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NETWORK-NORM ERROR ESTIMATES OF THE NUMERICAL SOLUTION OF EVOLUTIONARY EQUATIONS

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Network-norm error estimates of integer and fractional order of convergence are derived for a Cauchy boundary problem for some evolutionary equations. To this end a combined technique using average moduli of smoothness and interpolation of Besov spaces is developed.

Consider the following Cauchy boundary problem for evolutionary equations:

$$\frac{\partial}{\partial t} u(x, t)-H(\frac{\partial}{\partial x}, x, t)u(x, t)=0,$$

$$x \in (-\infty, \infty)=\mathsf{R}, \quad 0<t\leqq T<\infty, \tag{1}$$

$$u(x, 0)=f(x), \ x \in \mathsf{R}, \ f \in L_p(\mathsf{R}), \ 1\leqq p\leqq \infty.$$

Here L_p is defined as usually, with a norm $\|f\|_{L_p}=(\int_R |f(x)|^p\,dx)^{1/p}$.

The differential operator H is supposed to allow a unique representation of the solution, which is to be denoted by G: $u(x, t)=[G(t)f](x)$.

Problem (1) is being solved numerically via the approximating problem (2) [1, 2, 10, 11, 12]. $u_h(x, t+d) = \sum_{\alpha \in I} c_\alpha u_h(x+\alpha h, t)$, α-integer I — finite, $h\leqq 1$ with no loss of generality,

$$u_h(x, 0)=f(x), \tag{2}$$

$$x \in \Sigma_h=\{x_\mu : x_\mu=\mu h, \ \mu=0, \pm1, \pm2, \dots\},$$

$$t \in \Omega_d=\{t_\nu : t_\nu=\nu d, \ \nu=0, 1 \dots, N, \ Nd=T\},$$

h — step along x, d — step along t.

Here c_α and I may depend on x and t. The solution operator of (2) is $G_h(t): u_h(x, t)=[G_h(t)f](x)$. The error operator is then $E_h=G_h-G$. All these operators may depend on x as well as on t.

A natural question arises: what must the properties of the operators G, G_h and the initial value f be, in order to ensure convergence of G_hf to Gf in a certain norm and of a certain order (fractional order of convergence is also considered)? This question is being treated in many works and the estimates, obtained there, may be distinguished into two major types:

(i) Error estimates in integral L_p-norms under the assumptions:

(a) stability in L_p of the error operator

(b) certain order of approximation for sufficiently smooth functions f (this order is denoted by $r: r>0$).

These integral-norm estimates yield the functional spaces to which the initial value should belong, in order to ensure a certain order of convergence $s: 0<s\leq r$. (For such estimates v., e. g. [10, 11]).

(ii) When applying a concrete difference scheme, defined in the points of a concrete network, the values of the initial value function f are supposed to be known only in the points of the network and thus a discrete (network) norm estimate is needed to ensure convergence of a certain order. Such estimates have been made, e. g., in [1, 2]. They require, like the integral — norm ones: a) stability (in discrete norms), b) approximation (in discrete norms) for sufficiently smooth functions and, additionally, c) f has derivatives of certain order which, as a rule, is higher than the necessary one for formulating the problem.

The purpose of the present work is to derive network-norm estimates, requiring stability and approximation only, i. e. without the assumption of additional smoothness of f. To this end a method is developed (v. also [16, 18, 19]), which is rather general and can be applied to varied problems for error estimation, not necessarily connected to differential equations, here considered. The applications presented in this paper are restricted to the model evolutionary problem (1, 2) with concrete, simple differential operators H only. The method uses the following functional moduli and spaces:

A. Average moduli of smoothness. A local modulus of smoothness of k-th order to the point x is $\omega_k(f, x; \delta)=\sup\{|\Delta_h^k f(t)|: t,\ t+kh\in[x-\frac{k\delta}{2},\ x+\frac{k\delta}{2}]\}$, where, as usual,

$$\Delta_h^k f(t)=\sum_{m=0}^{k}(-1)^{k+m}\binom{k}{m}f(t+mh),\quad h>0.$$

The average modulus of smoothness of k-th order of f is defined by

$$\tau_k(f;\ \delta)_{L_p}=\|\omega_k(f, \cdot\,;\ \delta)\|_{L_p},\ \text{for } f\in A_p, \text{ where}$$

$$A_p=\{f:\ \|f\|_{A_p,\delta}=\|f\|_{L_p}+\tau_1(f;\ \delta)_{L_p}<\infty,\ \delta>0\}.$$

As far as we know, the average moduli were first considered in [3, 5]. Below some of their properties are related (e. g., v. [4, 6, 7, 8, 9, 13, 14, 15, 17]).

A1. Monotonicity: $0\leq\delta_1\leq\delta_2$ implies $\tau_k(f;\ \delta_1)_{L_p}\leq\tau_k(f;\ \delta_2)_{L_p}$.

