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A NEW NONPARAMETRIC INDEPENDENCE TEST

A. FASSÒ, F. PESARIN

It is proposed a new nonparametric test for independance of two continuous random variables.

The test is shown to be consistent against a subset of the first and the second order regression alternatives, i. e. alternatives belonging to a well defined subset Q_1 of the continuous random variables for which at least one of the quantities $E(Y \mid X)$, $E(X \mid Y)$, $\mathrm{Var}(Y \mid X)$ $\mathrm{Var}(X \mid Y)$ is, with positive probability, not constant.

Moreover, it is shown that when the parent distribution is a bivariate normal, the test proposed is asymptotically equivalent to the usual one based on the sample correlation coefficient having Bahadur's ARE equal to one.

1. Introduction. Several nonparametric tests for independence of two continuous random variables (r. v. 's) are available in the literature. Some of them are based on linear rank statistics, for instance the Spearman rank correlation coefficient or the Fisher-Yates normal-scores correlation coefficient; for a full discussion see Hájek and Sidák [15].

Others are based on nonlinear rank statistics, for instance the well-known Kendall's tau, the Cucconi's [7] mean square successive differences of ranks, the Hoeffding's [16] D-statistic, or the Deheuvels' [10] Kolmogorov-Smirnov type statistic.

Bell and Doksum [5], studied systematically distribution free tests of independence. Pesarin [18], proposed a test for first and second order regression alternatives, i. e. alternatives for which at least one of the quantities $E(Y \mid X)$, $E(X \mid Y)$, $\mathrm{Var}(Y \mid X)$, $\mathrm{Var}(X \mid Y)$ is not constant.

In this paper, following the approach of [Pesarin 18], the asymptotic behaviour of an independence test is studied over a well defined subset Q of the continuous bivariate r. v. 's.

2. The Random Normal Ranks. The solution discussed here is based on the replacement of the original data by means of random normal ranks (r. n. r. 's), first introduced by Bell and Doksum [4].

To define this randomized ranks, let us consider a sample (X_i, Y_i), $i=1,\ldots,n$ from a bivariate continuous distribution; let $X(i)$ be the i-th ordered value in the increasing arrangement of the X's, and let $Y[i]$ be the concomitant of $X(i)$, that is the Y value paired with $X(i)$. Symmetrically, with the increasing arrangement of the Y's, let $X[i]$ be the concomitant of $Y(i)$.

Furthermore, let $R_i = R(Y[i])$ be the rank of $Y[i]$, that is the position of $Y[i]$ in the increasing arrangement of $Y[1], \ldots, Y[n]$; analogously let $R'_i = R(X[i])$. Asymptotic properties of concomitants and their ranks are studied from different points of view by David and Galambos [9], Bhattacharya [6], Sen [20,21] and Fassò [13, 14].

Let us consider, apart, two independent simple random samples $S_1, \ldots, S_n$ and $S'_1, \ldots, S'_n$ from a standard normal distribution; let $S(1) \leq \cdots \leq S(n)$ and $S'(1) \leq \cdots \leq S'(n)$ be the related order statistics.

The randomized transformation: $Y[i] \to S(R_i)$ is here named i-th random normal rank of the concomitant $Y[i]$; $S'(R'_i)$ is defined similarly.

As shown by Bell and Doksum [4] when X and Y are independent, the r. n. r. 's $S(R_i)$ are independent and identically r. v. 's with standard normal distribution.

3. The Bell and Doksum statistic in the subset Q. Bell and Doksum [4, 5] suggested, for testing independence, the randomized sample covariance

$$W' = \frac{1}{n} \sum_{i=1}^{n} S(R_i)S'(i) = \frac{1}{n} \Sigma S'(R'_i)S(i).$$

Its exact null distribution is as the difference of two independent *chi* square with n degrees of freedom (see Bell & Doksum, [4], theorem 5.1). Moreover, the test based on W' has the following remarkable property: its Pitman's asymptotic relative efficiency (ARE) is one with respect to that one based on the usual sample correlation coefficient r, when the parent distribution is a bivariate normal. So it is asymptotically equivalent to the optimum solution in the normal case. In the present context we prefer the slight modification of W' defined by $W = (1/n)\Sigma S(R_i)\Phi^{-1}(i/(n+1))$ where Φ^{-1} is the inverse of the standard normal distribution function.

