Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

ON THE COVERING OF TRIPLES BY EIGHT BLOCKS

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Let $n>k>t$ be positive integers, and let $\mathscr{X}$ be a set of n elements. Let $C(n, k, t)$ denote the number of k-tuples (subsets of $\mathscr{X}$ having k elements each) in a minimal system of k-tuples such that every t-tuple is contained in at least one k-tuple of the system. It is known when $C(n, k, 3)=i$ for $i\leqq 7$ [1, 2]. In this paper a complete description is given of those n and k for which $C(n, k, 3)=8$.

1. Introduction. Let $\mathscr{X}$ be a set of n elements, i. e. $|\mathscr{X}|=n$, and $n>k\geq t$ be positive integers. A collection $F=\{B_1, \ldots, B_m\}$ of k-tuples of $\mathscr{X}$ is called (n, k, t)-covering if every t-tuple of $\mathscr{X}$ is contained in at least one B_i. The elements of F are called blocks. Let $C(n, k, t)$ denote the smallest integer m such that there exists an (n, k, t)-covering (called covering design) having m blocks. For $\{\alpha_1, \ldots, \alpha_p\}\subseteq\mathscr{X}$, and $G\subseteq F$ let $G(\alpha_1, \ldots, \alpha_p)$ denote the number of blocks of G containing $\{\alpha_1, \ldots, \alpha_p\}$, i. e.

$$G(\alpha_1, \ldots, \alpha_p)=|\{B_i\in G:\{\alpha_1, \ldots, \alpha_p\}\subseteq B_i\}|.$$

If $G(\alpha_1, \ldots, \alpha_p)=0$ we shall write $(\alpha_1, \ldots, \alpha_p)\overline{c}\, G$. The following inequalities hold (cf. [1]):

$$F(\alpha)\geqq C(n-1, k-1, t-1), \tag{1}$$

$$C(n, k, t)\geqq]nC(n-1, k-1, t-1)/k[, \tag{2}$$

where $]x[$ denotes the smallest integer that is at least x.

For any $\alpha\in\mathscr{X}$ let $SF(\alpha)$ denote the set of all $\beta\in\mathscr{X}$ that are contained in exactly the same blocks $B_i\in F$ as $\alpha(\alpha\in SF(\alpha))$.

Further on, let $\mathscr{X}(i)=\{\alpha\in\mathscr{X}:F(\alpha)=i\}$. It follows from (1) that if $i<C(n-1, k-1, t-1)$ then $\mathscr{X}(i)=\emptyset$. Let $A\subseteq\mathscr{X}$ $G\subseteq F$, $G=\{B_{i_1}, \ldots, B_{i_s}\}$. We say that A is "broken" in G if $A\subseteq\bigcup_{j=1}^{s} B_{i_j}$, $A\cap B_{i_j}\neq\emptyset$, $A\not\subseteq B_{i_j}$, $j=1, 2, \ldots, s$.

Let m, t be positive integers. It was shown in [1] that in the set of all ordered pairs (n, k) such that $C(n, k, t)\leqq m$ there exists a pair with n/k maximal. The notation $\Gamma_{m,t}$ was introduced for this maximal ratio. Particularly, a complete description was obtained of those n and k for which $C(n, k, 2)=i$, $3\leqq i\leqq 12$, and $C(n, k, 3)=j$, $4\leqq j\leqq 6$ (we note that $C(n, k, t)\geqq t+1$ [1]). The case $C(n, k, 3)=7$ was studied in [2]. Combining the results in [1] and [2] we get the following theorem:

Theorem 1. (a) $C(n, k, 3)=4$ *if and only if* $1<n/k\leqq 4/3$.

(b) $C(n, k, 3)=5$ *if and only if* $4/3<n/k\leqq 7/5$.

(c) $C(n, k, 3)=6$ *if and only if* $7/5<n/k\leqq 3/2$,

$$(n, k) \neq (6t+3, 4t+2), \ t \geqq 1.$$

(d) $C(n, k, 3)=7$ *if and only if either* $(n, k)=(6t+3, 4t+2)$, $t \geqq 1$ *or* $3/2 < n/k \leqq 17/11$, $(n, k) \neq (17t+3, 11t+2)$.

It follows from this theorem that $\Gamma_{4,3}=4/3, \Gamma_{5,3}=7/5, \Gamma_{6,3}=3/2, \Gamma_{7,3}=17/11$.

2. The Case $C(n, k, 3)=8$

Lemma. *Let* $n > k \geqq 3$ *be integers such that either*

(i) $n/k > 8/5$, *or*

(ii) $n=8t+3$, $k=5t+2$, $t>1$.

Then $C(n, k, 3)>8$.

Proof. Suppose on the contrary. Let $|\mathscr{X}|=n$ and $F=\{B_1, \ldots, B_8\}$ be an $(n, k, 3)$-covering. Denote $\overline{B}_i=\mathscr{X}\setminus B_i$, $i \leqq 8$ ($\overline{B}_i$ are also called blocks). It seems more convenient to write $\overline{B}_{ij\ldots s}$ instead of $B_i \cap B_j \cap \ldots \cap B_s$ ($\overline{B}_i \cap \overline{B}_j \cap \ldots \cap \overline{B}_s$), $1 \leqq i, j, \ldots, s \leqq 8$.

If $n/k > 8/5$ then $(n-1)/(k-1) > 8/5$, and $C(n-1, k-1, 2) \geqq 4$([1, 3]). Therefore $F(\alpha) \geqq 4$ for every $\alpha \in \mathscr{X}$ due to (1). We distinguish two cases:

Case 1. $\mathscr{X}(4) \neq \emptyset$.

Note that if (i) holds then $\mathscr{X}(4) \neq \emptyset$ since

$$\sum_{\alpha \in \mathscr{X}} |F(\alpha)| = k\,|F|. \tag{3}$$

Now let $\alpha \in \mathscr{X}(4)$, $\alpha \in B_{1234}$, and let $|SF(\alpha)|$ be maximal, i. e. if $\beta \in \mathscr{X}(4)$ then $|SF(\beta)| \leqq |SF(\alpha)|$. Denote $F_1=\{B_1, B_2, B_3, B_4\}$. Obviously the block $B_i \setminus SF(\alpha)$, $i \leqq 4$, form an $(n-|SF(\alpha)|, k-|SF(\alpha)|, 2)$-covering, and since $C(n, k, 2)=4$ if and only if $3/2 < n/k \leqq 5/3$ (cf.[1]) then

$$|SF(\alpha)| \leqq [(5k-3n)/2]. \tag{4}$$

We can suppose that $SF(\alpha)=B_{1234}$ (if $\gamma \in B_{12345}$ then γ can be removed from B_5). Further on, there exists $\beta \in \mathscr{X} \setminus SF(\alpha)$, $F_1(\beta)=2$. Let $\beta \in B_{12}$, $A=B_{12} \setminus SF(\alpha)$, $B=\overline{B}_1 \cup \overline{B}_2$ ($\overline{B}_{12}=\emptyset$, since if $\gamma \in \overline{B}_{12}$ then $(\alpha, \beta, \gamma) \,\overline{c}\, F$). Let $A_i=A \cap B_i$ $i=3,4$ ($A_3 \cap A_4=\emptyset$), $A'=A \setminus (A_3 \cup A_4)$. Since the pairs $(\overline{B}_1, \overline{B}_2)$ have to be covered in B_3, B_4, and since this cannot be done in one of these blocks ($2(n-k)>k$) then without loss of generality we can suppose that $\overline{B}_1$ is "broken" in $Q=\{B_3, B_4\}$, and $\overline{B}_2 \subset B_{34}$. Let $\overline{B}_1'=\overline{B}_1 \cap B_{34}$, $w=|\overline{B}_1'|$, $\mathscr{X}_1=A_3 \cup A_4 \cup \overline{B}_1'$, $\mathscr{X}_2=A' \cup (\overline{B}_1 \setminus \overline{B}_1')$
If $b \in \mathscr{X}_1$ then $F(\alpha, b)=3$, and there exists $\gamma \in \mathscr{X}$ such that $F(\alpha, \gamma)=F(\alpha, \gamma, b)=2$
If $b \in \mathscr{X}_2$ then $F(\alpha, b)=2$. Obviously $|\mathscr{X}_1|=|A_3 \cup A_4|+w=5k-3n-2\,|SF(\alpha)|$, $|SF(\alpha)|+|A|=2k-n$, $|X_2|=3n-4k+2\,|SF(\alpha)|$.