A2. Subadditivity: if $f,\ g\in A_p$, then

$$\tau_k(f+g;\ \delta)_{L_p}\leq\tau_k(f;\ \delta)_{L_p}+\tau_k(g;\ \delta)_{L_p}.$$

A3. Estimation of the higher order modulus by a lower order one

$$\tau_k(f;\ \delta)_{L_p}\leq 2\tau_{k-1}(f;\ \delta)_{L_p},\quad k>1.$$

A4. Estimation of the average modulus of order k of the function via the $(k-l)$-th integral modulus of smoothness of the l-th derivative, $0\leq l\leq k-1$, $k\geq 2$

$$\tau_k(f;\ \delta)_{L_p}\leq c(k,\ l)\delta^l\omega_{k-l}(f;\ \delta)_{L_p},$$

where the integral modulus $\omega_m(f;\delta)_{L_p}$ is defined in R, as usually, by

$$\omega_m(f;\delta)_{L_p}=\sup_{0\leq h\leq\delta}\left(\int_R|\Delta_h^m f(x)|^p\,dx\right)^{1/p},\quad 1\leq p\leq\infty.$$

A5. $\tau_k(f;\lambda\delta)_{L_p}\leq(2(\lambda+1))^{k+1}\tau_k(f;\delta)_{L_p},\quad \lambda>0.$

A6. $\lim_{\delta\to+0}\tau_1(f;\delta)_{L_p}=0$ iff f is integrable in the Riemannian sense.

A7. If $f\in W_p^1(W_p^1(R))$ is the Sobolev space

$$W_p^m(R)=\{f\colon R\to R,\ f,\ f^{(m)}\in L_p,\ f^{(m-1)}\in AC(R),\ m\geq 1\}$$

($AC(R)$ — the space of all absolutely continuous functions: $R\to R$), then $\tau_1(f;\delta)_{L_p}\leq\delta\|f'\|_{L_p}$.

A8. If $Vf<\infty$ (Vf — the variation of f), then $\tau_1(f;\delta)_{L_1}\leq\delta Vf$.

B. Besov spaces, interpolation and embedding results (the one-dimensional case)

$$B_{pq}^s(R)=\{f\colon \|f\|_{B_{pq}^s}=\|f\|_{L_p}+\left(\int_0^\infty (t^{-s}\omega_r(f;t)_{L_p})^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty\},$$

where $1\leq p,\ q\leq\infty,\ 0\leq s<r$.

C. A — spaces (v. [13]) **(for the one-dimensional case)**

$$A_{pq}^s(R)=\{f\colon \|f\|_{A_{pq}^s}=\|f\|_{L_p}+\left(\int_0^\infty (t^{-s}\tau_r(f;t)_{L_p})^q\frac{dt}{t}\right)^{1/q}<\infty\},$$

for the same $p,\ q,\ s$, as in **B**. Let us note [13], that for $\frac{1}{p}<s<r$, $A_{pq}^s=B_{pq}^s$ (equivalence of norms), for $0\leq s\leq\frac{1}{p}$, $A_{pq}^s\subsetneqq B_{pq}^s$.

D. The spaces M_p, $1\leq p\leq\infty$, of Fourier multipliers on $L_p(R)$

$M_p(R)=\{\rho\colon \rho$ — tempered distribution, $\|\rho\|_{M_p}=\sup_{\|f\|_{L_p}=1}\|\check\rho*f\|_{L_p}<\infty\}$, (We denote by $\hat\varphi$ or $F(\varphi)$ the Fourier transform of φ on R, and by $\check\varphi$ or $F^{-1}(\varphi)$ the inverse Fourier transform of φ on R).

Here and henceforth every function $f\in L_p$ is distinguished from its class of equivalence and considered defined by a concrete value in every point of R.

Let us consider the following two functional spaces:

$$l_h^p(\Sigma_h)=\{f\colon R\to R,\ \|f\|_{l_h^p(\Sigma_h)}<\infty\},$$

where

$$\|f\|_{l_h^p(\Sigma_h)}=\left(\sum_{\mu=-\infty}^{\infty}h|f(x_\mu)|^p\right)^{1/p},\quad x_\mu\in\Sigma_h(\text{see 2})),\ 1\leq p\leq\infty;$$

$$\tilde l_h^p(\Sigma_h)=\{f\colon R\to R,\ \|f\|_{\tilde l_h^p(\Sigma_h)}<\infty\},$$

where $\|f\|_{\tilde{l}^p_h(\Sigma_h)}=\left(\sum_{\mu=-\infty}^{\infty} h\sup\{|f(x)|^p,\quad x\in[x_\mu-\frac{h}{2},\ x_\mu+\frac{h}{2}]\}\right)^{1/p}$.

For $f\in l^p_h(\Sigma_h)$ the operator P_{Σ_h} is defined by

$$[P_{\Sigma_h}f](x)=f(x_\mu),\ x\in[x_\mu-h/2,\ x_\mu+h/2),\quad \mu=0,\ \pm1,\ \ .$$

For $f\in\tilde{l}^p_h(\Sigma_h)$ the operator $\tilde{P}_{\Sigma_h}$ is defined by

$$[\tilde{P}_{\Sigma_h}f](x)=\sup\{|f(\xi)|,\ x,\ \xi\in[x_\mu-h/2,\quad x_\mu+h/2)\}.$$

It is easy to realize that P_{Σ_h} is a linear operator and that $[P_{\Sigma_h}f(\,.+\alpha h)](x)=[P_{\Sigma_h}f(\,.\,)]\,(x+\alpha h)$, α-integer.

