Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Bulgaricae Mathematicae Publicationes
and its new series Serdica Mathematical Journal
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

A NEW ESTIMATE OF THE DEGREE OF MONOTONE INTERPOLATION

G. ILIEV, S. TASHEV

Let $x_0<x_1<\dots<x_n$, $y_0<y_1<\dots<y_n$, $A=\max\{\Delta y_i : i=1, 2, \dots, n\}$, $B=\min\{\Delta y_i : y_i=1, 2, \dots, n\}$, $C=\min\{\Delta x_i : i=1, 2, \dots, n\}$, $D=\max\{|\Delta\alpha_i| : i=2, 3, \dots, n\}$, where $\alpha_i=\operatorname{arctg}(\Delta y_i/\Delta x_i)$, $\Delta y_i=y_i-y_{i-1}$, $i=1, 2, \dots, n$. We prove the existence of an algebraic polynomial P and an absolute constant c_0 such that: i) $P(x_i)=y_i$, $i=0, 1, \dots, n$; ii) $P'(x)\geqq 0$, $x\in(x_0, x_n)$; iii) $\deg P\leqq c_0\frac{x_n-x_0}{C}\ln(\frac{AD}{B}+\frac{CD}{B}+\frac{AD}{C}+e)$.

1. Introduction. Given a set of real points $\{x_i, y_i\}_{i=0}^n$ (which we shall further refer to as data set), such that $x_0<x_1<\dots<x_n$ and $y_i\neq y_j$, $i\neq j$, there exists [1, 2, 3] an algebraic polynomial P with the properties:

i) $P(x_i)=y_i$, $i=0, 1, \dots, n$,

ii) $P(x)$ is monotone decreasing on $[x_{i-1}, x_i]$ if $y_{i-1}>y_i$ and monotone increasing on $[x_{i-1}, x_i]$ if $y_{i-1}<y_i$. The polynomial P is called a partially monotone interpolation polynomial. In the situation when $y_{i-1}<y_i$, $i=1, 2, \dots, n$, P is called monotone interpolation polynomial (m. i. p.). The problem of estimating the degree of m. i. p. is recently considered by many authors, using certain characteristics of the data set. In what follows we shall restrict our considerations to the degree of m. i. p. First of all, we note that if $y_{i-2}<y_{i-1}=y_i$ at least for some i, then there exists no monotone increasing interpolation polynomial. In this situation we may assume that the degree of m. i. p is infinity ($\deg P=\infty$). This means that the characteristics

$$B=\min\{\Delta y_i : i=1, 2, \dots, n\} \tag{1}$$

($\Delta y_i=y_i-y_{i-1}$) is essential for the adequate estimation of $\deg P$. Similarly, it turns out that the characteristics

$$\begin{aligned} A&=\max\{\Delta y_i : i=1, 2, \dots, n\},\\ C&=\min\{\Delta x_i : i=1, 2, \dots, n\}\end{aligned} \tag{2}$$

are also important. All estimates in [4—11] make use of the characteristic A, B, C.

There are two approaches to the estimation of the degree of m. i. p. The first one used estimates for the uniform approximation of a monotone continuous or differentiable function by means of monotone algebraic polynomials. G. Lorentz and K. Zeller proved [8], that the order of approximation of a monotone function f by monotone algebraic polynomial of degree $\leqq n$ is $O(\omega(f; n^{-1}))$, where $\omega(f; \delta)$ is modulus of continuity of f. Later on R. Devor [12] proved that the order of approximation of a monotone function f with k-th derivative by monotone algebraic polynomials of degree $\leqq n$ is $O(\omega(f^{(k)};$

$n^{-1})/n^k$). Thus, making use of the result of [8] E. Passow and L. Raymon [4] prove the following estimate for degree of m. i. p.

$$\deg P = O(\frac{A}{BC}). \tag{3}$$

This estimate is exact in respect to the order in the situation when the ratio A/B is uniformly bounded with respect to n. Otherwise, this estimate implies in general much larger degrees of m. i. p.