1) For every $\beta \in \mathscr{X}_2$, $F(\beta) \geqq 5$.

Conversely, let $\beta \in \overline{B}_{3456}$. Reversing the roles of α and β we see that one of the $\overline{B}_i$, $i=1,2$, say $\overline{B}_1$, is "broken" in $\{B_7, B_8\}$, and the other ($\overline{B}_2$) is contained in B_{78}. If $\gamma \in A' \cap \overline{B}_{78}$ then from $\overline{B}_i$, $i \geqq 5$:

$$4\,|SF(\alpha)|+2(n-k)+2\,|A|+|A_3 \cup A_4| \geqq 4(n-k),$$

which is a contradiction. Therefore $A' \cap \overline{B}_{78}=\emptyset$, and from $\overline{B}_7$, $\overline{B}_8$

$$2\,|SF(\alpha)|+|A \setminus \{\beta\}|+|A_3 \cup A_4|+(n-k) \geqq 2(n-k).$$

This yields $5n-8k+2 \leqq 0$ contradicting (i) as well as (ii). Thus it is proved that if $\beta \in A'$ then $F(\beta) \geqq 5$ but obviously our considerations are valid for $\beta \in \overline{B}_1 \setminus \overline{B}_1'$.

2) For every $a \in \mathscr{X}_1$, $F(a) \geqq 5$.

We shall prove the assertion for $a \in A_3$ but the proof will be valid for all elements in $\mathscr{X}_1$.

Suppose on the contrary, and let $a \in A \cap \bar{B}_{4567}$. Therefore $\bar{B}_8 \cap (\bar{B}_2 \cup \bar{B}_{13}) = \varnothing$ (if either $b_1 \in \bar{B}_{28}$, $b_2 \in \bar{B}_{13}$, or $b_1 \in \bar{B}_{138}$, $b_2 \in \bar{B}_2$ then $(a, b_1, b_2) \bar{c} F$). If $A' \cap \bar{B}_8 = \varnothing$ then from $\bar{B}_8$:

$$|SF(\alpha)| + |(A_3 \cup A_4) \setminus \{a\}| + |\bar{B}_{14}| + w \geqq n-k,$$

which is a contradiction since from B_3 we have $|\bar{B}_{14}| \leqq 2k-n-|SF(\alpha)|-w$ Thus $A' \cap \bar{B}_8 \neq \varnothing (|A' \cap \bar{B}_8| \geqq 2|SF(\alpha)|)$ and consequently $\bar{B}_{2567} = \bar{B}_{1567} = \varnothing$.

2a) Let $\bar{B}_{2ij} \neq \varnothing$, $5 \leqq i < j \leqq 7$. From 1), and from $\bar{B}_i$, $i=3, 5, 6, 7, 8$, we get (note that $\bar{B}_{1j} = \varnothing$, $j=5, 6, 7$):

$$4|SF(\alpha)| + 3(n-k) + |A| + |A \setminus \{a\}| + 2|A_3 \cup A_4| \geqq 5(n-k).$$

This gives $10n-16k+1+2|SF(\alpha)| \leqq 0$, a contradiction.

2b) Suppose that $\bar{B}_{256} = \varnothing$, $\bar{B}_{257} \neq \varnothing$, $\bar{B}_{267} \neq \varnothing$. From $\bar{B}_5$, $\bar{B}_6$ (note that $\bar{B}_{15} = \bar{B}_{16} = \varnothing$):

$$2|SF(\alpha)| + |A \setminus (A' \cap \bar{B}_8)| + |A_3 \cup A_4| + (n-k) \geqq 2(n-k),$$

which is a contradiction since $A' \cap \bar{B}_8 \neq \varnothing$.

2c) Let $\bar{B}_{256} = \bar{B}_{257} = \varnothing$, $\bar{B}_{267} \neq \varnothing$. It is clear that $|\bar{B}_{267}| \leqq |SF(\alpha)| + |A \setminus (A' \cap \bar{B}_8)|$ (we have $\bar{B}_{15} = \varnothing$ and then $|\bar{B}_{25}| \geqq n-k-|SF(\alpha)|-|A \setminus (\bar{A}' \cap B_8)|$). Counting the elements in $\bar{B}_i$, $i \geqq 5$, we obtain ($\bar{B}_{167} = \varnothing$):

$$4|SF(\alpha)| + |A| + 2(n-k) + |\bar{B}_{256}| + 2|\mathscr{X}_1| \geqq 4(n-k).$$

Since $|A' \cap \bar{B}_8| \geqq 2|SF(\alpha)|$ this gives $10n-16k+3|SF(\alpha)| \leqq 0$.

2d) $\bar{B}_{2ij} = \varnothing$, $5 \leqq i < j \leqq 7$. From $\bar{B}_i$, $i \leqq 5$ and from 1) we get $4|SF(\alpha)| + |A| + 2(n-k) + 2|\mathscr{X}_1| \geqq 4(n-k)$, a contradiction.

It is proved that for every $a \in \mathscr{X}_1$, $F(a) \geqq 5$.

3) For every $\beta \in \mathscr{X}_2$, $F(\beta) \geqq 6$.

Let us denote $F_2 = F \setminus F_1 = \{B_5, B_6, B_7, B_8\}$, $A_i' = \{a \in A' : F_2(a) = 3, a \in \bar{B}_{4+i}\}$, $i=1, 2, 3, 4$.

3a) $A_i' \neq \varnothing$, $i \leqq 4$.

First we shall prove that $\bar{B}_{1i} = \varnothing$, $5 \leqq i \leqq 8$. Conversely let $\bar{B}_{15} \neq \varnothing$. Therefore $\bar{B}_{2ij} = \varnothing$, $6 \leqq i < j \leqq 8$, since if $\bar{B}_{267} \neq \varnothing$ selecting $b_1 \in \bar{B}_{15}$, $b_2 \in \bar{B}_{267}$, $a \in A_4'$ we obtain $(a, b_1, b_2) \bar{c} F$. Now using 1) and 2) we get from $\bar{B}_6$, $\bar{B}_7$, $\bar{B}_8$:

$$3|SF(\alpha)| + |\bar{B}_2| + |\bar{B}_1 \setminus \{b_1\}| + |A \setminus A_1'| + |\mathscr{X}_1| \geqq 3(n-k)$$

($b_1 \in \bar{B}_{15}$), which is a contradiction. Hence $\bar{B}_{1i} = \varnothing$, $5 \leqq i \leqq 8$.

Further on, for every $b \in \bar{B}_2$, $F_2(b) \geqq 2$ since if $b \in \bar{B}_{2678}$, $a \in A_1'$, $b_1 \in \bar{B}_1$ then $(a, b, b_1) \bar{c} F$. Now from $\bar{B}_i$, $i=5, 6, 7, 8: 4|SF(\alpha)| + 2(n-k) + |A| + |A_3 \cup A_4| \geqq 4(n-k)$, a contradiction. Thus at least one of the A_i' is empty.

3b) $A_1' = \varnothing$, $A_i' \neq \varnothing$, $i=2, 3, 4$.