Lemma 1. *The following estimates hold:*

a. $|\|f\|_{l^p_h(\Sigma_h)}-\|f\|_{L_p}|\leq\tau_1(f;\ h)_{L_p},\ f\in l^p_h(\Sigma_h)\cap L_p.$

b. $|\|f\|_{\tilde{l}^p_h(\Sigma_h)}-\|f\|_{L_p}|\leq12\tau_1(f;\ h)_{L_p},\quad f\in\tilde{l}^p_h(\Sigma_h)\cap L_p.$

Proof:

(a) For the proof of a. v. [16], Lemma 1.

(b) $\|f\|_{\tilde{l}^p_h(\Sigma_h)}=\|\tilde{P}_{\Sigma_h}f\|_{L_p}\leq\|\,|f|\,\|_{L_p}+\|\tilde{P}_{\Sigma_h}f-|f|\,\|_{L}$

$$=\|f\|_{L_p}+\left(\sum_{\mu=-\infty}^{\infty}\int_{x_\mu-h/2}^{x_\mu+h/2}|\sup\{|f(\xi)|,\ \xi\in[x_\mu-\frac{h}{2},\ x_\mu+\frac{h}{2})\}-|f(x)|\,|^p\,dx\right)^{1/p}$$

$$=\|f\|_{L_p}+\left(\sum_{\mu=-\infty}^{\infty}\int_{x_\mu-h/2}^{x_\mu+h/2}|\sup\{(|f(\xi)|-|f(x)|),\quad \xi\in[x_\mu-\frac{h}{2},\ x_\mu+\frac{h}{2})\}|^p dx\right)^{1/p}$$

$$\leq\|f\|_{L_p}+\left(\sum_{\mu=-\infty}^{\infty}\int_{x_\mu-h/2}^{x_\mu+h/2}(\sup\{|f(\xi)-f(x)|,\quad \xi\in[x_\mu-\frac{h}{2},\ x_\mu+\frac{h}{2}\}|^p\,dx\right)^{1/p}$$

$$\leq\|f\|_{L_p}+\left(\sum_{\mu=-\infty}^{\infty}\int_{x_\mu-h/2}^{x_\mu+h/2}\omega_1(f,\ x_\mu;\ h)^p\,dx\right)^{1/p}$$

$$\leq\|f\|_{L_p}+\tau_1(f;\ 2h)_{L_p}\leq\|f\|_{L_p}+12\tau_1(f;\ h)_{L_p}.$$

An analogue of property A5. of the average moduli of smoothness for λ-integer is used (e. g. v. [4]) to yield the last inequality. The inequality $\|f\|_{L_p}\leq\|f\|_{\tilde{l}^p_h(\Sigma_h)}+12\tau_1(f;\ h)_{L_h}$ is evident.

The case $p=\infty$ (with ess sup instead of an integral norm) can be proved similarly. The lemma is proved.

Lemma 2. *For every* $\xi\in\mathbf{R}$, $\tilde{l}^p_h(\Sigma_h)=\tilde{l}^p_h(\Sigma_h+\xi)$ *(with equivalence of norms).*

Proof: Evidently, for $\xi_1 \equiv \xi_2 \pmod h$, $\tilde{l}_h^p(\Sigma_h+\xi_1)=\tilde{l}_h^p(\Sigma_h+\xi_2)$ (equality of norms) is fulfilled. Therefore, one may assume with no loss of generality that $\xi \in [-\frac{h}{2}, \frac{h}{2})$. For two such ξ_1, ξ_2 the following holds:

$$\sup\{|f(t)| : t\in [x_\mu-\frac{h}{2}+\xi_1,\ x_\mu+\frac{h}{2}+\xi_1)\}$$

$$\leq \sup\{|f(t)| : t\in [x_\mu-\frac{h}{2}+\xi_2,\ x_\mu+\frac{h}{2}+\xi_2)\}+\sup\{|f(t)| : t\in [x_{\mu-\operatorname{sgn}(\xi_2-\xi_1)}-\frac{h}{2}+\xi_2,\ x_{\mu-\operatorname{sgn}(\xi_2-\zeta_1)}+\frac{h}{2}+\xi_2)\},$$

where, as usual, $\operatorname{sgn} x=\frac{x}{|x|}$, $x\neq 0$; $\operatorname{sgn} 0=0$.

Hence, it is easy to realize that $\frac{1}{2}\|f\|_{\tilde{l}_h^p(\Sigma_h+\xi_1)}\leq\|f\|_{\tilde{l}_h^p(\Sigma_h+\xi_2)}\leq 2\|f\|_{\tilde{l}_h^p(\Sigma_h+\xi_1)}$

The lemma is proved. Let us consider the space $A_{p,h}(\mathbf{R})$, $h>0$ fixed:

$$A_{p,h}(\mathbf{R})=\{f:\ \|f\|_{A_{p,h}}=\|f\|_{L_p}+\tau_1(f;\ h)_{L_p}<\infty,\ h>0\quad \text{fixed}\}.$$

Corollary 1. (a) For every $\xi\in\mathbf{R}$ $\tilde{l}_h^p(\Sigma_h+\xi)=A_{p,h}(\mathbf{R})$ (equivalence of norms with constants independent of h). (b) For every h_1, $h_2>0$, $A_{p,h_1}=A_{p,h_2}$ (equivalence of norms with constants depending on h_1, h_2).