In order to study the consistency of the modified Bell & Doksum test, as well as those considered in the following sections, we need to define a subset Q of the continuous bivariate r. v. 's. For this let us consider, first, the following "regularity" definition:

Def. 3.1. *A function* $g: \mathbf{R}^2 \to \mathbf{R}$ *is called regular (with respect to the measures* P_X *and* P_U*) iff it satisfies the following continuity conditions:*
i) $g(.\,, z)$ *is* P_X*-a. s. continuous, for* P_U*-almost every fixed* $z \in \mathbf{R}$;
ii) $P_U[g(x, .) = c] = 0$, $\forall c \in \mathbf{R}$, *for* P_X*-almost every fixed* $x \in \mathbf{R}$.
We are now ready to define Q.

Def. 3.2. *A bivariate continuous r. v.* (X, Y) *is a member of* Q *iff there exist two regular functions* $g_1, g_2: \mathbf{R}^2 \to \mathbf{R}$ *and two continuous r. v. 's* U_1 *and* U_2*, for which:*
i) $Y = g_1(X, U_1)$,
ii) $X = g_2(Y, U_2)$,
where X *and* U_1 *are independent, as well as* Y *and* U_2.

Equivalently, the set Q can be defined by means of continuity conditions on the conditional distributions $F_{Y|X}(y, x) = P(Y \leq y \mid X = x)$ and $F_{X|Y}$.

Def. 3.3. *A bivariate continuous r. v.* (X, Y) *is a member of* Q *iff:*
i) $F_{Y|X}(y, .)$ *is* P_X*-a. s. continuous,* $\forall$ *fixed* $y \in \mathbf{R}$;
ii) $F_{Y|X}(.\,, x)$ *is continuous for* P_X*-almost every fixed* $x \in \mathbf{R}$;
iii) $F_{X|Y}(x, .)$ *is* P_Y*-a. s. continuous,* $\forall$ *fixed* $x \in \mathbf{R}$*, and*
v) $F_{X|Y}(.\,, y)$ *is continuous for* P_X*-almost every fixed* $y \in \mathbf{R}$.

The proof of the equivalence of the two definitions is ommitted because immediate.

It's now useful to note that:

Theorem 3.1. *If $(X, Y) \in Q$, then both W' and W converge in probability to cov $(Z(X), Z(Y))$ as $n \to \infty$, where $Z(X) = \Phi^{-1} \circ F_X(X)$, $Z(Y) = \Phi^{-1} \circ F_Y(Y)$ and Φ is the standard normal distribution function.*
For the proof see Fassò, [13], p. 79, art. 1.

Hence it is easy to see that the Bell and Doksum test is not consistent for certain nonmonotonic regression alternatives.

4. First order regression alternatives. Let us consider the elements of Q for which the conditional r. v.'s $(Y \mid X = x,\ x \in \mathbf{R})$, are stochastically ordinable in some sense, at least for x in some P_X-non-null set. To be more precise, let us consider the following generalized regression model:

Def. 4.1. *$Y = g(X, U)$ where $(X, Y) \in Q$ and there exist disjoint and P_X-non-null sets: E_1, E_2 for which, $x \in E_1$, $y \in E_2$, imply that:*
i) *either $g(x, .) \leq g(y, .)\ P_U$-a. s. or*
ii) *$g(x, .) \geq g(y, .) P_U$-a. s.,*
both with strong inequality over a P_U-non-null set.

