$ such that

$$\| g-P\|_{C[0,a]}\leqq K_{r+1}.\omega_{r+1}(g; [0, a]) \text{ and } P(air^{-1})=g(air^{-1})$$

for $i=0, 1,\dots, r$. Therefore $|P(air^{-1})|\leqq M$ for $i=0, 1,\dots, r$. Now Lemma 3.1 in [4] gives $\| P\|_{C[0,a]}\leqq C(r)M$. Hence $\| g\|_{C[0,a]}\leqq\| P\|_{C[0,a]}+\| g-P\|_{C[0,a]}\leqq C(r)M+K_{r+1}\omega_{r+1}(g; [0, a])$.

Let $h_1, h_2,\dots, h_n$ be vectors in R^m. By $S_r=S_r(h_1,\dots, h_n; z)$ we denote the set of all points in R^m of the type $z+\sum_{i=1}^{n}\alpha_i h_i$, $\alpha_i=0, 1, 2,\dots$, $\sum_{i=1}^{n}\alpha_i\leqq r$. Then S_r^* stands for the closed hull of S_r.

Lemma 5. *There is a constant $c(r, n)$ such that for each $F\in C(S_r^*)$, $F(x)=0$ for $x\in S_r$ we have $\| F\|_{C(S_r^*)}\leqq c(r, n)\,\omega_{r+1}(F; S_r^*)$.*

Proof. We prove the lemma by induction with respect to n. If $n=1$ then Lemma 5 coincides with Theorem A. Now let Lemma 5 be true for n. The idea for proving it for $n+1$ is the following. Let us assume that

$$(17)\qquad |F(x)|\leqq c_1(r, n)\omega_{r+1}(F; S_r^*(h_1,\dots, h_{n+1}; z))$$

For $x\in S_i^*(h_1,\dots, h_n; z+(r-i)h_{n+1})$, $i=1, 2,\dots, r$. Then for each $y\in S_r^*(h_1,\dots, h_{n+1}; z)$ we consider the restriction of F on the line passing through y and $z+rh_{n+1}$. This line intersects $\{z+rh_{n+1}\}$ and $S_i^*(h_1,\dots, h_n; z+(r-i)h_{n+1})$ $(i=1, 2,\dots, r)$ in equidistant points. So in view of (17) Lemma 4 proves Lemma 5.

To prove (17) we consider the functions

$$(18)\qquad g_i(x)=\Delta^{r-i}(h_{n+1})F(x),\ i=1, 2,\dots, r$$

for $x\in S_i^*(h_1,\dots, h_n; z)$. Obviously $g_i(x)=0$ for $x\in S_i(h_1,\dots, h_n; z)$ and by induction we get

$$(19)\qquad |g_i(x)|\leqq c_2(i, n)\omega_{i+1}(g_i; S_i^*(h_1,\dots, h_n; z))$$

for $x \in S_i^*(h_1, \dots, h_n; z)$. But $\Delta^{i+1}(h)g_i = \Delta^{i+1}(h)\Delta^{r-i}(h_{n+1})F$ and Theorem 1 gives

(20) $$\omega_{i+1}(g_i;\ S_i^*(h_1, \dots, h_n; z)) \leqq c_3(i, r)\omega_{r+1}(F;\ S_r^*)(h_1, \dots, h_{n+1}; z).$$

Now combining (18) with (19) and (20) we get (17) step by step for $i=r$, $r-1, \dots, 1$.

Proof of Theorem 3. For each $x \in \Gamma$ we have $h \in \mathbb{R}^m$ and $V \ni 0$, V — an open set in $\mathbb{R}^m$. The set $\Gamma(x, h; 0)$ is closed and non-empty $(x \in \Gamma(x, h; 0))$ and hence there are numbers t_1, t_2, such that $t' < t_1 < 0 < t_2 < t''$ and $y_i = x + t_i h \bar{\in} \Gamma$. Then y_1 and y_2 belong to $\operatorname{Int} D$ or $\operatorname{Ext} D$. It is impossible both points to belong to one of these sets. Really, if we assume that $y_1, y_2 \in \operatorname{Ext} D$, then there is $\varepsilon > 0$ such that $y_i + B(\varepsilon) = \{y_i + y\colon y \in B(\varepsilon)\} \subset \operatorname{Ext} D \cap \{x+z+th\colon z \in V \cap h^{\perp}, t' < t < t''\}$ for $i=1, 2$. But $x \in \overline{\operatorname{Int} D}$ and hence there is $y_3 \in \operatorname{Int} D$, $y_3 \in x + B(\varepsilon)$. If $y_3 = x + z_3 + t_3 h (z_3 \in V \cap h^{\perp})$ then we have $t' < t_1 < -\varepsilon < t_3 < \varepsilon < t_2 < t''$ and hence $\Gamma(x, h; z_3)$ is not connected, which is a contradiction. Therefore $y_1 \in \operatorname{Int} D$ and $y_2 \in \operatorname{Ext} D$ (we change h to $-h$ if necessary).