Proof: (a) is implied by

$$\sup_{t\in[x-\frac{h}{2},\,x+\frac{h}{2}]}|f(t)|\leq|f(x)|+\omega_1(f,x;h)\leq 3\sup_{t\in[x-\frac{h}{2},\,x+\frac{h}{2})}|f(t)|,$$

which is easily yielded from the definition of $\omega(f, x; h)$. (b) is implied by property A5, twice used, with $\lambda_1=\frac{h_2}{h_1}$ and $\lambda_2=\frac{1}{\lambda_1}$. Note that for any $h>0$ $A_{p,h}$ consists of the same functions as A_p and in this sense $\tilde{l}_h^p(\Sigma_h+\xi)=A_p$ for any $h>0$, $\xi\in\mathbf{R}$. The corollary is proved.

Lemma 3. Assume that $\rho\in S$ (Schwartz' space), $c\in\mathbf{R}$, $c\neq 0$.

(a) Let $f\in L_p(\mathbf{R})$, then,

$$\|\rho(\breve{c\,\cdot})*f\|_{L_p(\mathbf{R})}\leq c_a\|\rho(\breve{\cdot})\|_{L_1(\mathbf{R})}\|f\|_{L_p(\mathbf{R})},$$

(b) Let $f\in A_p(\mathbf{R})$, then,

$$\|\rho(\breve{c\,\cdot})*f\|_{l_h^p(\Sigma_h)}\leq c_b\|\rho(\breve{\cdot})\|_{L_1(\mathbf{R})}\|f\|_{A_{p,h}(\mathbf{R})},$$

c_a, c_b — abs. constants.

Proof: For the proof of (a) v. e. g. [10, 11]. It can also be proved similarly to (b). (b) Note first, that if $\rho\in S$, then $\breve{\rho}\in S$. Let $1\leq p<\infty$.

$$\|\rho(\breve{c\,\cdot})*f\|_{l_h^p(\Sigma_h)}=\Big(\sum_{k=-\infty}^{\infty}h\,|\rho(\breve{c\,\cdot})*f(x_k)|^p\Big)^{1/p}$$

$$=\Big(\sum_{k=-\infty}^{\infty} h\Big|\int_{-\infty}^{\infty} f(x_k-t)\int_{-\infty}^{\infty}\rho(c\xi)e^{i\xi t}d\xi dt\Big|^p\Big)^{1/p}$$

$$=\Big(\sum_{k=-\infty}^{\infty} h\Big|\int_{-\infty}^{\infty}\frac{1}{c}\check{\rho(.)}\Big(\frac{t}{c}\Big)f(x_k-t)dt\Big|^p\Big)^{1/p}$$

$$=\Big(\sum_{k=-\infty}^{\infty} h\Big|\int_{-\infty}^{\infty}\check{\rho(.)}(\eta)f(x_k-c\eta)d\eta\Big|^p\Big)^{1/p}$$

$$\leq\int_{-\infty}^{\infty}|\check{\rho(.)}(\eta)|\Big(\sum_{k=-\infty}^{\infty} h|f(x_k-c\eta)|^p\Big)^{1/p}d\eta$$

$$=\int_{-\infty}^{\infty}|\check{\rho(.)}(\eta)|\,\|f\|_{l_h^p(\Sigma_h-c\eta)}d\eta$$

$$\leq\int_{-\infty}^{\infty}|\check{\rho(.)}(\eta)|\,\|f\|_{\mathcal{T}_h^p(\Sigma_h)}d\eta=\|\check{\rho(.)}\|_{L_1(\mathbf{R})}\|f\|_{\mathcal{T}_h^p(\Sigma_h)}$$

$$\leq c\|\check{\rho(.)}\|_{L_1(\mathbf{R})}\|f\|_{A_{p,h}(\mathbf{R})}.$$

In the proof we used the generalized Minkowski inequality and the fact tha

$$\|f\|_{l_h^p(\Sigma_h+\xi)}\leq\|f\|_{\mathcal{T}_h^p(\Sigma_h)}.$$

For $p=\infty$ the proof is almost the same. The lemma is proved.

The next theorem is a useful generalization of a theorem of V. Popov [6, 13].

Theorem 1 [16, 18, 19]. Let E be a Lipschitz operator: $A_p\to l_h^p(\Sigma_h)$ with the properties:

(A) For every $f,\ g\in A_p$, $\|Ef-Eg\|_{l_h^p(\Sigma_h)}\leq c_1\|f-g\|_{A_{p,h^{\sigma/r}}}$.

(B) For every $f\in W_p^r$, $\|Ef\|_{l_h^p(\Sigma_h)}\leq c_2h^\sigma(\|f\|_{L_p}+\|f^{(r)}\|_{L_p})$, $r>0$, where $0<\sigma\leq r$; c_1, c_2 do not depend on h; $h\leq 1$ (without loss of generality). Then for every $f\in A_p$, $0<s\leq r$,

$$\|Ef\|_{l_h^p(\Sigma_h)}\leq c(r,\ c_1,\ c_2,\ p)\,(h^\sigma\|f\|_{L_p}+\tau_r(f;\ h^{\sigma/r})_{L_p}),$$

and, hence,
$$\|Ef\|_{l_h^p(\Sigma_h)}\leq c\,(r,\ c_1,\ c_2,\ p)h^{\frac{\sigma s}{r}}\|f\|_{A_{p\infty}^s}.$$

Proof: It is similar to the proof of V. Popov for the case $E=I-L$ (I — identity, L — arbitrary linear operator with properties similar to (A) and (B)). A main fact here is the existence for any function $f\in L_p$ of an intermediate appproximating function $f_{r,h_1}\in W_p^r$ (function of Steklov) (v., e. g., [4]). f_{r,h_1} has the following properties ($f\in A_{p,h_1}$ — the choice of h_1 will be additionally specified):

(i) $|f(x)-f_{r,h_1}(x)|\leq\omega_k(f,\ x;\ 2h_1)$.