For example: $Y = m(X) + U$, with $m(.)$ P_X-a. s. continuous, is a generalized omoschedastic regression model, being $E(Y \mid X) = m(X) + a$ and $\mathrm{Var}(Y \mid X) = b$; moreover, it satisfies the conditions of Def. 4.1 if $P_X(m(.) = c) < 1$, $\forall c \in \mathbf{R}$.

To test independence against regression of Y over X given by Def. 4.1, consider the sample autocorrelation coefficient of the r. n. r.'s $S(R_i)$

$$T_1 = \frac{1}{n-1} \sum_{i=1}^{n-1} S(R_i) S(R_{i+1}).$$

Under the null hypothesis, T_1 is asymptotically normal* with zero mean and variance $\frac{1}{n-1}$; the proof is a particular case of theorem 6.1.

Remark. The normal approximation is likely to be good even for small sample size, because $ET^k = 0\ \forall$ odd k.

Furthermore, in Fassò ([13], p. 67, art. 1) is proved that T_1 converges in probability to $\mathrm{Var}(E(Z(Y) \mid X))$ as $n \to \infty$, when $(X, Y) \in Q$ and $Z(Y) = \Phi^{-1} \circ F_Y(Y)$. Finally, the test based on the large significant values of T_1 is consistent against the set of alternatives given by Def. 4.1, because of the following theorem.

Theorem 4.1. *Under the conditions of Def. 4.1,* $\mathrm{Var}(E(Z(Y) \mid X))$ *is strictly positive.*

Proof. It is sufficient to show that $E(Z \mid X = x_1) \neq E(Z \mid X = x_2)\ \forall\ x_i \in E_i$, where E_i, $i = 1, 2$ are given in Def. 4.1. Let N be the set of the points where F_Y is constant, obviously $P_Y(N) = 0$. Let $B = B(x_1, x_2)$, $x_i \in E_i$, be any P_U-non-null set such that

i) $g_{x_1}(t) > g_{x_2}(t)$, $t \in B$, where $g_x(t) = g(x, t)$.

The existence of such a set B is ensured by Def. 4.1. Noting that Φ is an increasing function, and $P_X(E_i) > 0$, it is sufficient that

ii) (i) implies $F_Y \circ g_{x_1}(t) > F_Y \circ g_{x_2}(t)$, $t \in B'$, for some P_U-non-null B'.

Now let us suppose, ab absurdo, that

iii) $F_Y \circ g_{x_1}(.) = F_Y \circ g_{x_2}(.) P_U$-a. s.

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* By the statement: T_n is asymptotically normal with mean μ_n and variance σ_n^2, we mean that $(T_n - \mu_n)/\sigma_n$ converges in law to a normal r. v. with zero mean and variance one, as $n \to \infty$.

We have that (iii) and (i) imply $g_{x_1}(t) \in N$, $g_{x_2}(t) \in N$ when $t \in B$. Furthermore, $P[g_{x_1}(U) \in N,\ g_{x_2}(U) \in N,\ U \in B] \leq P(g_{x_1}(U) \in N)$. By definition, the right hand side is equal to $P(Y \in N \mid X = x_1)$ and this is zero for P_X-almost every x_1 because $O = P_Y(N) = \int P(Y \in N \mid X) dP_X$. Hence (ii) is true and the conclusion follows. QED

Therefore, the test based on T_1 is a more general independence test than the Bell and Doksum's one. Nevertheless it has a local lack of efficiency.

In fact, if the parent distribution is a bivariate normal, the ARE (both the Pitman's one and the approximate Bahadur's one [2, 3]) of T_1 with respect to the sample correlation coefficient r is zero. So with large normal samples T_1 need infinitely more observations than r to have the same power.