Now for each $x \in \Gamma$ we set (h, ε, t_1, t_2 given above) $U(x) = \{y \in \mathbb{R}^m\colon y = x + z + th,\ z \in B(\varepsilon/2) \cap h^{\perp},\ t_1 - \varepsilon/2 < t < t_2\}$ and $G(x) = \{y \in \mathbb{R}^m\colon y = x + z + th,\ z \in \overline{B(\varepsilon/2)} \cap h^{\perp},\ |t_1 - t| \leqq \varepsilon/2\}$. Then $G(x)$ is closed, $G(x) \subset \operatorname{Int} D$, $u(x)$ is open, $u(x) \ni x$. Γ is compact and hence there is a finite subset $\{x_i : i = 1, 2, \dots, N\} \subset \Gamma$ such that $\bigcup_{i=1}^{N} U(x_i) \supset \Gamma$. Let $D_1 = (D \setminus \bigcup_{i=1}^{N} U(x_i)) \cup \bigcup_{i=1}^{N} G(x_i)$. D_1 is a close subset of the open set $\operatorname{Int} D$. Therefore we can find a finite number of closed cubs $\{T_i\colon i = 1, 2, \dots, M\}$ with edges parallel to the co-ordinate lines such that $T = \bigcup_{i=1}^{M} T_i$ is connected and $D_1 \subset T \subset \operatorname{Int} D$.

We choose x_0 and linear independent vectors $h_1, \dots, h_m \in \mathbb{R}^m$ such that $S_{n-1}^*(h_1, \dots, h_m; x_0) \subset T$. Then there is unique $P \in H_{n-1}^m$ such that $P(y) = F(y)$ for $y \in S_{n-1}(h_1, \dots, h_m; x_0)$. Applying Lemma 5 for $F-P$ we get

(21) $$|F(x) - P(x)| \leqq c(n, m)\omega_n(F;\ D)$$

for $x \in S_{n-1}^*(h_1, \dots, h_m; x_0)$. Now using (21) and Lemma 3 we can pass from one to another cube from $\{T_i\colon i = 1, \dots, M\}$ and we get that (21) is true (with another constant) for $x \in T$. At the last using $G(x_i) \subset T$ and Lemma 3 we prove the theorem.

Remark 4. Without any modification of the proof we can prove Theorem 3 for bounded functions if we use instead of Theorem A its generalization given in [5] by Whitney.

Now we give a generalization of Theorem B in the case $1 \leqq p < \infty$.

Theorem 4. *Let D be a segment-graph domain in $\mathbb{R}^m$ and $1 \leqq p < \infty$. Then for each n there exists $c = c(n, D)$ such that for each $F \in L_p(D)$ there exists $P \in H_{n-1}^m$ such that $\|F - P\|_{L_p(D)} \leqq c\omega_n(F;\ D)_p$.*

To prove this theorem we need an analog of Lemma 3 for $L_p (1 \leqq p < \infty)$. Let $h \in \mathbb{R}^m$ and $E \subset h^1$ be a measurable set, $0 < \varepsilon < \varkappa < \infty$ and $\alpha(z)$ be a function defined on E and satisfying the inequalities $\varepsilon \leqq \alpha(z) \leqq \varkappa$. We set $V(\alpha) = \{x \in \mathbb{R}^m\colon x = z + th,\ z \in E,\ 0 \leqq t \leqq \alpha(z)\}$ and $V(c) = V(\alpha)$ when $\alpha(z) = c = \text{const}$.

Lemma 6. *If $g \in L_p(V(\alpha))$ and $\|g\|_{L_p(V(\varepsilon))} \leqq K\omega_n(g;\ V(\varepsilon))_p$, then*

$$\|g\|_{L_p(V(\alpha))} \leqq (2^{rn}K + 2^{rn})\omega_n(g;\ V(\alpha))_p,$$

where $r \in \mathbb{N}$, $r - 1 < (\ln \varkappa - \ln \varepsilon)/(\ln(n+1) - \ln n) \leqq r$.