(ii) $\|f-f_{r,h_1}\|_{L_p}\leq\tau_k(f;\ 2h_1)_{L_p}$,

(iii) $f_{r,h_1} \in W_p^r$ and

$$\| f_{r,h_1}^{(s)} \|_{L_p} \leq c(r) h_1^{-s} \tau_s(f;\ h_1)_{L_p}, \quad s=1,\ 2, \ldots, r.$$

Let for $f \in A_{p,h_1}$ f_{r,h_1} be constructed. Then $\| Ef \|_{l_h^p(\Sigma_p)} \leq \| Ef_{r,h_1} \|_{l_h^p(\Sigma_h)} +$ $\| Ef - Ef_{r,h_1} \|_{l_h^p(\Sigma_h)}$.

Properties (B), (iii), (ii), A5 yield

$$\| Ef_{r,h_1} \|_{l_h^p(\Sigma_h)} \leq c_2 h^\sigma (\| f_{r,h_1}^{(r)} \|_{L_p} + \| f_{r,h_1} \|_{L_p}) \leq c_2 h^\sigma (\| f_{r,h_1}^{(r)} \|_{L_p} + \| f \|_{L_p} + \| f_{r,h_1} - f \|_{L_p})$$

$$\leq c_2 h^\sigma (\| f_{r,h_1}^{(r)} \|_{L_p} + \| f \|_{L_p} + c_2'(r)\, \tau_r(f;\ h_1)_{L_p})$$

$$\leq c_2 h^\sigma (\| f \|_{L_2} + c_2'(r) \tau_r(f; h_1)_{L_p} + c(r) h_1^{-r} \tau_r(f;\ h_1)_{L_p})$$

$h \leq 1$ implies $h^{\sigma/r} \geq h$ for $\sigma \leq r$. Let $h_1 = h^{\sigma/r}$. Property (A), Lemma 2, Corollary 1, A5 now yield:

$$\| Ef - Ef_{r,h_1} \|_{l_h^p(\Sigma_h)} \leq c_1 \| f - f_{r,h_1} \|_{A_{p,h^{\sigma/r}}} = c_1 \| f - f_{r,h_1} \|_{A_{p,h_1}}$$

$$\leq c_1'(p) \tau_r(f,\ 2h^{\sigma/r})_{L_p} \leq c_1''(r,\ p) \tau_r(f;\ h^{\sigma/r})_{L_p}.$$

Hence, the following estimate holds:

$$\| Ef \|_{l_{h(\Sigma_h)}^p} \leq c(r,\ c_1,\ c_2,\ p)(h^\sigma \| f \|_{L_p} + \tau_r(f;\ h^{\sigma/r})_{L_p})$$

$$\leq c(r,\ c_1,\ c_2,\ p) h^{s \frac{\sigma}{r}} \| f \|_{A^s_{p\infty}}, \quad 0 \leq s \leq r.$$

Since, according to Corollary 1, $A_{p,h^{\sigma/r}}$ coincides as a set with A_p, the above estimate holds for any $f \in A_p$. The theorem is proved.

Theorem 1 is valid for a bounded interval instead of **R** and *a* non-uniform network, too. It is the main result used in obtaining network-norm interpolation estimates in the following applications.

A p p l i c a t i o n I. The parabolic equation of arbitrary order (v. [10, 11]). For denotations, see (1, 2). In this case $H(\frac{\partial}{\partial x},\ x,\ t) = \frac{\partial^{2m}}{\partial x^{2m}}$; $m > 0$, integer. Here we may consider $H = H(\xi) = \xi^{2m}$. The operators G and G_h are defined in the network points (via Fourier multipliers):

$$G(\nu d) f = F^{-1} (\exp(-\nu d H(\cdot))) * f$$

$$G_h(\nu d) f = F^{-1} (\exp(-\nu d H_h(\cdot))) * f, \text{ where}$$

(3) $$H_h(\xi h) = \frac{\ln e(h\xi)}{d}, \text{ where}$$

$$e(h\xi) = \sum_{\alpha \in I} c_\alpha e^{i\alpha h \xi},\ c_\alpha,\ I \text{ — constants } (cf.\ (2)), \text{ and}$$

$$F(G_h(\nu d) f)(\xi) = [e(h\xi)]^\nu \hat{f}(\xi) . (v.\ [10]).$$

We need the following definition in order to formulate further results (v. [10]).

Let $\alpha = k\,|\,H(h)\,| = kh^{2m}$, $k>0$ constant. $H_h(\xi)$ is said to be an approximation of $H(\xi)$ of order $s>0$, if

(4) $$H_h(\xi) - H(\xi) = h^s\,|\,\xi\,|^{m+s}Q(h\xi),$$

where $Q \in C^\infty(A)$ with bounded derivatives in A and $|\,Q(\eta)\,| \geq Q_0 > 0$ for $\eta \in A'$ where $A = \{\eta : 0 < |\,\eta\,| < \varepsilon_0\}$. The next theorem yields an integral-norm estimate for this problem.