Let $\bar{B}_{15} \neq \emptyset$, and $b_1 \in \bar{B}_{15}$. As before $\bar{B}_{2ij} = \emptyset$, $6 \leq i < j \leq 8$. Suppose that $b_2 \in \bar{B}_{256}$. If $b_3 \in \bar{B}_{17}$, $a \in A_4'$ then $(a, b_2, b_3) \bar{c} F$. Thus $\bar{B}_{17} = \bar{B}_{18} = \emptyset$, and since $\bar{B}_{278} = \emptyset$ then from $\bar{B}_7, \bar{B}_8$ we get

$$2|SF(\alpha)| + |A \setminus A_2'| + |A_3 \cup A_4| + (n-k) \geq 2(n-k),$$

a contradiction. This shows that $\bar{B}_{25i} = \emptyset$, $i=6, 7, 8$. Now from $\bar{B}_i$, $i=5, 6, 7, 8$: $4|SF(\alpha)| + |\mathscr{X}_1| + |A| + 2(n-k) \geq 4(n-k)$ which yields $7n - 11k \leq 0$. Thus, we have proved that $\bar{B}_{15} = \emptyset$. Suppose that $\bar{B}_{16} \neq \emptyset$, $\bar{B}_{17} \neq \emptyset$. Clearly $\bar{B}_{25i} = \emptyset$, $i=6, 7, 8$. From $\bar{B}_5$: $|\bar{B}_{25}| \geq n-k-|SF(\alpha)| - |A_3 \cup A_4|$ and therefore $|\bar{B}_{26} \cup \bar{B}_{27} \cup \bar{B}_{28}| \leq |SF(\alpha)| + |A_3 \cup A_4|$. By our choice of α we have $|\bar{B}_{2678}| \leq |SF(\alpha)|$. Now from $\bar{B}_i$, $i=5, 6, 7, 8$:

$$6|SF(\alpha)| + 2(n-k) + |A| + 2|A_3 \cup A_4| + w \geq 4(n-k),$$

which gives $10n - 16k + 3 \leq 0$ (note that $|SF(\alpha)| = 2k - n - |A| \leq 2k - n - 3$ since $A_i' \neq \emptyset$, $i=2, 3, 4$). Thus, not more than one of the sets $\bar{B}_{1i}$, $i=6, 7, 8$ is non-empty. Let $\bar{B}_{16} \neq \emptyset$. Therefore $\bar{B}_{257} = \emptyset$, and from $\bar{B}_5$, $\bar{B}_7$:

$$2|SF(\alpha)| + (n-k) + |A \setminus (A_4' \cup A_2')| + |A_3 \cup A_4| \geq 2(n-k),$$

but this gives $5n - 8k + 3 \leq 0$, showing that $\bar{B}_{1i} = \emptyset$, $5 \leq i \leq 8$. Obviously $|\bar{B}_{25ij}| = 0$, $6 \leq i < j \leq 8$, $|\bar{B}_{2678}| \leq |SF(\alpha)|$, and from $\bar{B}_i$, $5 \leq i \leq 8$:

$$5|SF(\alpha)| + 2(n-k) + |A| + |A_3 \cup A_4| \geq 4(n-k).$$

a contradiction. Therefore at least two of the A_i', $i \leq 4$ are empty.

3c) $A_1' = A_2' = \emptyset$, $A_i' \neq \emptyset$, $i=3, 4$.

Suppose that $\bar{B}_{256} \neq \emptyset$. Then $\bar{B}_{17} = \bar{B}_{18} = \emptyset$ since if $b_1 \in \bar{B}_{17}$, $b_2 \in \bar{B}_{256}$, $a \in A_4$ then $(b_1, b_2, a) \bar{c} F$. We shall prove that $\bar{B}_{2ij} = \emptyset$, $i=5, 6$, $j=7, 8$.

Conversely, let $\bar{B}_{267} \neq \emptyset$. Therefore $\bar{B}_{15} = \emptyset$, and $\bar{B}_{257} = \emptyset$ since if $\bar{B}_{257} \neq \emptyset$ then $\bar{B}_{16} = \emptyset$, and using that $\bar{B}_{2567} = \bar{B}_{2568} = \emptyset$, $|\bar{B}_{2578}| \leq |SF(\alpha)|$, $|\bar{B}_{2678}| \leq |SF(\alpha)|$ we get from $\bar{B}_i$, $5 \leq i \leq 8$: $6|SF(\alpha)| + 2(n-k) + |A| + |A_3 \cup A_4| \geq 4(n-k)$, but this yields (see (4)) $7n - 11k \leq 0$. Now from $\bar{B}_5$, $\bar{B}_7$: $2|SF(\alpha)| + |A \setminus A_4'| + |A_3 \cup A_4| + n - k \geq 2(n-k)$, a contradiction.

Thus, $\bar{B}_{2ij} = \emptyset$, $i=5, 6$, $j=7, 8$ and $\bar{B}_{278} \neq \emptyset$ (from $\bar{B}_7$, $\bar{B}_8$: $2|SF(\alpha)| + (n-k) + |A| + |A_3 \cup A_4| - |(\bar{B}_5 \cup \bar{B}_6) \cap \bar{B}_2| \geq 2(n-k)$ which is a contradiction since $|(\bar{B}_5 \cup \bar{B}_6) \cap \bar{B}_2| > 0$). Now if $b_1 \in \bar{B}_{135}$, $b_2 \in \bar{B}_{146}$, $b_3 \in \bar{B}_{278}$ then $(b_1, b_2, b_3) \bar{c} F$ and thus either $\bar{B}_{15} \subseteq \bar{B}_1'$ $(\bar{B}_{16} \subseteq \bar{B}_1')$, or $\bar{B}_1 \cap (\bar{B}_5 \cup \bar{B}_6) \subset \bar{B}_3 \cup \bar{B}_1'$ $(\bar{B}_4 \cup \bar{B}_1')$. If for example $\bar{B}_{16} \subset \bar{B}_1'$ then from $\bar{B}_6$, $\bar{B}_7$: $2|SF(\alpha)| + |A \setminus A_4'| + w + (n-k) + |A_3 \cup A_4| \geq 2(n-k)$, which is impossible. If $\bar{B}_1 \cap (\bar{B}_5 \cup \bar{B}_6) \subseteq \bar{B}_3 \cup \bar{B}_1'$ then from $\bar{B}_i$, $i = 4, 5, 6, 7, 8$: $4|SF(\alpha)| + 2|A| + |A_3 \cup A_4| + 3(n-k) + w \geq 5(n-k)$, which yields $7n - 11k \leq 0$.

It is proved that $\bar{B}_{256} = \emptyset$.

Further on, we shall prove that $\bar{B}_{2ij} = \emptyset$, $i=5, 6$, $j=7, 8$. Suppose that $\bar{B}_{267} \neq \emptyset$. Hence $\bar{B}_{15} = \emptyset$, $\bar{B}_{257} = \bar{B}_{258} = \emptyset$ (if $\bar{B}_{257} \neq \emptyset$ then $\bar{B}_{16} = \emptyset$ and from $\bar{B}_5$,

$\bar{B}_6$ we get a contradiction) and $|\bar{B}_{25}| \geq n-k-|SF(\alpha)|-|A_3 \cup A_4|$. Obviously $|\bar{B}_{2578}|=|\bar{B}_{2568}|=|\bar{B}_{2567}|=\emptyset$, $|\bar{B}_{2678}| \leq |SF(\alpha)|$, and from $\bar{B}_i$, $i=5, 6, 7, 8$:

$$6|SF(\alpha)|+2|A_3 \cup A_4|+|A|+w+2(n-k) \geq 4(n-k)$$

(only the elements of $\bar{B}_2 \setminus \bar{B}_5$ are contained in more than one of the sets $\bar{B}_i$, $i=5, 6, 7, 8$), which is a contradiction. Thus, $\bar{B}_{267}=\emptyset$ and similarly $|\bar{B}_{2ij}|=\emptyset$, $i=5, 6$, $j=7, 8$. Now from $\bar{B}_5$, $\bar{B}_6$, $\bar{B}_7$:

$$3|SF(\alpha)|+2(n-k)+|A \setminus A_4'|+|A_3 \cup A_4|+w \geq 3(n-k).$$

If this inequality gives no contradiction then (ii) holds, $A_3'=A' \setminus A_4'$ $|A_4'|=1$. It is easy to see that $|A'| \geq t+|SF(\alpha)|>2$ $(t>1)$. Therefore $|A_3'| \geq 2$, and a contradiction will be received from $\bar{B}_5$, $\bar{B}_6$, $\bar{B}_8$.

3d) Finally suppose that $A_i'=\emptyset$, $i=1, 2, 3$, $A_4' \neq \emptyset$.