P r o o f. Let us denote $V_k=V((\alpha)\cap V(n-1)^k\varepsilon n^{-k})$. We shall establish for each $k=0, 1, \ldots, r$ the inequality

$$\|g\|_{L_p(V_k)}\leqq(2^{nk}K+2^{nk}-1)\omega_n(f;\ V_k)_p. \tag{22}$$

The assertion of the lemma follows from (22) because of $V_r=V(\alpha)$. For $k=0$ (22) coincides with the condition for g. Let us assume that (22) holds true for k. For $x\in V_{k+1}\setminus V_k$ we have $x-i\varepsilon(n+1)^k n^{-k-1}\in V_k$ for $i=1, 2, \ldots, n$ and hence

$$\|g\|_{L_p(V_{k+1})}\leqq\|g\|_{L_p(V_k)}+\|g\|_{L_p(V_{k+1}\setminus V_k)}$$

$$\leqq\|g\|_{L_p(V_k)}+\sum_{i=1}^{n}\binom{n}{i}\|g(.-i\frac{\varepsilon}{n}(\frac{n+1}{n})^k h)\|_{L_p(V_{k+1}\setminus V_k)}$$

$$+\|\Delta^n(\frac{\varepsilon}{n}(\frac{n+1}{n})^k h)\,g(.-\varepsilon(\frac{n+1}{n})^k h)\|_{L_p(V_{k+1}\setminus V_k)}$$

$$\leqq(1+(2^n-1))\|g\|_{L_p(V_k)}+\omega_n(g;\ V_{k+1})_p\leqq(2^n(2^{kn}K+2^{kn}-1)+1)\omega_n(g;\ V_{k+1})_p.$$

This completes the proof the Lemma.

We may prove Theorem 4 by the same method as Theorem 3. For that purpose it is necessary to use Lemma 6 instead of Lemma 3 and to replace Lemma 5 with its analog for $1\leqq p<\infty$, which can be obtained using the modified Steklov function. To reduce the proof we use directly Theorem B. Obviously the proof is also valid for the case $p=\infty$.

P r o o f of T h e o r e m 4. Under the notations of the proof of Theorem 3 the set T is Lipschitz-graph domain. Applying Theorem B for it we get that there exists $P\in H^m_{n-1}$ such that $\|F-P\|_{L_p(T)}\leqq K\omega_n(f;\ T)_p$. Using Lemma 6 for the sets $u(x_i)\cap D_0$ and function $g=F-P$ we obtain that the above inequality holds true for D (with a bigger constant). The theorem is proved.

E x a m p l e 1. Let $D_a=\{(x_1,\ x_2)\in \mathrm{R}^2:\ 0\leqq x_1\leqq 1,\ ax_1^2\leqq x_2\leqq x_1^2\}$ for $a<1$. For $a\leqq 0$ D_a is a segment-graph domain (but not a Lipschitz-graph domain) and Theorem 3 is valid. But a theorem for $D_a(0<a<1)$ as Theorem 3 can not be true. For example if $f_\varepsilon(x_1,\ x_2)=\sin\varepsilon|\ln x_1|(\varepsilon>0)$ then $\inf\{\|f_\varepsilon-P\|_{C(D_a)}:P\in H^2_{n-1}\}=1$ (n fixed) and $\omega_n(f_\varepsilon;\ D_a)\leqq c(a,\ n)\varepsilon^n$ if $\varepsilon\to 0$.

E x a m p l e 2. Let $D=\{(x_1,\ x_2,\ x_3)\in\mathrm{R}^3:\ x_2^2+x_3^2\leqq 1,\ |x_1|\leqq x_2^2+x_3^2\}$. Then D is not a segment-graph domain but Theorem 3 is true for D. We can easily prove this proceeding as in the proof of Theorem 3: first we consider the subdomain $\{(x_1,\ x_2,\ x_3):\ 1/2\leqq x_2^2+x_3^2\leqq 1,\ |x_1|\leqq 1/2\}$ and after that the remain of D. But for the two-dimensional analog of $D-D'=\{(x_1,\ x_2)\in\mathrm{R}^m:\ |x_2|\leqq 1,\ |x_1|\leqq x_2^2\}$ (D' is not a domain) Theorem 3 is not true. If f is given by $f(x_1,\ x_2)=x_1$ if $x_2\geqq 0$ and $f(x_1,\ x_2)=-x_1$ if $x_2<0$ then we have $\omega_n(f;D')=0$ for each $n=2, 3\ldots$ but f is not a polynomial.

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