Theorem A. (*v.* [10]). *Let the following conditions hold.*

(i) $G_h(t)$ is stable in L_p (uniformly with respect to t).

(This condition contains the case $\sum_{\alpha \in I} |\,c_\alpha\,| \leq 1 + cd$, $c>0$)

(ii) H_h is an approximation of H (see (4)) of order $r>0$. Then, $\|\,E_h(t)f\,\|_{L_p} \leq ch^s\,\|\,f\,\|_{B^s_{p\infty}}$, $t = \nu d = \nu kh^{2m}$; $\nu = 0, 1, \ldots, N$; $0 \leq s \leq r$. We shall prove the following theorem.

Theorem 2. *Let $f \in A_p$ and the following conditions hold:*

(i) $G_h(t)$ is stable: $A_{p,h} \to l^p_h(\Sigma_h)$ (uniformly with respect to t).

(This contains also the case $\sum_{\alpha \in I} |\,c_\alpha\,| \leq 1 + cd$, $c>0$. See also Remark 1 below).

(ii) Condition (ii) of theorem **A** is fulfilled.

Then [16],

$$\|\,E_h(t)f\,\|_{l^p_h(\Sigma_h)} \leq ch^s\,\|\,f\,\|_{A^s_{p\infty}},$$

$t = \nu kh^{2m}$, $0 \leq s \leq r$.

Proof: We shall first check the conditions of theorem 1. Condition (i) for $G_h(t)$ coincides with condition (A) of theorem 1. For $G(t)$ condition (i) is implied by Lemma 3b. Let $f \in W^r_p$, $r \geq 1$. Then, from Lemma 1,

(5) $$\|\,E_h(t)f\,\|_{l^p_h(\Sigma_h)} \leq \|\,E_h f\,\|_{L_p} + \tau_1(E_h f;\, h)_{L_p}.$$

Since $f \in W^r_p$, it is easy to check that $G(t)f$, $G_h(t)f$ and, hence, $E_h(t)f$ belong to the space W^1_p and commute with $\frac{\partial}{\partial x}$: $[E_h(t)f]'_x(x) = [E_h(t)f'](x)$.

Therefore, property A7. of the average moduli implies

$$\tau_1(E_h f;\, h)_{L_p} \leq h\,\|\,(E_h f)'_x\,\|_{L_p} = h\,\|\,(E_h f')\,\|_{L_p}.$$

Since $r-1 \geq 0$ and $f \in W^r_p$, $f' \in W^{r-1}_p$.

Theorem A. now implies for $s = r$ and $s = r-1$, resp.

(6) $$\|\,E_h(t)f\,\|_{L_p} \leq ch^r\,\|\,f\,\|_{B^r_{p\infty}} \leq ch^r\,\|\,f\,\|_{W^r_p}$$

(7) $$\|\,E_h(t)f'\,\|_{L_p} \leq ch^{r-1}\,\|\,f'\,\|_{B^{r-1}_{p\infty}} \leq ch^{r-1}\,\|\,f'\,\|_{W^{r-1}_p}$$

$$\leq ch^{r-1}(\|\,f\,\|_{W^1_p} + \|\,f\,\|_{W^r_p}) \leq 2ch^{r-1}\,\|\,f\,\|_{W^r_p}.$$

Here we used that $\|f'\|_{L_p} \leq \|f\|_{L_p} + \|f'\|_{L_p} = \|f\|_{W_p^1}$ and the embedding $W_p^r \subset W_p^1$ for $r \geq 1$ (5), (6), (7) now yield $\| E_h(t) f \|_{l_h^p(\Sigma_h)} \leq ch^r \|f\|_{W_p^r} + 2ch \,.\, h^{r-1} \|f\|_{W_p^r} = 3ch^r \|f\|_{W_p^r}$. Therefore, condition (B) of theorem 1 is fulfilled and theorem 1 holds. Applying it yields

$$\| E_h(t) f \|_{l_h^p(\Sigma_h)} \leq c(r,\ c_1,\ c_2,\ p)(h^r \|f\|_{L_p} + \tau_r(f;\ h)_{L_p}) \,.$$

The theorem is proved.

Using theorem 2 and the properties of the average moduli of smoothness, the following two corollaries can be easily proved.

Corollary 2. If $f^{(s)} \in L_p$, $1 \leq s < r$, then

$$\| E_h(t) f \|_{l_h^p(\Sigma_h)} = O(h^s \omega_{r-s}(f^{(s)};\ h)_{L_p} + h^r).$$

Corollary 3. If $Vf^{(s-1)} < \infty$, $1 \leq s \leq r$, then

$$\| E_h(t) f \|_{l_h^p(\Sigma_h)} = O(h^{s-1+1/p}).$$

The constants here are independent of t.