To obtain a test both consistent and locally asymptotically efficient, let us consider both statistics W and T_1. For this let $D = \max(T_0, T_1)$, where $T_0 = |W| \frac{n}{n-1}$. The null distribution of D is asymptotically the same of the maximum between a normal r. v., with zero mean and variance $\frac{1}{n-1}$, and the absolute value of another independent normal, with zero mean and variance $\frac{1}{(n-1)^2}$; the proof is a particular case of theorem 6.1.

Obviously the limit in probability of D is strictly positive iff $\mathrm{Var}[E(Z \mid X)] > 0$, and the test based on large significant values of D, is consistent against alternatives given by Def. 4.1.

If the parent distribution is a bivariate normal with correlation coefficient ρ, then D converges in probability to $|\rho|$; furthermore the Bahadur's large deviations condition is satisfied being $\ln[1 - \Phi(x)(2\Phi(x) - 1)] = \ln[(1 - \Phi(x))(1 + 2\Phi(x))] = -\frac{x^2}{2}(1 + o(1))$. Hence the local* Bahadur's ARE of D with respect to r is one; in other words, to test $\rho = 0$ against $\rho \neq 0$, D is asymptotically locally equivalent to the optimum unbiased solution in the normal case.

5. Second order regression alternatives. To describe the nonconstant conditional variability, let us consider a generalized regression model, for which the deviations of the r. v.'s $(Y \mid X = x)$, $x \in \mathbf{R}$, from a median value ξ of Y have the same sign and are stochastically discriminable, in absolute value, at least for x in some non-null set. To be more precise, let us consider the following second order regression model:

Def. 5.1. *$Y = g(X, U)$, where $(X, Y) \in Q$, and there exist disjoint and P_X-non-null sets C_1, C_2, and a median value ξ of Y, for which: $x \in C_1$, $y \in C_2$ imply*

i) *$(g(x, .) - \xi)(g(y, .) - \xi) \geq 0$ P_U-a. s.; moreover for the above values*

ii) *either $|g(x, .) - \xi| \geq |g(y, .) - \xi|$ P_U-a. s.*

iii) *or $|g(x, .) - \xi| \leq |g(y, .) - \xi|$ P_U-a. s., both* (ii) *and* (iii) *hold with strong inequality over a P_U-non-null set.*

For example, $Y = \sigma(X)U + \xi$, where σ is a P_X-a. s. continuous and nonzero function, U symmetric about zero, is a regular second order regression model, such that $\mathrm{Var}(Y \mid X) = a\, \sigma^2(X)$. In this model the regression functions $E(Y \mid X)$ and $E(Z(Y) \mid X)$ are constant and the test based on T_1 is not consistent.

Note that, in this model, $E(Z(Y)^2 \mid X) = 1$ a. s. iff $\sigma(X) = c$ a. s. for some $c \in \mathbf{R}$. In general the following theorem holds.

* We call "local" Bahadur's ARE the limit of the approximate Bahadur's ARE $\psi(\rho)$ as $\rho \to 0$, see [2].

Theorem 5.1. *Under the conditions of Def. 5.1 the quantity* $\mathrm{Var}[E(Z^2 \mid X)] = E\{[E(Z^2 \mid X) - 1]^2\} = E[E^2(Z^2 \mid X)] - 1$, $Z = Z(Y)$, *is strictly positive.*

Proof. Def. 5.1 implies that, $\forall$ fixed $x \in C_1$, $y \in C_2$, there exist disjoint sets A_1 and A_2 with $P_U(A_1 \text{ or } A_2) = 1$ such that

$$g_x(t) - \xi \geqq g_y(t) - \xi \geqq 0, \quad t \in A_1,$$

$$\xi - g_x(t) \leqq \xi - g_y(t) \leqq 0, \quad t \in A_2;$$

in other words

$$F_Y \circ g_x(t) \geqq F_Y \circ g_y(t) \geqq 1/2, \quad t \in A_1,$$

$$F_Y \circ g_x(t) \leqq F_Y \circ g_y(t) \leqq 1/2, \quad t \in A_2.$$

Hence one has $[\Phi^{-1} \circ F_Y \circ g_x(.)]^2 \geqq [\Phi^{-1} \circ F_Y \circ g_y(.)]^2$ P_U-a. s. and the proof is thus concluded in a very similar way as in theorem 4.1. QED