Clearly $\bar{B}_{2567}=\emptyset$. Suppose that $\bar{B}_{2ij} \neq \emptyset$, $5 \leq i<j \leq 7$. Therefore $\bar{B}_{1i}=\emptyset$, $i=5, 6, 7$, and from $\bar{B}_5$, $\bar{B}_6$, $\bar{B}_7$ we get

$$3|SF(\alpha)|+2|A_3 \cup A_4|+2(n-k) \geq 3(n-k),$$

which is a contradiction. Let $\bar{B}_{256}=\emptyset$, $\bar{B}_{257} \neq \emptyset$, $\bar{B}_{267} \neq \emptyset$. Then $\bar{B}_{15}=\bar{B}_{16}=\emptyset$, and from $\bar{B}_5$, $\bar{B}_6$:

$$2|SF(\alpha)|+2|A_3 \cup A_4|+(n-k) \geq 2(n-k),$$

a contradiction. Therefore we can suppose that $\bar{B}_{256}=\bar{B}_{257}=\emptyset$, $\bar{B}_{267} \neq \emptyset$. This yields $\bar{B}_{15}=\emptyset$ and then $|\bar{B}_{25}| \geq n-k-|SF(\alpha)|-|A_3 \cup A_4|$. Now $|\bar{B}_{267}| \leq |SF(\alpha)| +|A_3 \cup A_4|$ and from $\bar{B}_5$, $\bar{B}_6$, $\bar{B}_7$:

$$4|SF(\alpha)|+2(n-k)+3|A_3 \cup A_4|+w \geq 3(n-k). \tag{5}$$

If (5) gives no contradiction then (ii) holds and $\bar{B}_8 \subseteq \bar{B}_2 \cup A' \cup SF(\alpha)$. Suppose that $\bar{B}_{258} \neq \emptyset$. Therefore either $\bar{B}_{16} \subseteq \bar{B}_1'$ $(\bar{B}_{17} \subseteq \bar{B}_1')$, or $\bar{B}_1 \cap (B_6 \cup B_7) \subseteq \bar{B}_3 \cup \bar{B}_1'$ $(\bar{B}_4 \cup \bar{B}_1')$ but in the both cases the inequality (5) will be strong. Thus, $\bar{B}_{258}=\emptyset$. Now from $\bar{B}_5$, $\bar{B}_8$: $2|SF(\alpha)|+|A|+(n-k) \geq 2(n-k)$, which is a contradiction. Therefore $\bar{B}_{256}=\bar{B}_{257}=\bar{B}_{267}=\emptyset$, and from $\bar{B}_5$, $\bar{B}_6$, $\bar{B}_7$: $3|SF(\alpha)|+2(n-k)+2|A_3 \cup A_4|+w \geq 3(n-k)$, which gives $7n-11k \leq 0$.

Thus it is proved that for every $\beta \in \mathscr{X}_2$, $F(\beta) \geq 6$.

Since $F(a) \geq 4$ for every $a \in \mathscr{X}$, and $|\mathscr{X}_2|=n-|\mathscr{X}_1|-|\bar{B}_2|-|SF(\alpha)|=3n-4k+|SF(\alpha)|$ then a simple counting yields $4n+|\mathscr{X}_1|+2|\mathscr{X}_2| \leq 8k$, which gives $7n-11k \leq 0$, a contradiction.

CASE 1 has been eliminated.

Case 2. $\mathscr{X}(4)=\emptyset$.

Clearly (ii) holds, and $|\mathscr{X}(5)|=8t+2$, $|\mathscr{X}(6)|=1$. The element of $\mathscr{X}(6)$ will be denoted by ω.

1) Let $\alpha \in \mathscr{X}(5)$, $\alpha \in \bar{B}_{123}$. Suppose that one of the blocks $\bar{B}_i$, $i>3$ has no common elements with the union of three of them. For example let $\bar{B}_4 \cap (\bar{B}_5 \cup \bar{B}_6 \cup \bar{B}_7)=\emptyset$. Therefore $\bar{B}_{567} \neq \emptyset$.

Indeed if $\bar{B}_{567}=\emptyset$, then the blocks $B_i\setminus\{\alpha\}$, $i=4, 5, 6, 7$, form a $(8t+2, 5t+1, 2)$-covering and then all triples containing α are covered by the blocks B_4, B_5, B_6, B_7. Consequently α can be removed from B_8, which yields $\alpha\in\mathscr{X}(4)$, contradicting CASE 1.

Now we shall prove that for every pair (α_1, α_2), $F(\alpha_1, \alpha_2)\geqq 3$.

Conversely let $\alpha_1, \alpha_2\in\mathscr{X}(5)$, $F(\alpha_1, \alpha_2)=2$, $\alpha_1\in\bar{B}_{678}$, $\alpha_2\in\bar{B}_{345}$ and let $|SF(\alpha_1)|+|SF(\alpha_2)|$ be maximal, i. e. if $\beta_1, \beta_2\in\mathscr{X}(5)$, $F(\beta_1, \beta_2)=2$ then $|SF(\beta_1)|+|SF(\beta_2)|\leqq|SF(\alpha_1)|+|SF(\alpha_2)|$. We introduce the following notations: $D=\bar{B}_1\cup\bar{B}_2$ (obviously $\bar{B}_{12}=\emptyset$) $R_1=\{B_6, B_7, B_8\}$, $\bar{R}_1=\{\bar{B}_6, \bar{B}_7, \bar{B}_8\}$, $R_2=\{B_3, B_4, B_5\}$, $\bar{R}_2=\{\bar{B}_3, \bar{B}_4, \bar{B}_5\}$, $D_i^1=(\alpha\in D\colon R_1(\alpha)=1,\ \alpha\in\bar{B}_i\}$, $i=6, 7\ 8$, and similarly $D_i^2=\{\alpha\in D\colon R_2(\alpha)=1,\ \alpha\in\bar{B}_i\}$, $i=3, 4, 5$, $A=B_{12}\setminus(SF(\alpha_1)\cup SF(\alpha_2))$. Note that $\bar{B}_i\cap(D\setminus\{\omega\})\neq\emptyset$.

2) $D_i^1=D_j^2=\emptyset$, $i=6, 7, 8$, $j=3, 4, 5$.

Let $b_1\in D_i^1\cap\bar{B}_1$. If $b_2\in\bar{B}_{2i}$ then $(b_1, b_2, \alpha_2)\,\bar{c}\,F$ so $D\cap\bar{B}_i\subset\bar{B}_1$ and this yields $D_i^1\subset\bar{B}_1$. Thus, either $\bigcup_{i=6}^{8} D_i^1\subseteq\bar{B}_1$, or $\bigcup_{i=6}^{8} D_i^1\subseteq\bar{B}_2$, and obviously this holds also for the D_j^2.

2a) First we shall prove that not more than one of the D_i^1, and one of the D_i^2 are nonempty. Conversely let $D_6^1\neq\emptyset$, $D_7^1\neq\emptyset$, $D_6^1\subset\bar{B}_1$. This yields $\bar{B}_{26}=\bar{B}_{27}=\emptyset$, and therefore $D_8^1\neq\emptyset$ due to 1) ($\bar{B}_2$ has no common elements with $\bar{B}_1\cup\bar{B}_6\cup\bar{B}_7$). Thus, $\bar{B}_{28}=\emptyset$. Since $F(\alpha)=5$ for all $\alpha\in\mathscr{X}\setminus\{\omega\}$ then $D_i^2\neq\emptyset$, $i=3, 4, 5$, (if $D_3^2=D_4^2=\emptyset$, then $\bar{B}_2\setminus\{\omega\}\subset\bar{B}_3\cap\bar{B}_4$ and thus, $\bar{B}_3=\bar{B}_4$ which shows that at, least two of the D_i^2 are nonempty), $\bigcup_{i=3}^{5} D_i^2=\bar{B}_2\setminus\{\omega\}$. If there exists a pair (i, j) $i\in\{3, 4, 5\}$, $j\in\{6, 7, 8\}$ such that $\bar{B}_{ij}\neq\emptyset$ then selecting $b_1\in D_j^1$, $b_2\in D_i^2, a\in\bar{B}_{ij}$ we obtain $(a, b_1, b_2)\,\bar{c}\,F$, a contradiction. This shows that $\bar{B}_{ij}=\emptyset$, $i=3, 4, 5$, $j=6, 7, 8$, and consequently $A=\{\omega\}$, or $A=\emptyset$. Without loss of generality we can suppose that $\omega\notin\bar{B}_1$ and $\omega\notin\bigcup_{i=6}^{8}\bar{B}_i$. Counting the elements in the blocks of $\bar{R}_1$ we get $3|SF(\alpha_1)|+2|\bar{B}_1|=3(n-k)$, which gives $3|SF(\alpha_1)|=3t+1$, a contradiction. Thus it is proved that not more than one of the sets D_i^1, and one of the sets D_j^2 are nonempty.