The second approach make use of the estimate for the Hausdorff approximation of the jump-function

$$f(x) = \begin{cases} -1, & x \in [-1, 0] \\ 1, & x \in (0, 1] \end{cases}$$

by means of monotone algebraic polynomials [13]. Namely, for every δ, $0 < \delta < 1/2$ there exists an odd monotone increasing algebraic polynomial P, such that:
i) $\deg P \leq n$; ii) $0 < P(x) < 1$ if $x \in (0, \delta)$; iii) $1 - \exp(-c_1 n\delta) \leq P(x) \leq 1$ for $x \in [\delta, 1]$, where c_1 is a constant independent of n and δ. This fact has been successfully used in the papers of M. Nikolčeva [6] and G. Iliev [7]. It is proved in [6] that if the knotes $\{x_i\}_{i=0}^n$ are equidistant and $A/B \asymp n^\alpha$, where $\alpha \geq 1$, then

$$\deg P = O(n \ln n). \tag{4}$$

In this case it follows immediately from (3) that $\deg P = O(n^{1+\alpha})$. The estimate (4) has been improved in [7] by

$$\deg P = O\left(\frac{1}{C} \ln\left(\frac{A}{B} + e\right)\right). \tag{5}$$

The estimates (3) and (4) follow as special cases of (5). Moreover, (5) is order-exact for more general classes of data sets. However, there are data sets for which the estimate (5) is not order exact. For instance, if data set is $\{\frac{i}{n}, f_\alpha(\frac{i}{n})\}_{i=0}^n$, where $f_\alpha(x) = |x - \frac{1}{2}| \operatorname{sign}(x - \frac{1}{2})$, $x \in [0, 1]$, $0 < \alpha < 1$ taking into account that $A/B \asymp n^{1-\alpha}$ and $C = 1/n$, then (5) implies $\deg P = O(n \ln n)$. On the other hand, it is known [14], that for this data set we have $\deg P = O(n)$.

Purpose of the present work is to improve the estimate (5) in such a way that it will be order-exact for wirder class of data sets, which wil in particular include the above mentioned data set. To this end we shall first introduce an additional characteristics of the data set $\{x_i, y_i\}_{i=0}^n$, which is discrete analogue of the modulus of continuity of the function $\theta_f(x) = \operatorname{arctg} D(f; x)$ [15, 16]

$$D = \max\{|\Delta\alpha_i|;\ i = 2, 3, \ldots, n\}, \tag{6}$$

where $\alpha_i = \operatorname{arctg}(\Delta y_i/\Delta x_i)$, $i = 1, 2, \ldots, n$. Evidently α_i is the angle between the line passing through the points (x_{i-1}, y_{i-1}), (x_i, y_i) and the real axes.

2. Main result. Theorem. *Given a data set* $\{x_i, y_i\}_{i=0}^n (x_0<x_1<\ldots<x_n,$ $y_0<y_1<\ldots<y_n)$ *there exists an algebraic polynomial* P *and an absolute constant* c_0, *such that*

i) $P(x_i)=y_i, \quad i=0, 1, \ldots, n,$

ii) $P'(x)\geqq 0, \quad x\in(x_0, x_n),$

iii) $\deg P\leqq c_0 \dfrac{x_n-x_0}{C} \ln\left(\dfrac{AD}{B}+\dfrac{AD}{C}+\dfrac{CD}{B}+e\right),$

where A, B, C, D *are the characteristics defined in (1), (2) and (6).*

The proof of the theorem is based on several auxiliary propositions. The first one gives sufficient conditions for the existence of positive solution of the system of linear algebraic equations.

Lemma 1. *The system*

$$t_i+\sum_{j=1}^n \varepsilon_{ij}t_j=b_i, \quad i=1, 2, \ldots, n, \tag{7}$$

where ε_{ij} $(i, j=1, 2, \ldots, n)$ *are real numbers and* $b_i>0$, $i=1, 2, \ldots, n$ *has an unique positive solution, if there exists a monotone sequence of real numbers* $x_0<x_1<\ldots<x_n$ *and* $q>0$, *such that:*

$$3_q+|\varepsilon_{kk}|\leqq 1, \quad k=1, 2, \ldots, n, \tag{8}$$

$$\sum_{j=1}^n |\varepsilon_{kj}|\,\Delta x_j\leqq qC, \quad k=1, 2, \ldots, n, \tag{9}$$