Thus we can use $T_3' = \frac{1}{n-1} \Sigma(S(R_i)^2 - 1)(S(R_{i+1})^2 - 1)$, with large significant values, to obtain a test consistent against (5.1). In fact, under H_0, T_3' is asymptotically normal distributed with mean zero and variance $4/n$, and, in Q, is a consistent estimator of $\mathrm{Var}(E(Z^2 \mid X))$. The normality follows as a particular case of theorem (6.1); the convergence in probability is proved similarly to that of T_1 if one notes that $E(S^k) < \infty$ for every integer k.

6. The test against first and second order regression alternatives. Until now alternatives of first and second regression have been considered separately. To combine them, let us consider the following generalized regression model:

Def. 6.1. $Y = g(X, U) = g_1(X, U) + g_2(X, U)$, $(X, Y) \in Q$, where the regular functions g_1 and g_2 satisfy the conditions of Def.,'s 4.1 and 5.1 respectively, under the additional condition that the sets E_1 E_2 and C_1, C_2, are all disjoint and such that at least one pair is P_X-non-null i. e.

$$P_X(E_1)P_X(E_2) + P_X(C_1)P_X(C_2) > 0.$$

Remark. Such a definition don't exclude completely a regular first and second order regression model like

$$Y = m(X) + \sigma(X)U \tag{6.2}$$

with $E(Y \mid X) = m(X)$ and $\mathrm{Var}(Y \mid X) = \sigma^2(X) \subset$ two independent functions, bu Def. 6.1 restricts the domain of such combined regression to a set with $P_X < 1$. More clearly, Def. 6.1 requires that in (6.2) there exist a P_X-non-null set A such that, if $x \in A$, then $m(x)$ is not constant and $\sigma(x) = \sigma$, or $\sigma(x)$ is not constant and $m(x) = 0$.

Let us consider now the set $Q_1 \subset Q$ of the bivariate r. v. 's such that $Y = g(X, U)$ is given by Def. 6.1, and analogously $X = g'(Y, U')$, where g' satisfies the regression conditions corresponding to Def. 6.1.

In order to obtain a test consistent against Q_1 one can proceede as follows: Draw two independent auxiliary random samples $S_1, \ldots, S_n$ and $S_1', \ldots, S_n'$ from the standard normal distribution. Calculate from the actual data (X_i, Y_i), $i = 1, \ldots, n$ the ranks R_i and R_i' of the concomitants $Y[i]$'s and $X[i]$'s, respectively. Then calculate

$$T=\max[T_i, \quad i=0,\ldots,4]$$

where T_0 and T_1 are as in section 4,

$$T_2=\frac{1}{n-1}\Sigma S'(R_i')S'(R_{i+1}'),$$

$$T_3=\frac{1}{2(n-1)}(S(R_i)^2-1)(S(R_{i+1})^2-1)=T_3'/2,$$

$$T_4=\frac{1}{2(n-1)}(S'(R_i')^2-1)(S'(R_{i+1}')^2-1)$$

where summations range over $i=1,\ldots,n-1$.

If $\sqrt{n-1}\,T\geq k_{\alpha'}$, with k_α such that $C(k_\alpha)=\Phi(k_\alpha)^4[2\Phi(k_\alpha)-1]=1-\alpha$, then reject the null hypothesis.

In fact, under H_0, T is asymptotically the maximum among four independent normal r. v. 's, with zero mean and variance $(1/n-1)$, and the absolute value of another independent normal with zero mean and variance $n/(n-1)^2$ as it is stated in the following theorem.