2b) Let $D_8^1\neq\emptyset$, $D_5^2\neq\emptyset$, $D_8^1\cup D_5^2\subseteq\bar{B}_1$.

Clearly $\bar{B}_{25}=\bar{B}_{28}=\bar{B}_{23i}=\bar{B}_{267}=\emptyset$. We shall prove that $\bar{B}_{2i}\neq\emptyset$, $i=3, 4, 6, 7$. Let $\bar{B}_{23}=\emptyset$. Since $\bar{B}_{25}=\emptyset$ then, according to 1), $\bar{B}_{135}\neq\emptyset$ which contradicts $D_4^2=\emptyset$. Thus, $\bar{B}_{2i}\neq\emptyset$, $i=3, 4, 6, 7$. Moreover, $\bar{B}_{23}\cup\bar{B}_{24}\supseteq\bar{B}_2\setminus\{\omega\}$, $\bar{B}_{26}\cup\bar{B}_{27}\supseteq\bar{B}_2\setminus\{\omega\}$, and $\bar{B}_{2i}\setminus\{\omega\}\neq\emptyset$, $i=3, 4, 6, 7$ (if $\bar{B}_{23}=\{\omega\}$ then $|\bar{B}_{24}|=n-k-1$, and since $\alpha_2\in\bar{B}_4$ then $\bar{B}_{14}=\emptyset$, a contradiction).

Let $b_1\in\bar{B}_{23i}$, $b_2\in\bar{B}_{24j}$, where $\{i, j\}=\{6, 7\}$ (since $\bar{B}_{2i}\setminus\{\omega\}\neq\emptyset$ such a choice is always possible). If $b_3\in\bar{B}_{158}$ then $(b_1, b_2, b_3)\,\bar{c}\,F$. Hence $\bar{B}_{158}=\emptyset$. Let $a\in A\cap\mathscr{X}(5)\cap\bar{B}_{58}$. Suppose that $a\in\bar{B}_3$. Selecting $b_1\in D_8^1$, $b_2\in\bar{B}_{24}$ we get $(a, b_1, b_2)\,\bar{c}\,F$. Therefore $A\cap\mathscr{X}(5)\cap\bar{B}_{58}=\emptyset$, and from $\bar{B}_5, \bar{B}_8$:

$$|SF(\alpha_1)|+|SF(\alpha_2)|+|A|+1+(n-k)-|D_8^1\cup D_5^2|\geqq 2(n-k),$$

which is a contradiction since $|SF(\alpha_1)|+|SF(\alpha_2)|+|A|=2k-n$.

2c) Let $D_8^1 \neq \emptyset$, $D_5^2 \neq \emptyset$, $D_8^1 \subset \bar{B}_1$, $D_5^2 \subset \bar{B}_2$.

Obviously $\bar{B}_{58} = \emptyset$. According to our choice of α_1, α_2, we have $|D_8^1| \leqq |SF(\alpha_1)|$, $|D_5^2| \leqq |SF(\alpha_2)|$. As before $\bar{B}_{1i} \neq \emptyset$, $i = 3, 4$, $\bar{B}_{134} = \emptyset$, $\bar{B}_{2j} \neq \emptyset$, $j = 6, 7$, $\bar{B}_{267} = \emptyset$ due to 1).

(α) First suppose that $\bar{B}_{138} \neq \emptyset$, $\bar{B}_{148} \neq \emptyset$, $\bar{B}_{256} \neq \emptyset$, $\bar{B}_{257} \neq \emptyset$. This yields $\bar{B}_{ij} = \emptyset$, $i = 3, 4$, $j = 6, 7$ (if $b \in \bar{B}_{36}$, $b_1 \in \bar{B}_{148}$, $b_2 \in \bar{B}_{257}$ then $(b, b_1, b_2) \bar{c} F$) and therefore either $A = \{\omega\}$, or $A = \emptyset$.

(α_1) Let $A = \{\omega\}$. Then $|SF(\alpha_1)| + |SF(\alpha_2)| = 2t$. Suppose that $\omega \notin \bar{B}_{34}$, $\omega \notin \bar{B}_{67}$. From $\bar{B}_i$, $i = 3, 4, 6, 7$ we get

$$2(|SF(\alpha_1)| + |SF(\alpha_2)|) + |D_8^1| + |D_5^2| + 2(n-k) + 1 \geqq 4(n-k),$$

which is a contradiction since $|D_8^1| + |D_5^2| \leqq |SF(\alpha_1)| + |SF(\alpha_2)|$. Thus without loss of generality $\omega \in \bar{B}_{34}$. Now from $\bar{B}_i$, $i = 3, 4, 6, 7$: $|D_8^1| + |D_5^2| = 2t$ (and therefore $|D_8^1| = |SF(\alpha_1)|$, $|D_5^2| = |SF(\alpha_2)|$) and $\bar{B}_1 \setminus D_8^1 \subset \bar{B}_3 \cup \bar{B}_4$, $\bar{B}_2 \setminus D_5^2 \subset \bar{B}_6 \cup \bar{B}_7$. Thus, from $\bar{B}_3$, $\bar{B}_4$: $|\bar{B}_1 \setminus D_8^1| = 2(n-k) - 4|SF(\alpha_2)| - 2$, and from $\bar{B}_6$, $\bar{B}_7$: $|\bar{B}_2 \setminus D_5^2| = 2(n-k) - 4|SF(\alpha_1)|$. This gives $5|SF(\alpha_2)| + 2 = 5|SF(\alpha_1)|$, a contradiction.

(α_2) $A = \emptyset$. Let $\omega \in \bar{B}_1$. If $\omega \in \bar{B}_{18}$ then $|D_8^1| = |SF(\alpha_1)|$ and $|D_5^2| = |SF(\alpha_2)|$ (if $|D_8^1| < |SF(\alpha_1)|$ then there exists $a \in \bar{B}_1 \setminus \bar{B}_8$ for which either $a \in \bar{B}_{34}$, or $a \in \bar{B}_{ij}$, $i \in \{3, 4\}$, $j \in \{6, 7\}$, a contradiction). Now from the blocks of $\bar{R}_1$: $4|SF(\alpha_1)| - |SF(\alpha_2)| = n-k$ but from $\bar{R}_2$: $4|SF(\alpha_2)| - |SF(\alpha_1)| = n-k+1$, a contradiction. If $\omega \in \bar{B}_{17}$ then $|D_8^1| = |SF(\alpha_1)| - 1$, $|D_5^2| = |SF(\alpha_2)|$ and a contradiction is obtained as before. If $\omega \in \bar{B}_{13}$ then $|D_8^1| = |SF(\alpha_1)| - 1$, $|D_5^2| = |SF(\alpha_2)|$ and from $\bar{R}_1$: $4|SF(\alpha_1)| - |SF(\alpha_2)| = n-k+2$ but from $\bar{R}_2$: $4|SF(\alpha_2)| - |SF(\alpha_1)| = n-k-1$. Thus, it is proved that at least one of the sets $\bar{B}_{138}$, $\bar{B}_{148}$, $\bar{B}_{256}$, $\bar{B}_{257}$ is empty.

(β) Let $\bar{B}_{138} = \emptyset$, $\bar{B}_{256} \neq \emptyset$, $\bar{B}_{257} \neq \emptyset$.