$$\sum_{j=1}^n |\varepsilon_{kj}|\,|x_j-x_k|\,\Delta x_j\leqq q\,\frac{\min(C^3, BC^2)}{D(A+C)}, \quad k=1, 2, \ldots, n, \tag{10}$$

where

$$A=\max\{b_i:\ i=1, 2, \ldots, n\},\ B=\min\{b_i:\ i=1, 2, \ldots, n\},$$

$$C=\min\{\Delta x_i:\ i=1, 2, \ldots, n\},\quad D=\max\{|\Delta\alpha_i|:\ i=2, 3, \ldots, n\},$$

$$\alpha_i=\operatorname{arctg}(b_i/\Delta x_i), \quad i=1, 2, \ldots, n.$$

Remark. If $D=0$, then (8) and (9) are sufficient for the existence of unique positive solution.

Proof. It is easily seen that the system (7) has a dominating main diagonal. Indeed, from the inequalities $\Delta x_i/C\geqq 1$, $i=1, 2, \ldots, n$, together with (8) and (9) imply

$$\sum_{j=1}^n |\varepsilon_{kj}|\leqq \frac{1}{C}\sum_{j=1}^n |\varepsilon_{kj}|\,\Delta x_j\leqq q\leqq \frac{1}{3}\cdot \tag{11}$$

Hence, there exists an unique solution. In order to prove that $t_{k_1}\geqq 0$ $(1\leqq k_1 \leqq n)$ in the k_1-th equation $t_{k_1}=b_{k_1}-\Sigma_{k_2=1}^n \varepsilon_{k_1k_2}t_{k_2}$. We replace t_{k_2} $(k_2=1, 2, \ldots, n)$ by $t_{k_2}=b_{k_2}-\Sigma_{k_3=1}^n \varepsilon_{k_2k_3}t_{k_3}$
e. t. c. Thus we have $t_{k_1}=b_{k_1}+S_{k_1}^{km}+R_{k_1}^{km}$, where we have denoted

$$S_{k_1}^{k_m} = -\sum_{k_2=1}^{n} \varepsilon_{k_1k_2} b_{k_2} + \sum_{k_2=1}^{n} \varepsilon_{k_1k_2} \sum_{k_3=1}^{n} \varepsilon_{k_2k_3} b_{k_3} - \dots$$
$$+(-1)^{m-1} \sum_{k_2=1}^{n} \varepsilon_{k_1k_2} \sum_{k_3=1}^{n} \varepsilon_{k_2k_3} \dots \sum_{k_m=1}^{n} \varepsilon_{k_{m-1}k_m} b_{k_m},$$

$$R_{k_1}^{k_m} = (-1)^m \sum_{k_2=1}^{n} \varepsilon_{k_1k_2} \sum_{k_3=1}^{n} \varepsilon_{k_2k_3} \dots \sum_{k_{m+1}=1}^{n} \varepsilon_{k_m k_{m+1}} t_{k_{m+1}}, \quad m=2, 3, \dots$$

Then, from the relations

(12) $$R_{k_1}^{k_m} \to 0, \quad m \to \infty,$$

(13) $$|S_{k_1}^{k_m}| \leq b_{k_1}, \quad m=2, 3, \dots$$

it follows that $t_{k_1} \geq 0$, $k_1 = 1, 2, \dots, n$. We shall first prove that (12) holds true.

Denote $K = \sum_{i=1}^{n} |t_i|$. Since $|\varepsilon_{ki}| \leq 1$, $k, i = 1, 2, \dots, n$, we have $\sum_{k_{m+1}} |\varepsilon_{k_m k_{m+1}}| \, |t_{k_{m+1}}| \leq K$. This inequality together with (11) imply

$$|R_{k_1}^{k_m}| \leq K \sum_{k_2=1}^{n} |\varepsilon_{k_1k_2}| \sum_{k_3=1}^{n} |\varepsilon_{k_2k_3}| \dots \sum_{k_m=1}^{n} |\varepsilon_{k_{m-1}k_m}| \leq K q^{m-1}.$$

Since $q \leq 1/3$ (12) follows.