Remark 1. It can be easily proved, that, since G_h is defined as in (2), condition (i) of theorem 2 is weaker than condition (i) of theorem A. It should be only noted that P_{Σ_h} commutes with all G_h defined as in (2), because P_{Σ_h} is linear and

$$[P_{\Sigma_h} f(\cdot + \alpha h)](x) = (P_{\Sigma_h} f(\cdot)](x + \alpha h),\ \alpha \text{ — integer}$$

(see the definition of P_{Σ_h} and below). Hence, it is easy to see that

$$\| G_h(t) f \|_{l_h^p(\Sigma_h)} = \| P_{\Sigma_h} G_h f \|_{L_p} = | G_h(P_{\Sigma_h} f) \|_{L_p} \leq c \| P_{\Sigma_h} f \|_{L_p} = c \|f\|_{l_h^p(\Sigma_h)}.$$

Here we used that, if $\varphi \in l_h^p(\Sigma_h)$, then $P_{\Sigma_h} \varphi \in L_p$; the commutation of P_{Σ_h} and G_h and condition (i) of theorem A.

Application 2. The hyperbolic equation of first order. Using the denotations of (1), (2) and (3), we have $H(\frac{\partial}{\partial x},\ x,\ t) = \frac{\partial}{\partial x}$, i. e. $H(\xi) = \xi$, and a correspondingly modified definition (4) with $d = k\,|\,H(h)\,| = kh$, $k > 0$.
The following integral-norm estimate holds.

Theorem B. (v. [11]). *Let the conditions of theorem A. be fulfilled (for $G(t)$, corresponding to $H(\xi) = \xi$). Then*

$$\| E_h(t) f \|_{L_p} \leq ch^{\frac{sr}{r+1}} \|f\|_{B_{p\infty}^s},\quad 0 \leq s < r+1.$$

The following network-norm estimate holds.

Theorem 3. *Let $f \in A_p$ and the conditions of theorem 2 hold (with theorem B instead of theorem A)*

Then, $\| E_h(t) f \|_{l_h^p(\Sigma_h)} \leq ch^{\frac{sr}{r+1}} \|f\|_{A_{p\infty}^s}$.

Proof: We shall consider two cases: $0 \leq s \leq r$ and $1 < s < r+1$. (1) $0 \leq s \leq r$. The proof here is analogous to theorem 2. Let us first check the conditions of theorem 1. Condition (i) for $G_h(t)$ coincides with condition A of theorem 1. $G(t)f$ for the concrete problem has a simple representation: $[G(t)f](x) = f(x+t)$. Hence, it is easy to see that $G(t)$ satisfies (uniformly with respect to t) the condition (A) of theorem 1 ($f \in A_p$):

$$\|f(\cdot+t)\|_{l^p_h(\Sigma_h)} \leq \|f\|_{l^p_h(\Sigma_h)} \leq \|f\|_{A_{p,h}} \leq \|f\|_{A_{p,h}^{\sigma/r}}.$$

Let $f \in W^r_p$, $r \geq 1$. Then, from Lemma 1, $\|E_h(t)f\|_{l^p_h(\Sigma_h)} \leq \|E_h f\|_{L_p} + \tau_1(E_h f;\ h)_{L_p}$.
Since $f \in W^r_p$, it is easily checked that $E_h(t)f \in W^1_p$ and $[E_h(t)f]'_x(x) = [E_h(t)f'](x)$. Hence, $\tau_1(E_h f;\ h)_{L_p} \leq h \| E_h f' \|_{L_p}$.

Hence, by Theorem B and some embedding results, already mentioned in Theorem 1, we obtain

$$\|E_h(t)f\|_{l^p_h(\Sigma_h)} \leq ch^{\frac{r^2}{r+1}} \|f\|_{W^r_p} + 2ch \, . \, h^{\frac{(r-1)r}{r+1}} \|f\|_{W^r_p}.$$

Hence, $h \leq 1$ and $1 + \frac{(r-1)r}{r+1} > \frac{r^2}{r+1}$ for $r \geq 1$ imply

$$\|E_h(t)f\|_{l^p_h(\Sigma_h)} \leq 3ch^{\frac{r^2}{r+1}} \|f\|_{W^r_p}.$$

Hence, theorem 1 holds, with $\sigma = r^2/(r+1)$.
Applying theorem 1 yields

$$\|E_h(t)f\|_{l^p_h(\Sigma_h)} \leq c(r,\ c_1,\ c_2,\ p)(h^r \|f\|_{L_p} + \tau_r(f;\ h^{\frac{r}{r+1}})_{L_p})$$

$$\leq c(r,\ c_1,\ c_2,\ p) h^{s\frac{r}{r+1}} \|f\|_{A^s_{p\infty}}, \quad 0 \leq s \leq r.$$

Case (1) is proved.
(2) $1 < s < r+1$. Since the conditions of theorem B hold and, evidently, $f \in W^1_p$, we may obtain

$$\|E_h f\|_{l^p_h(\Sigma_h)} \leq \|E_h f\|_{L_p} + \tau_1(E_h f;\ h)_{L_p} \leq \|E_h f\|_{L_p} + h\|Ef'\|_{L_p}$$

$$\leq ch^{\frac{sr}{r+1}} \|f\|_{B^s_{p\infty}} + h^{1+\frac{(s-1)r}{r+1}} \|f'\|_{B^{s-1}_{p\infty}} \leq ch^{\frac{sr}{r+1}} (\|f\|_{B^s_{p\infty}} + \|f'\|_{B^{s-1}_{p\infty}}),$$

since $h \leq 1$ and $\frac{sr}{r+1} < 1 + \frac{(s-1)r}{r+1}$, $1 < s < r+1$, $r \geq 1$. We shall estimate $\|f'\|_{B^{s-1}_{p\infty}}$ by means of $\|f\|_{B^s_{p\infty}}$:

$$\|f'\|_{B^{s-1}_{p\infty}} = \|f'\|_{L_p} + \sup_{0<t<\infty} (t^{-s+1} \omega_{r+1}(f;t)_{L_p})$$

$$\leq (\|f\|_{L_p} + \|f'\|_{L_p}) + (\|f\|_{L_p} + \sup_{0<t<\infty} = (t^{-s+1}\omega_{r+1}(f;\ t)_{L_p}) = \|f\|_{W^1_p} + \|f\|_{B^s_{p\infty}}.$$