Clearly $\bar{B}_{18} \setminus \{\omega\} \subseteq \bar{B}_{14}$, $\bar{B}_{36} = \bar{B}_{37} = \emptyset$. Moreover $\bar{B}_3 \cap A \cap \mathscr{X}(5) = \emptyset$ since if $a \in \bar{B}_3 \cap A \cap \mathscr{X}(5)$, then $a \in \bar{B}_8$, $\bar{R}_1(a) = 1$ (obviously $\bar{R}_1(a) \geq 1$, and if $a \in \bar{B}_7$, $b \in B_{148}$, $c \in \bar{B}_{256}$ then $(a, b, c)\, \bar{c} F$). If $a \in \bar{B}_5$, $b \in D_8^1$, $c \in D_5^2$ then $(a, b, c)\, \bar{c} F$. Therefore $a \in \bar{B}_{48}$, and if $b \in \bar{B}_{25}$, $c \in D_8^1$ then $(a,b,c)\, \bar{c} F$. Similarly $\bar{B}_8 \cap A \cap \mathscr{X}(5) = \emptyset$. Since $\bar{B}_{36} = \bar{B}_{37} = \emptyset$, and $B_{13} \neq \emptyset$ then $B_{13} = \{\omega\}$. This yields $|D_5^2| = |\bar{B}_{23}| = n-k-1-|SF(\alpha_2)|$, and then $|\bar{B}_{14}| = 1$, a contradiction.

(γ) Let $\bar{B}_{138} = \bar{B}_{257} = \emptyset$

Obviously $\bar{B}_{18} \setminus \{\omega\} \subseteq \bar{B}_{14}$, $\bar{B}_{25} \setminus \{\omega\} \subseteq \bar{B}_{26}$, $\bar{B}_{37} = \emptyset$. Now from $\bar{B}_3$, $\bar{B}_5$, $\bar{B}_7$, $\bar{B}_8$ we obtain $2(|SF(\alpha_1)| + |SF(\alpha_2)| + |A|) + 2(n-k) \geqq 4(n-k)$ which is a contradiction.

2d) Let $D_8^1 \neq \emptyset$, $D_8^1 \subset \bar{B}_1$.

We have $\bar{B}_{2i} \neq \emptyset$, $i = 6, 7$ due to 1). Clearly

$$\bigcup_{i=3}^{5} \bar{B}_{2i} \supseteq \bar{B}_2 \setminus \{\omega\}, \quad \bigcup_{i=3}^{5} \bar{B}_{1i} \supseteq \bar{B}_1 \setminus (D_8^1 \cup \{\omega\}), \quad \bigcup_{i=6}^{8} \bar{B}_{1i} \supseteq \bar{B}_1 \setminus \{\omega\},$$

$\bar{B}_{26} \cup \bar{B}_{27} \supseteq \bar{B}_2 \setminus \{\omega\}$, $D \cap \bar{B}_8 \subset \bar{B}_1$. Now select $b_1 \in \bar{B}_{2i6}$, $b_2 \in \bar{B}_{2j7}$ where $i \neq j$, $\{i, j\} \subset \{3, 4, 5\}$. We shall prove that this choice is possible. First suppose that

$(\bar{B}_{26} \cup \bar{B}_{27}) \setminus \{\omega\} \subseteq \bar{B}_{25}$. Then $|\bar{B}_{25}| = n-k-1$, $|\bar{B}_{15}| = 0$, $\bar{B}_{23} \cup \bar{B}_{24} = \{\omega\}$ ($\bar{B}_{23} \cup \bar{B}_{24} = \emptyset$ yields $D_5^2 \neq \emptyset$ due to 1)). If $\bar{B}_{138} = \emptyset$ then for example $\bar{B}_{136} \neq \emptyset$ and since $\bar{B}_{257} \neq \emptyset$ then $\bar{B}_{48} = \emptyset$ which gives $\bar{B}_{18} = \emptyset$, a contradiction. Thus, $\bar{B}_{138} \neq \emptyset$, and similarly $\bar{B}_{148} \neq \emptyset$. Therefore $\bar{B}_{ij} \neq \emptyset$, $i = 3, 4$, $j = 6, 7$, and from $\bar{B}_i$, $i = 3, 4, 6, 7$:

$$2(|SF(\alpha_1)| + |SF(\alpha_2)| + |A|) + |D_8^1| + 2(n-k) \geqq 4(n-k).$$

This gives $|D_8^1| \geqq 2t$, but $|SF(\alpha_1)| \geqq 2t$, and $3t+1 = |\bar{B}_6| \geqq 4t$.

Now suppose that $\bar{B}_{23} \cup \bar{B}_{24} \cup \bar{B}_{25} \subseteq \bar{B}_{26}$. Therefore $\bar{B}_{26} = \bar{B}_2 \setminus \{\omega\}$, $\bar{B}_{27} = \{\omega\}$ but this yields $\bar{B}_{16} = \emptyset$, contradicting $D_8^1 \neq \emptyset$.

Thus without loss of generality we can suppose that $b_1 \in \bar{B}_{256}$, $b_2 \in \bar{B}_{247}$. Therefore $\bar{B}_{138} = \emptyset$. If $\bar{B}_{148} \neq \emptyset$, $\bar{B}_{158} \neq \emptyset$ then $\bar{B}_{36} = \bar{B}_{37} = \emptyset$ and then $\bar{B}_{13} \cup \bar{B}_{23} = \{\omega\}$ which gives $|SF(\alpha_2)| + |A| + 1 \geqq 3t+1$, a contradiction. Therefore we can suppose that $\bar{B}_{148} = \emptyset$ and then $\bar{B}_{18} \setminus \{\omega\} \subseteq \bar{B}_{15}$. Consequently $\bar{B}_{36} = \emptyset$. Furthermore $\bar{B}_{246} = \emptyset$ (if $\bar{B}_{246} \neq \emptyset$ then $\bar{B}_{37} = \emptyset$ and $\bar{B}_{13} \cup \bar{B}_{23} = \{\omega\}$). Now consider $\bar{B}_{146}$. Clearly the blocks $B_i \setminus \bar{B}_{146}$, $i = 2, 3, 5, 7, 8$, form an $(n - |\bar{B}_{146}|, k - |\bar{B}_{146}|, 2)$-covering having 5 blocks and therefore $(n - |\bar{B}_{146}|)/(k - |\bar{B}_{146}|) \leqq 9/5$ (cf. [1]). Thus, $|\bar{B}_{146}| \leqq [(9k-5n)/4]$. It is obvious that the elements of $D \setminus \bar{B}_{146}$ are contained in not more than one of the blocks $\bar{B}_3$, $\bar{B}_4$, $\bar{B}_6$, $\bar{B}_8$. If $a \in A$ is contained in three of them then $a \in \bar{B}_{468}$ (if $a \in \bar{B}_{368}$, $b \in \bar{B}_{25}$, $c \in D_8^1$ then $(a, b, c) \subset F$; $a \notin \bar{B}_{368}$, $a \notin \bar{B}_{346}$ since $\bar{B}_{36} = \emptyset$). Further on, $\bar{B}_{137} = \bar{B}_{237} = \emptyset$ since $\bar{B}_{15} \neq \emptyset$, $\bar{B}_{25} \neq \emptyset$ but this gives $\bar{B}_{13} \cup \bar{B}_{23} = \{\omega\}$, a contradiction. Thus every element $a \in A$ is contained in two of the blocks $\bar{B}_3$, $\bar{B}_4$, $\bar{B}_6$, $\bar{B}_8$ at the most. Therefore

$$2(|SF(\alpha_1)| + |SF(\alpha_2)| + |A|) + 2(n-k) + |\bar{B}_{146}| \geqq 4(n-k),$$

which is impossible.

The proof of 2) is complete.

3) $\bar{B}_{ij} \neq \emptyset$, $i = 1, 2$, $j = 3, \ldots, 8$.