In order to prove (13) it is sufficient to prove the inequalities

(14) $$\sum_{i=1}^{n} |\varepsilon_{kj}| b_j \leq q b_k, \quad k=1, 2, \dots, n.$$

Indeed, (14) implies

$$|S_{k_1}^{k_m}| \leq \sum_{k_2=1}^{n} |\varepsilon_{k_1k_2}| b_{k_2} + \dots + \sum_{k_2=1}^{n} |\varepsilon_{k_1k_2}| \sum_{k_3=1}^{n} |\varepsilon_{k_2k_3}| \dots \sum_{k_m=1}^{n} |\varepsilon_{k_{m-1}k_m}| b_{k_m}$$

$$\leq q b_{k_1} + q \sum_{k_2=1}^{n} |\varepsilon_{k_1k_2}| b_{k_2} + \dots + q \sum_{k_2=1}^{n} |\varepsilon_{k_1k_2}| \dots \sum_{k_{m-1}=1}^{n} |\varepsilon_{k_{m-2}k_{m-1}}| b_{k_{m-1}}$$

$$\leq q b_{k_1} + q^2 b_{k_1} + \dots q^{m-1} b_{k_1} < \frac{q}{1-q} b_{k_1} < b_{k_1} \text{ because of } q \leq 1/3.$$

In order to prove (14) we shall need the following inequalities:

(15) $$b_j \leq b_k \left[\frac{\Delta x_j}{C} + (AD+DC)\left(\frac{1}{C^3} + \frac{1}{BC^2}\right) |x_j - x_k| \Delta x_j\right], \quad j=1, 2, \dots, n.$$

For this, we shall prove the inequalities:

(16) $$b_j \leq \Delta x_j |j-k| D\left(1 + \frac{b_j}{\Delta x_j}\right)\left(1 + \frac{b_k}{\Delta x_k}\right) + \frac{\Delta x_j}{\Delta x_k} b_k, \quad j=1, 2, \dots, n.$$

We have

$$\left|\frac{b_j}{\Delta x_j} - \frac{b_k}{\Delta x_k}\right| = |\operatorname{tg} \alpha_j - \operatorname{tg} \alpha_k| = |\sin(\alpha_j - \alpha_k)| \sqrt{1+\operatorname{tg}^2 \alpha_j} \sqrt{1+\operatorname{tg}^2 \alpha_k}$$

$$\leq |\alpha_j-\alpha_k| \, (1+|\operatorname{tg}\alpha_j|)\,(1+|\operatorname{tg}\alpha_k|)\leq |j-k|\,D(1+\frac{b_j}{\Delta x_j})(1+\frac{b_k}{\Delta x_k}).$$

The last inequality and

$$b_j\leq \Delta x_j\,|\frac{b_j}{\Delta x_j}-\frac{b_k}{\Delta x_k}|+\frac{\Delta x_j}{\Delta x_k}b_k$$

imply (16). Since $C\leq \Delta x_j$, $j=1, 2, \ldots, n$, $1\leq B^{-1}b_k$, and $|j-k|\leq |x_j-x_k|/C$ (16) implies (15).

It remains to prove (14). Using (8, 9, 10) and (15) we obtain

$$\sum_{j=1}^{n} |\varepsilon_{kj}|b_j=|\varepsilon_{kk}|\,b_k+\sum_{j\neq k}|\varepsilon_{kj}|b_j$$

$$\leq |\varepsilon_{kk}|\,b_k+\frac{b_k}{C}\sum_{j\neq k}|\varepsilon_{kj}|\Delta x_j$$

$$+b_k(DC+AD)\,(C^{-3}+B^{-1}C^{-2})\sum_{j\neq k}|\varepsilon_{kj}|\,|x_j-x_k|\,\Delta x_j$$

$$\leq b_k(|\varepsilon_{kk}|+3_q)\leq b_k.$$

This proves the lemma 1.