Theorem 6.1. *$P(\sqrt{n-1}\,T\leq x\mid H_0)\rightarrow G(x)=\Phi(x)^4[2\Phi(x)-1]$ as $n\rightarrow\infty$.*

Proof. By Bell and Doksum's theorem it follows that, under H_0, $S(R_1)$, $\ldots, S(R_n)$, $S'(R_1'),\ldots,S'(R_n')$ are distributed as $S_1,\ldots,S_n$, $S_1',\ldots,S_n'$. Hence, putting $T_5=(n/(n-1))W$, we can write

$$(n-1)T^*=(n-1)\sum_{j=1}^{5}a_jT_j$$

$$=\sum_{i=1}^{n-1}[a_1S_iS_{i+1}+a_2S_i'S_{i+1}'+(a_3/2)(S_i^2-1)(S_{i+1}^2-1)$$

$$+(a_4/2)(S_i'^2-1)(S_{i+1}'^2-1)+S_i\Phi^{-1}(i/(n+1))]+o_p(n^{1/2})$$

$$=\sum_{i=1}^{n-1}A_{n,i}+o_p(n^{1/2})$$

say, intending equality in distribution.

Using the central limit theorem for triangular arrays (see [1], p. 349), we can easily extend the Diananda's theorem for stationary m-dependent sequences (see [12]) to include $\{A_{n,i}\}$ which is 1-dependent but not stationary, and conclude that $nT^*/(\sum_{i=1}^{n}E(A_{n,i}^2))^{1/2}$ is asymptotically normal for every $a_1,\ldots,a_5$, not all zero. Furthermore, note that $E(T_i)=0$ and $E(T_iT_j)=0$ for every $i\neq j$ $=1,\ldots,5$. Hence $n^{1/2}(T_1,\ldots,T_5)'$ is asymptotically jointly normal with independent components. The conclusion follows noting that $G(.)$ is the distribution function of the maximum among 4 independent standard normal r.v.'s and the absolute value of another independent standard normal variable. QED

The test based on $T=\max[T_0,\ldots,T_4]$ is consistent against Q_1 because, from sections 3 to 5, T converges in probability to $\max\{|\mathrm{cov}[Z(X),Z(Y)]|$, $\mathrm{Var}[E(Z(Y)\mid X)]$, $\mathrm{Var}[E(Z(X)\mid Y)]$, $\frac{\mathrm{Var}[E(Z(Y)^2\mid X)]}{2}$, $\frac{\mathrm{Var}[E(Z(X)^2\mid Y)]}{2}\}$ if $(X,Y)\in Q$; and this quantity is strictly positive when $(X,Y)\in Q_1$.

Finally note that the Bahadur's large deviations condition is satisfied by the distribution function $G(.)$, and $\ln[1-G(x)] = -\frac{x^2}{2}(1+o(1))$ as $x\to\infty$. Hence if (X, Y) is bivariate normal r. v. then the local Bahadur's ARE of T with respect to unbiased optimum solution $|r|$ is one.

7. Conclusion. We have obtained a randomized test consistent against the wide non usually parametrized set of alternatives Q_1.

From a practical point of view a question of interest is wheter one can obtain approximate or equivalent solutions using scores like $\Phi^{-1}(\frac{R_i}{n+1})$ or $E(S(R_i)|R_i)$ instead of r. n. r 's $S(R_i)$.

Bell & Doksum [5] proved that W' and $W^* = E(W|R_1, \ldots, R_n, R'_1, \ldots, R'_n)$ are asymptotically equivalent in the sense that $nE[(W'-W^*)^2|H_0]\to 0$ as $n\to\infty$. Hence W' and W^* have the same asymptotic null distribution as well as the same set of consistency and ARE.

The problem of finding a normal scores test against Q_1 will be studied in a forthcoming paper of Fassò.

Thus it will be apparent the value of r. n. r. 's procedures in providing good tools to find approximate and asymptotic distribution of normal scores procedures.

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University of Padova
Department of Statistics
35121 Padova, Via S. Francesco 33
Italy