Since $s>1$, $B^s_{p\infty}\subset W^1_p$ holds. This implies $\|f'\|_{B^{s-1}_{p\infty}}\leq c'\|f\|_{B^s_{p\infty}}$ and, hence,

$\|E_h f\|_{l^p_h(\Sigma_h)}\leq c''h^{s\frac{r}{r+1}}\|f\|_{B^s_{p\infty}}$, $1<s<r+1$.

Case (2) is proved. It remains to note that when $s\in(1, r]$ — the common interval for both cases (1) and (2), the estimates of case (1) and (2) are equivalent, since $A^s_{pq}=B^s_{pq}$ (equivalence of norms) for $s>1/p$ (see the definition C. of the A-spaces). This allows us to write

$$\|E_h(t)f\|_{l^p_h(\Sigma_h)}\leq c(r, c_1, c_2, p)h^{s\frac{r}{r+1}}\|f\|_{A^s_{p\infty}}.$$

The theorem is proved.

Theorem 3 and some of the properties of the average moduli of smoothness now easily imply:

Corollary 4. If $f^{(s)}\in L_p$, $1\leq s<r+1$, then

$$\|E_h(t)f\|_{l^p_h(\Sigma_h)}=O(h^{\frac{sr}{r+1}}\,\omega_{r+1-s}(f^{(s)};\, h^{\frac{r}{r+1}})_{L_p}+h^r).$$

Corollay 5. If $Vf^{(s-1)}<\infty$, $1\leq s\leq r$, then

$$\|E_h(t)f\|_{l^1_h(\Sigma_h)}=O(h^{\frac{sr}{r+1}}).$$

Finally, we should like to illustrate the difference between estimates in theorems A. and B., and those in theorems 2 and 3, respectively. We shall consider the following lemma (the uniformity of the network is essential in this lemma).

Lemma 4. *Let* $f\in L_p$, $1\leq p\leq\infty$. *Then*

$$\|f\|_{L_p(\mathbf{R})}=\Big(\frac{1}{h}\int_{-h/2}^{h/2}\|f\|^p_{l^p_h(\Sigma_h+\xi)}\,d\xi\Big)^{1/p},\quad 1\leq p\leq\infty$$

$$\|f\|_{L_\infty(\mathbf{R})}=\operatorname*{ess\,sup}_{\xi\in\left[-\frac{h}{2},\,\frac{h}{2}\right)}\|f\|_{l^\infty_h(\Sigma_h+\xi)},\quad p=\infty.$$

Proof: Consider the case $1\leq p<\infty$.

$$\|f\|^p_{L_p(\mathbf{R})}=\int_{-\infty}^{\infty}|f(x)|^p dx=\sum_{k=-\infty}^{\infty}\int_{x_k-\frac{h}{2}}^{x_k+\frac{h}{2}}|f(x)|^p\,dx.$$

For every k: $k=0, \pm1,\ldots$ the variable χ is being replaced by ξ: $x=x_k+\xi$. We obtain

$$\|f\|^p_{L_p(\mathbf{R})}=\sum_{k=-\infty}^{\infty}\int_{-h/2}^{h/2}|f(x_k+\xi)|^p\,d\xi$$

$$=\int_{-h/2}^{h/2}\Big(\sum_{k=-\infty}^{\infty}|(f(x_k+\xi)|^p\Big)d\xi=\frac{1}{h}\int_{-h/2}^{h/2}\|f\|^p_{l^p_h(\Sigma_h+\xi)}\,d\xi.$$

The proof for $p=\infty$ is almost the same. The lemma is proved.

Lemma 4 shows that the estimates of theorems A. and B. can be represented in the following way ($\varphi(u)$ — a linear function of u).

$$\Big(\frac{1}{h}\int_{-h/2}^{h/2}\|E_h f\|^p_{l^p_h(\Sigma_h+\xi)}\,d\xi\Big)^{\frac{1}{p}}\leqq ch^{\varphi(s)}\|f\|_{B^p_{p\infty}},\quad 1\leqq p<\infty$$

(for $p=\infty$ — analogously). From here it can be seen that these estimates are, in a way, mean values of network-norm estimates requiring data about f on $\Sigma_h+\xi$ for almost every $\xi\in\left[-\frac{h}{2},\frac{h}{2}\right)$. In the case of theorems 3. and 4. the corresponding estimates are of the form:

$$\|E_h f\|_{l^p_h(\Sigma_h)}\leqq ch^{\varphi(s)}\|f\|_{A^s_{p\infty}}.$$

They require data about f on Σ_h only and, thus, are suitable for deriving a priori estimates of the error for a concrete difference scheme, defined on a concrete Σ_h.

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