D e f i n i t i o n. *A 2π-periodic function f is called bell-shaped function if it is even and non-decreasing for $x\in[-\pi, 0]$.*

Let us consider the generalized Jackson operator

$$U_{m,r}(\sigma_T;\, x)=\mu_{m,r}\int_{-\pi}^{\pi}\sigma_T(x+t)\,(\frac{\sin mt/2}{m\sin t/2})^{2r}\,dt,$$

where $\mu_{m,r}\int_{-\pi}^{\pi}(\frac{\sin mt/2}{m\sin t/2})^{2r}\,dt=1$ applied to the function

$$\sigma_T(x)=\begin{cases}0, & x\in[-\pi,\ -\pi/2),\\ 1, & x\in[-\pi/2,\ \pi/2],\\ 0, & x\in(\pi/2,\ \pi].\end{cases}$$

In [13] it is proved that $U_{m,r}(\sigma_T;\, x)$ is a trigonometric polynomial of order $(m-1)r$ and if m is even then $U_{m,r}(\sigma_T;\, x)$ is a bell-shaped function. In what follows we shall suppose m even and $r\geq 2$.

We need the inequalities

$$\text{(17)}\qquad \sum_{0}^{\pi/2-\delta}|\sigma_T(x)-U_{m,r}(\sigma_T;\, x)|\,dx\leq \delta(\frac{\pi^2}{2m\delta})^{2r-1}$$

$$\text{(18)}\qquad \int_{0}^{\pi/2-\delta}\cos x\,|\sigma_T(x)-U_{m,r}(\sigma_T;\, x)|\,dx\leq \delta^2(\frac{\pi^2}{2m\delta})^{2r-1},$$

where $0<\delta<\pi/2$. We shall prove only the inequality (18).

Since

$$\mu_{m,r}^{-1}=\int_{-\pi}^{\pi}(\frac{\sin mt/2}{m\sin t/2})^{2r}\,dt\geq 2\int_{0}^{\pi/m}(\frac{\sin mt/2}{m\sin t/2})^{2r}\,dt\geq 2\int_{0}^{\pi/m}(\frac{mt/\pi}{mt/2})^{2r}\,dt=\frac{\pi 2}{m}(\frac{2}{\pi})^{2r}$$

then

$$\int_0^{\pi/2-\delta} \cos x \,|\, \sigma_T(x) - U_{m,r}(\sigma_T;\ x) \,|\, dx$$

$$\leq \mu_{m,r} \int_0^{\pi/2-\delta} \cos x \,[\int_{-\pi}^{\pi} |\, \sigma_T(x+t) - 2\sigma_T(x) + \sigma_T(x-t) \,|\, (\frac{\sin mt/2}{m \sin t/2})^{2r} dt] dx$$

$$\leq \frac{m}{\pi} . (\frac{\pi}{2})^{2r} \int_0^{\pi/2-\delta} \cos x \int_{\pi/2-x}^{\pi} (\frac{\sin mt/2}{m \sin t/2})^{2r} dt \,.\, dx$$

$$\leq \frac{m}{\pi} \,.\, (\frac{\pi}{2})^{2r} \int_0^{\pi/2-\delta} (\frac{\pi}{2} - x) \int_{\pi/2-x}^{\pi} (\frac{\pi}{mt})^{2r} dt \,.\, dx$$

$$\leq \frac{m}{\pi(2r-1)} \cdot (\frac{\pi^2}{2m})^{2r} \int_0^{\pi/2-\delta} (\frac{\pi}{2} - x)^{2-2r} dx$$

$$\leq \frac{m}{\pi(2r-1)(2r-3)} \cdot (\frac{\pi^2}{2m})^{2r} \delta^{3-2r} \leq \delta^2 (\frac{\pi^2}{2m\delta})^{2r-1}.$$

L e m m a 2. *For any positive integers* m *and* r (m-even, $r \geq 2$) *there exists a positive algebraic polynomial* P *of degree* $\leq (m-1)r$, *monotone increasing in* $[-1,\ 1]$ *and such that for any* $0 < \delta < 1$ *the following relations hold*:

$$\int_{-1}^{-\delta} |\, \sigma(x) - P(x) \,|\, dx = \int_{\delta}^{1} |\, \sigma(x) - P(x) \,|\, dx \leq \delta \,(\frac{\pi^2}{2m\delta})^{2r-1}, \tag{19}$$

$$-\int_{-1}^{-\delta} x \,|\, \sigma(x) - P(x) \,|\, dx = \int_{\delta}^{1} x \,|\, \sigma(x) - P(x) \,|\, dx \leq \delta^2 (\frac{\pi^2}{2m\delta})^{2r-1}, \tag{20}$$

$$1 - P(x) = P(-x), \quad -1 \leq x \leq 1, \tag{21}$$

where

$$\sigma(x) = \begin{cases} 0, & x \in [-1,\ 0] \\ 1, & x \in (0,\ 1]. \end{cases}$$