Conversely let $\bar{B}_{28} = \emptyset$. Therefore $\bar{B}_{26} \cup \bar{B}_{27} \supseteq \bar{B}_2 \setminus \{\omega\}$. We shall prove that it is possible to select $b_1 \in \bar{B}_{2i6}$ $b_2 \in \bar{B}_{2j7}$ where $i \neq j$, $\{i, j\} \subset \{3, 4, 5\}$. If this choice is not possible then either $(\bar{B}_{26} \cup \bar{B}_{27}) \setminus \{\omega\} \subseteq \bar{B}_{2i}$ for some $i \in \{3, 4, 5\}$, or $\bar{B}_{23} \cup \bar{B}_{24} \cup \bar{B}_{25} \subseteq \bar{B}_{2j}$ for some $j \in \{6, 7\}$. In the first case we get a contradiction in the way shown in 2d). Let $\bar{B}_{23} \cup \bar{B}_{24} \cup \bar{B}_{25} \subseteq \bar{B}_{26}$. Therefore $\bar{B}_{27} = \{\omega\}$, $\bar{B}_{26} = \bar{B}_2 \setminus \{\omega\}$, $\bar{B}_{16} = \emptyset$. Since $(\bar{B}_{26} \cup \bar{B}_{27}) \setminus \{\omega\} \not\subseteq \bar{B}_{2i}$, $i = 3, 4, 5$, we can suppose that $\bar{B}_{24} \neq \emptyset$, $\bar{B}_{25} \neq \emptyset$. Now $\bar{B}_{13} = \emptyset$ (if $b \in \bar{B}_{13j}$ then $\bar{B}_{14} \cup \bar{B}_{15} \subset \bar{B}_{1j}$, $j = 7, 8$, showing that either $\bar{B}_{18} = \bar{B}_1$, or $\bar{B}_{17} = \bar{B}_1$), and then $\bar{B}_{23} \neq \emptyset$. Using that $\bar{B}_{23} \neq \emptyset$, $\bar{B}_{25} \neq \emptyset$ we can prove in the same way that $\bar{B}_{15} = \emptyset$, which is a contradiction.

Thus without loss of generality we can suppose that $b_1 \in \bar{B}_{256}$, $b_2 \in \bar{B}_{247}$. Obviously $\bar{B}_{138} = \emptyset$. If $\bar{B}_{148} \neq \emptyset$, $\bar{B}_{158} \neq \emptyset$ then $\bar{B}_{36} = \bar{B}_{37} = \emptyset$. Hence $\bar{B}_{13} \cup \bar{B}_{23} = \{\omega\}$, a contradiction. Now suppose that $\bar{B}_{148} = \emptyset$ ($\bar{B}_{18} \setminus \{\omega\} \subset \bar{B}_{15}$). Therefore $\bar{B}_{36} = \emptyset$. If $\bar{B}_{246} \neq \emptyset$ then $\bar{B}_{37} = \emptyset$, which is impossible. Hence $\bar{B}_{246} = \emptyset$, and it is proved that every element of $D\setminus\overline{B}_{146}$ is contained in one of the blocks $\overline{B}_3$, $\overline{B}_4$, $\overline{B}_6$, $\overline{B}_8$ at the most.

Now we shall prove that the elements of A are contained in not more than two of these blocks. Let $a\in A\cap\overline{B}_{348}$. Therefore $\overline{B}_{17}=\emptyset$ and then $\overline{B}_{13}\subsetneq\{\omega\}$. If $\overline{B}_{13}\cup\overline{B}_{14}\subseteq\{\omega\}$ then $\overline{B}_{25}=\emptyset$, a contradiction. Thus, $\overline{B}_{146}\neq\emptyset$ and since $\overline{B}_{158}\neq\emptyset$ then $\overline{B}_{237}=\emptyset$ which shows that $\overline{B}_{13}\cup\overline{B}_{23}=\{\omega\}$ ($\overline{B}_{28}=\emptyset$). Let $a\in A\cap\overline{B}_{468}$. Since $\overline{B}_{15}\neq\emptyset$, $\overline{B}_{25}\neq\emptyset$ then $\overline{B}_{137}=\overline{B}_{237}=\emptyset$ but this gives $\overline{B}_{13}\cup\overline{B}_{23}=\{\omega\}$. Thus, $A\cap\overline{B}_{468}=\emptyset$. Since $\overline{B}_{36}=\emptyset$ then $A\cap\overline{B}_{368}=A\cap\overline{B}_{346}=\emptyset$. Now we obtain a contradiction from the blocks $\overline{B}_3$, $\overline{B}_4$, $\overline{B}_6$, $\overline{B}_8$.

4) $\overline{B}_{ij}\neq\{\omega\}$, $i=1,\ 2,\ j=3,\ldots,\ 8$.

Let $\overline{B}_{28}=\{\omega\}$. Then $\overline{B}_{26}\cup\overline{B}_{27}=\overline{B}_2\setminus\{\omega\}$. Now we can select a pair $(i,\ j)$, $i\neq j$, $\{i,\ j\}\subset\{3,\ 4,\ 5\}$ such that $\overline{B}_{2i6}\neq\emptyset$, $\overline{B}_{2j7}\neq\emptyset$ (let $i=5$, $j=4$) since $\overline{B}_{2r}\neq\emptyset$, $r\leq 7$ due to 3). Further on, $\overline{B}_{138}=\emptyset$. If $\overline{B}_{148}\neq\emptyset$, $\overline{B}_{158}\neq\emptyset$ then $\overline{B}_{36}=\overline{B}_{37}=\emptyset$ contradicting $\overline{B}_{23}\neq\emptyset$. If $\overline{B}_{18}\subseteq\overline{B}_{15}$ ($\overline{B}_{148}=\emptyset$) then $\overline{B}_{36}=\emptyset$. Suppose that $\overline{B}_{246}\neq\emptyset$. This yields $\overline{B}_{37}=\emptyset$, a contradiction. Thus, $\overline{B}_{246}=\emptyset$ and we have proved that the elements of $D\setminus\overline{B}_{146}$ are contained in not more than one of blocks $\overline{B}_3$, $\overline{B}_4$, $\overline{B}_6$, $\overline{B}_8$.

Now if $a\in A\cap\overline{B}_{348}$ then $\overline{B}_{17}=\emptyset$ contradicting 3). If $a\in\overline{B}_{468}$ then $\overline{B}_{137}=\emptyset$ and $\overline{B}_{13}=\emptyset$ ($\overline{B}_{36}=\emptyset$, $\overline{B}_{138}=\emptyset$). Since $\overline{B}_{36}=\emptyset$ then $A\cap\overline{B}_{368}=B\cap\overline{B}_{346}=\emptyset$. This shows that the elements of A are contained in not more than two of the sets $\overline{B}_i$, $i=3,\ 4,\ 6,\ 8$, and from these blocks we obtain a contradiction.

According to 3) and 4), there exist two pairs $\{i,\ r\}$, $\{j,\ s\}$, $i\neq r$, $\{i,\ r\}\subset\{3,\ 4,\ 5\}$, $j\neq s$, $\{j,\ s\}\subset\{6,\ 7,8\}$ such that $\overline{B}_{2ij}\neq\emptyset$, $\overline{B}_{2rs}\neq\emptyset$. Without loss of generality we can assume that $i=5$, $j=6$, $r=4$, $s=7$. Therefore $\overline{B}_{138}=\emptyset$. If $\overline{B}_{148}\neq\emptyset$, $\overline{B}_{158}\neq\emptyset$ then $\overline{B}_{36}=\overline{B}_{37}=\emptyset$ and $\overline{B}_{13}\subseteq\{\omega\}$, contradicting either 3), or 4). Thus, $\overline{B}_{148}=\emptyset$, $\overline{B}_{18}\setminus\{\omega\}\subseteq\overline{B}_{15}$. Now $\overline{B}_{36}=\emptyset$ and if $\overline{B}_{246}\neq\emptyset$ then $\overline{B}_{137}=\emptyset$ but this yields $\overline{B}_{13}\subseteq\{\omega\}$. Therefore $\overline{B}_{246}=\emptyset$. Similarly $\overline{B}_{248}=\emptyset$. Let $\overline{B}_{238}=\emptyset$. As before it is proved that $A\cap\overline{B}_{348}=A\cap\overline{B}_{468}=A\cap\overline{B}_{368}=A\cap\overline{B}_{346}=\emptyset$, and from $\overline{B}_3$, $\overline{B}_4$, $\overline{B}_6$, $\overline{B}_8$ a contradiction is obtained. Hence $\overline{B}_{238}\neq\emptyset$, and $\overline{B}_{138}=\overline{B}_{147}=\overline{B}_{156}=\emptyset$. If $\overline{B}_{157}\neq\emptyset$ then $\overline{B}_{146}=\emptyset$ which gives $\overline{B}_{14}\subseteq\{\omega\}$. Therefore $\overline{B}_{17}\setminus\{\omega\}\subseteq\overline{B}_{13}$, and similarly $\overline{B}_{16}\setminus\{\omega\}\subseteq\overline{B}_{14}$, $\overline{B}_{18}\setminus\{\omega\}\subseteq\overline{B}_{15}$, $\overline{B}_{26}\setminus\{\omega\}\subseteq\overline{B}_{25}$, $\overline{B}_{27}\setminus\{\omega\}\subseteq\overline{B}_{21}$, $\overline{B}_{28}\setminus\{\omega\}\subseteq\overline{B}_{23}$. Denoting $\overline{B}_{ij}\setminus\{\omega\}=\tilde{B}_{ij}$ we have $\tilde{B}_{18}=\tilde{B}_{15}$, $\tilde{B}_{17}=\tilde{B}_{13}$, $\tilde{B}_{16}=\tilde{B}_{14}$, $\tilde{B}_{28}=\tilde{B}_{23}$, $\tilde{B}_{27}=\tilde{B}_{24}$, $\tilde{B}_{26}=\tilde{B}_{25}$. It is easy to verify that $A\subseteq\{\omega\}$.