P r o o f: The desired polynomial is $P(x) = U_{m,r}(\sigma_T;\ \arccos x)$. Indeed, $U_{m,r}(\sigma_T;\ t)$ is a bell-shaped positive trigonometrical polynomial which is mapped into a monotone and positive algebraic polynomial in the interval $[-1,\ 1]$ by means of the transformation $t = \arccos x$. The equality

$$1 - U_{m,r}(\sigma_T;\ \pi/2 - t) = U_{m,r}(\sigma_T;\ \pi/2 + t)$$

implies (21).

Further, put $x = \cos t$ in the integral

$$I = \int_{\delta}^{1} x \,|\, \sigma(x) - P(x) \,|\, dx \text{ we obtain}$$

$$I = -\int_{\arccos\delta}^{0} \cos t \,.\, \sin t \,|\, \sigma(\cos t) - P(\cos t) \,|\, dt$$

$$= \int_0^{\arccos\delta} \cos t \,.\, \sin t \,|\, \sigma_T(t) - U_{m,r}(\sigma_T;\ t) \,|\, dt$$

$$\leq \int_0^{\pi/2-\delta} \cos t \,|\, \sigma_T(t) - U_{m,r}(\sigma_T;\ t) \,|\, dt.$$

This inequality together with (18) imply (20). The inequality (19) is proved analogously.

Proof of the Theorem.

Without loss of generality we may assume that $x_0=0$, $x_n=1$, $y_0=0$. Form the polynomial $Q(x)=\sum_{i=1}^{n} t_i P(x-\frac{x_i+x_{i-1}}{2})$, where P is the monotone polynomial from Lemma 2 of degree $\leq(m-1)r$. Choose the coefficients t_i in such way that

$$Q(x_i)=y_i, \quad i=1, 2, \dots, n. \tag{22}$$

We shall show that the coefficients t_i are positive. Thus the polynomial Q will be desired because P is monotone. Write the system (22) in matrix form as follows:

$$\begin{bmatrix} 1-\Delta_{11} & \Delta_{12} & \Delta_{13} & \cdots & \Delta_{1n} \\ 1-\Delta_{21} & 1-\Delta_{22} & \Delta_{23} & \cdots & \Delta_{2n} \\ 1-\Delta_{31} & 1-\Delta_{32} & 1-\Delta_{33} & \cdots & \Delta_{3n} \\ \cdots & & & \cdots & \\ 1-\Delta_{n1} & 1-\Delta_{n2} & 1-\Delta_{n3} & \cdots & 1-\Delta_{nn} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \tag{23}$$

where the following notation has been used:

$$1-\Delta_{i,j}=P(x_i-\frac{x_j+x_{j-1}}{2}), \quad i=1, 2, \dots, n, \quad j=1, 2, \dots, i,$$

$$\Delta_{ik}=P(x_i-\frac{x_k+x_{k-1}}{2}), \quad i=1, 2, \dots, n, \quad k=i+1, \quad i+2, \dots, n.$$

From (23) we get the equivalent system

$$\begin{bmatrix} 1-\Delta_{11} & \Delta_{12} & \Delta_{13} & \cdots & \Delta_{1n} \\ -\Delta_{21}+\Delta_{11} & 1-\Delta_{22}-\Delta_{12} & \Delta_{23}-\Delta_{13} & \cdots & \Delta_{2n}-\Delta_{1n} \\ -\Delta_{31}+\Delta_{21} & -\Delta_{32}+\Delta_{22} & 1-\Delta_{33}-\Delta_{23} & \cdots & \Delta_{3n}-\Delta_{2n} \\ \cdots & & & \cdots & \vdots \\ -\Delta_{n1}+\Delta_{n-1,1} & -\Delta_{n2}+\Delta_{n-1,2} & -\Delta_{n3}+\Delta_{n-1,3} & \cdots & 1-\Delta_{nn}-\Delta_{n-1,n} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_n \end{bmatrix}$$

$$= \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \\ \vdots \\ \Delta y_n \end{bmatrix}$$