(α) Let $A=\{\omega\}$. Clearly $\overline{B}_{ij}=\tilde{B}_{ij}$, $i=1,\ 2,\ j=3,\ldots,\ 8$. If $R_1(\omega)=1$ then from the blocks of $\overline{R}_1$: $3|SF(\alpha_1)|+2(n-k)+2=3(n-k)$, a contradiction. Therefore $R_1(\omega)=R_2(\omega)=2$. From $\overline{R}_1$, $\overline{R}_2$ we obtain $|SF(\alpha_1)|=|SF(\alpha_2)|=t$. Let $\omega\notin\overline{B}_{56}\cup\overline{B}_{47}\cup\overline{B}_{38}\cup\overline{B}_{37}\cup\overline{B}_{46}\cup\overline{B}_{58}$. Without loss of generality we can suppose that $\omega\in\overline{B}_{57}$. Selecting $b_1\in\overline{B}_{28}$, $b_2\in\overline{B}_{16}$ we get $(b_1,\ b_2,\ \omega)\ \overline{\in}\ F$. Now let $\omega\in B_{56}$. Since $\overline{B}_{25}=\overline{B}_{26}$ then $|\overline{B}_{15}|=|\overline{B}_{16}|=|\overline{B}_{14}|=|\overline{B}_{18}|$ and consequently $|\overline{B}_{25}|+1=|\overline{B}_{26}|+1=|\overline{B}_{24}|=|\overline{B}_{29}|=|\overline{B}_{23}|=|\overline{B}_{27}|$. Denoting $|\overline{B}_{25}|=x$ we obtain from $|\overline{B}_{2i}|$, $i=3,\ldots,\ 8$: $6x+4=6t+2$, which is impossible.

(β) $A=\emptyset$. Let $\omega\in\overline{B}_{13}$. From $\overline{R}_2$ we get $3|SF(\alpha_2)|=3t+1$, a contradiction.

It is proved that for every pair (α_1, α_2), $F(\alpha_1, \alpha_2)\geqq 3$.

Let $\alpha, \beta \in \mathscr{X}$. We say that α is equivalent to β, and write $\alpha\sim\beta$ if and only if $\beta\in SF(\alpha)$. It is obvious that $\sim$ is an equivalence relation and thus $\mathscr{X}$ is divided into disjoint classes $S_1, \ldots, S_p$. Let $\alpha\in S_i$, $\beta\in S_j$, $R\subseteq F$. We define $R(S_i, S_j)=R(\alpha, \beta)$. Clearly if $S_i\neq\{\omega\}$, $S_j\neq\{\omega\}$ then $3\leqq F(S_i, S_j)\leqq 4$. Let $F(S_1)$, $F(S_2)$, $F(S_1, S_2)=3$, $S_1\subseteq\bigcap_{i=1}^{5} B_i$, $S_2\subseteq\bigcap_{j=3}^{7} B_j$. Denote $R_1=\{B_1, B_2\}$, $R_2=\{B_3, B_4, B_5\}$, $R_3=\{B_6, B_7\}$, $M_j=\{S_i:\ \forall\alpha\in S_i,\ R_2(\alpha)=j\}$. Now we shall prove that $M_1\setminus\{\omega\}=\emptyset$. Let $S_3\in M_1\setminus\{\omega\}$, $S_3\subset B_3$. If $S_{i_0}\in M_3\setminus\{\omega\}$ then $R_1(\alpha)+R_2(\alpha)=2$ for every $\alpha\in S_{i_0}$ since $F(S_{i_0}, S_3)\geqq 3$. Therefore $S_{i_0}\cap B_8=\emptyset$. Obviously if $S_{i_1}\in M_1\setminus\{\omega\}$ then $S_{i_1}\cap B_8=\emptyset$.

Further on, if $S_{i_2}\in M_2\setminus\{\omega\}$, $S_{i_2}\subset B_{45}$ then $S_{i_2}\cap B_8=\emptyset$ since $F(S_{i_2}, S_3)\geqq 3$. Thus if $S_i\subset B_8$ then either $S_i=\{\omega\}$, or $S_i\subset B_3$. Since $S_1\cup S_2\cup S_3\subset B_3$ then $|B_3|\geqq|B_8|+2$, a contradiction. It is proved that $M_1\setminus\{\omega\}=\emptyset$ and then $\bar{B}_{ij}\setminus\{\omega\}=\emptyset$, $3\leqq i<j\leqq 5$. Therefore $8t+3\geqq|\bigcup_{i=3}^{5}\bar{B}_i|\geqq 9t+2$ which contradicts $t>1$. It follows that $F(\alpha, \beta)\geqq 4$ for every pair $\{\alpha, \beta\}\subset\mathscr{X}$, but a simple counting shows that this is impossible.

The proof is complete.

Theorem 2. $C(n, k, 3)=8$ *if and only if either*

(*i*) $17/11<n/k\leqq 8/5$, $(n, k)\neq(8t+3, 5t+2)$, $t>1$, *or*

(*ii*) $n=17t+3$, $k=11t+2$, $t\geqq 1$.

Proof. First we shall prove that if either (i), or (ii) holds then $C(n, k, 3)\leqq 8$. It follows from Theorem 1 that $C(n, k, 3)>7$. Let $k=5t+j$, $0\leqq j\leqq 4$. Since $C(n_1, k, t)\leqq C(n, k, t)$ for $n_1\leqq n$ it is sufficient to prove that $C(n, k, 3)\leqq 8$ for $n=8t+i$, $k=5t+j$ where (i, j) is one of the ordered pairs (0, 0), (1, 1), (2, 2), (4, 3), (6, 4).

Let $S_1, \ldots, S_8$ be disjoint sets, and $A=S_1\cup S_3\cup S_5\cup S_7$, $B=S_2\cup S_4\cup S_6\cup S_8$. Clearly the blocks $A\cup S_i$, and $B\cup S_j$ $i=2, 4, 6, 8$, $j=1, 3, 5, 7$ cover all triples of $\mathscr{X}=A\cup B$. If $n=8t+i$ set $|S_v|=t+1$, $1\leqq v\leqq i$, $|S_w|=t$, $w>i$. This shows that if (i) holds, and $(n, k)\neq(11, 7)$ then $C(n, k, 3)\leqq 8$. Setting $|S_1|=3t$, $|S_i|=2t+1$, $i=2, 3, 4$, $|S_j|=2t$, $5\leqq j\leqq 8$ we obtain $C(17t+3, 11t+2, 3)\leqq 8$.

Finally, the blocks

$$\{1, 6, 7, 8, 9, 10, 11\}, \quad \{2, 6, 7, 8, 9, 10, 11\},$$
$$\{3, 6, 7, 8, 9, 10, 11\}, \quad \{4, 6, 7, 8, 9, 10, 11\},$$
$$\{5, 6, 7, 8, 9, 10, 11\}, \quad \{1, 2, 3, 4, 5, 6, 7\},$$
$$\{1, 2, 3, 4, 5, 8, 9\}, \quad \{1, 2, 3, 4, 5, 10, 11\}$$

form an (11, 7, 3)-covering. The Lemma shows that there are no other pairs (n, k) such that $C(n, k, 3)=8$.

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Received 10. 7. 1984
Revised 8. 10. 1985