The last system we write in the form

$$t_i+\sum_{j=1}^{n} \varepsilon_{ij} t_j=\Delta y_i, \quad i=1, 2, \dots, n. \tag{24}$$

where

$$-\Delta_{11}=\varepsilon_{11}\,;\ \Delta_{1j}=\varepsilon_{ij},\ j=2,\ 3,\ldots,\ n\,;$$

$$-\Delta_{ij}+\Delta_{i-1,j}=\varepsilon_{ij},\ i=2,\ 3,\ldots,\ n,\ j=1,\ 2,\ldots,\ i-1\,;$$

$$-\Delta_{ii}-\Delta_{i-1,i}=\varepsilon_{ii},\ i=2,\ 3,\ldots,\ n\,;$$

$$\Delta_{ij}-\Delta_{i-1,j}=\varepsilon_{ij},\ i=2,\ 3,\ldots,\ n-1,\ j=i+1,\ i+2,\ldots,\ n.$$

We shall prove that for suitable m and r the coefficients ε_{ij} satisfy the condition of Lemma 1.

At first we shall prove the inequalities

$$\sum_{i=1}^{n}|\varepsilon_{ki}|\Delta x_i\leqq 16\int_{C/4}^{1}|1-P(x)|\,dx, \tag{25}$$

$$\sum_{i=1}^{n}|\varepsilon_{ki}|\ |x_i-x_k|\Delta x_i\leqq 32\int_{C/4}^{1}|x|\ |1-P(x)|\,dx,\quad k=1,\ 2,\ldots,\ n. \tag{26}$$

We have:

$$\sum_{i=1}^{n}|\varepsilon_{ki}|\Delta x_i\leqq\sum_{i=1}^{n}(|\Delta_{ki}|+|\Delta_{k-1,i}|)\Delta x_i$$

$$=\sum_{i=1}^{k}|\Delta_{ki}|\Delta x_i+\sum_{i=k+1}^{n}|\Delta_{ki}|\Delta x_i+\sum_{i=1}^{k-1}|\Delta_{k-1,i}|\Delta x_i$$

$$+\sum_{i=k}^{n}|\Delta_{k-1,i}|\Delta x_i=S_1+S_2+S_3+S_4,$$

where $\Delta_{0i}=0,\ i=1,\ 2,\ldots,\ n.$

Let $\xi_i=x_k-(x_i+x_{i-1})/2,\ i=1,\ 2,\ldots,\ k,\ \xi_{k+1}=(x_k-x_{k-1})/4$. Then $x_i-x_{i-2}=2(\xi_{i-1}-\xi_i),\ i=2,\ 3,\ldots,\ k$ and $x_k-x_{k-1}=4(\xi_k-\xi_{k+1})$.
From here and the condition for monotonity of the polynomial P we find

$$\sum_{i=1}^{k}|\Delta_{ki}|\Delta x_i=\sum_{i=1}^{k}|1-P(x_k-\frac{x_i+x_{i-1}}{2})|(x_i-x_{i-1})$$

$$\leqq 4\sum_{i=1}^{k}|1-P(\xi_i)|(\xi_i-\xi_{i+1})\leqq 4\int_{\xi_{k+1}}^{1}|1-P(x)|\,dx$$

such that $\xi_{k+1}\geqq C/4$ we have $S_1\leqq 4\int_{C/4}^{1}|1-P(x)|\,dx$. In the same way

$$S_i\leqq 4\int_{C/4}^{1}|1-P(x)|\,dx,\quad i=2,\ 3,\ 4.$$

This implies (25). The inequality (26) is proved analogously.

The inequalities (19, 20, 25, 26) imply

$$\sum_{i=1}^{n}|\varepsilon_{ki}|\Delta x_i\leqq 4C\left(\frac{2\pi^2}{mC}\right)^{2r-1}, \tag{27}$$

$$\sum_{i=1}^{n}|\varepsilon_{ki}|\ |x_i-x_k|\Delta x_i\leqq 2C^2\left(\frac{2\pi^2}{mC}\right)^{2r-1}. \tag{28}$$

Now we set $m=2([2\pi^2 e/C]+1)$,

$$r=25([\ln(\frac{AD}{B}+\frac{AD}{C}+\frac{CD}{B}+e)]+1).$$

Then the inequalities (27, 28) imply (8, 9, 10) and Lemma 1 gives a positive solution of the system (22). Thus the Theorem is proved.

Corollary 1. *Let the data set* $\{x_i, y_i\}_{i=0}^n$ *be* $x_i=i/n$, $y_i=f_\alpha(x_i)$, $i=0, 1, 2, \dots, n$, *where* $f_\alpha(x)=|x-1/2|^\alpha . \operatorname{sign}(x-1/2)$, $x\in[0, 1]$ *and* $0<\alpha<1$. *Taking into account that* $A=O(n^{-\alpha})$, $B\asymp n^{-1}$, $C=n^{-1}$, $D=O(n^{\alpha-1})$, *the Theorem implies existence of a m. i. p. P such that deg* $P=O(n)$.

Corollary 2. *Let the data set* $\{x_i, y_i\}_{i=0}^n$ *be* $x_i=i/n$, $y_i=(1+\ln x_i^{-1})^{-1}$, $i=1, 2, \dots, n$ *and* $(x_0, y_0)=(0, 0)$. *Taking into account that* $A=O((\ln n)^{-1})$, $B\asymp n^{-1}$, $C=n^{-1}$, $D=O(n^{-1}\ln^2 n)$ *the Theorem implies existence of a m. i. p. P* such that deg $P=O(n\ln\ln n)$.

REFERENCES

1. W. Wolibner. Sur un polynome d'interpolation. *Coloq. Math.*, **2**, 1951, 136-137.
2. W. Kammeror. Polynomial approximation to finitely oscillating functions. *Math. Comp.*, **15**, 1961, 115-119.
3. S. Young. Piecewise monotone interpolations. *Bull. Amer. Math. Soc.*, **73**, 1967, 642-643.
4. E. Passow, L. Raymon. The degree of Piecewise Monotone Interpolation. *Proc. Amer. Math. Soc.*, **48**, 1975, 409-412.
5. E. Passow. An improved estimate of the degree of monotone interpolation. *J. Approximation Theory*, **17**, 1976, 115-118.
6. M. Nikolčeva. Monotone interpolation and its application to parametric approximation *C. R. Acad. Bulg. Sci.*, **29**, 1976, 463-473 (in Russian).
7. G. Iliev. Exact Estimates for Monotone Interpolation. *J. Approximation Theory*, **28**, 1980, 101-112.
8. G. Lorentz, K. Zeller. Degree of approximation by monotone polynomials. I. *J. Approximation Theory*, **1**, 1968, 501-504.
9. J. Roulier. Nearby monotone approximation. *Proc. Amer. Math. Soc.*, **47**, 1975, 84-88.
10. D. Newman, E. Passow, L. Raymon. Piecewise monotone polynomial approximation. *Trans. Amer. Math. Soc.*, **172**, 1972, 465-472.
11. E. Passow, L. Raymon. Monotone and comonotone approximation. *Proc. Amer Math. Soc.*, **42**, 1974, 390-394.
12. D. Devore. Monotone approximation by polynomial approximation. *SIAM J. Math. Anal.* 8, 1977, 906-921.
13. Bl. Sendov, V. Popov. Approximation of monotone functions by monotone polynomials in Hausdorff metric. *Rev. Anal. Num. Theorie Approx.*, **3**, 1974, 79-88.
14. G. Iliev, S. Tashev. Monotone interpolation. Proc. Int. Conf. on Constructive Function Theory, Varna, 1981.
15. S. Tashev. New estimates for the Hausdorff and local approximations of functions. Proc. Int. Conf. on Constructive Function Theory, Varna, 1981, 551—558.
16. P. P. Petrushev. Rational aproximation of function of the class V_r. *C. R. Acad. Bulg. Sci.*, **33**, 1980, 1607-1610.

Institute of Mathematics and Mechanics
Bulgarian Academy of Sciences
1090 Sofia P. O. Box 373
Bulgaria

Received 1.8. 1984