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ON THE THIRD BOUNDARY VALUE PROBLEM FOR A FULLY NONLINEAR CONVEX ELLIPTIC EQUATION IN TWO VARIABLES

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The paper considers the third boundary value problem for a fully nonlinear convex second order elliptic equation. Using Bernstein's method of a priori bounds, a theorem of Nirenberg for equations in two variables and the method of continuity a unique solution is proved to exist, belonging to $C^{2,\alpha}(\bar{\Omega})$, $\Omega\subset R^2$.

1. Introduction. Let Ω be a bounded domain in R^2, $\partial\Omega\in C^3$. Consider the problem

$$(1)\quad f(D^2u)+g(x,\ u,\ Du)=0 \quad \text{in } \Omega,$$

$$(2)\quad (\partial_\nu u+\sigma(x)u)|_{\partial\Omega}=0 \quad ,$$

where D^2u and Du denote as usually the Hessian matrix, resp. the gradient of u, and ν the inner normal to $\partial\Omega$; we shall also use the notation ∂_ν instead of $\partial/\partial\nu$. We shall suppose that the above problem (1)—(2) satisfies the following conditions:

1) $f\in C^2(R^4)$, $f(0)=0$, $\sigma\in C^2(R^2)$, $g\in C^2(R^2\times R\times R^2)$.

2) The equation is uniformly elliptic, i. e.

$$(3)\quad \theta|\xi|^2\leq\sum_{i,j=1}^{2}f_{ij}(r)\xi^i\xi^j\leq\Theta|\xi|^2,\quad \forall r\in R^4,\ \xi\in R^2,$$

where $f_{ij}=\partial f/\partial r_{ij}=f_{ji}$ and $0<\theta\leq\Theta<\infty$.

3) The functions $r\to f(r)$, $r\in R^4$, and $p\to g(x, z, p)$, $p\in R^2$ ((x, z) denotes an arbitrary point in $R^2\times R$), are convex.

According to the smoothness, the convexity is equivalent to the inequalities

$$(4)\quad \sum_{i,j,k,l}f_{ij,kl}(r)\xi^{ij}\xi^{kl}\geq 0,\quad \forall r\in R^4,\ \xi\in R^4,$$

$$(5)\quad \sum_{i,j}g_{p_ip_j}(x,\ z,\ p)\xi^i\xi^j\geq 0,\quad \forall x\in R^2,\ z\in R,\ p\in R^2,\ \xi\in R^2.$$

4) The functions g and σ satisfy the following inequalities:

(i) $g_z(x,\ z,\ p)\leq-\eta<0$; (ii) $\sigma(x)\leq-\delta<0$;

(iii) $\max\limits_{\substack{x\in\bar\Omega\\ |z|\leq K}}|g_{p_i}(x,\ z,\ p)|\leq G(K)$; $i=1,\ 2$;

(iv) $\max\limits_{\substack{x\in\bar\Omega\\ |z|\leq K}}|g_{x_k}(x,\ z,\ p)|\leq G+G|p|$, $G=G(K)$; $k=1,\ 2$.

During the past few years many mathematicians have actively studied the fully nonlinear elliptic equations. In [1] E v a n s considers Dirichlet's problem for the n — dimensional equation $f(D^2u)=0$ for convex f and proves existence and uniqueness of a solution belonging to $C^{2,\alpha}(\Omega)\cap C(\bar{\Omega})$, $\Omega\subset R^n$. Similar results were published by N. T r u d i n g e r [2] for the more general equation $F(D^2u, Du, u, x)=0$, where F satisfies some natural structure conditions. The paper [3] treats Dirichlet's problem for the two-dimensional equation (1) and establishes existence and uniqueness of a solution in the class $C^{2,\alpha}(\bar{\Omega})$, $\Omega\subset R^2$, $0<\alpha<1$.

The main result of this paper is contained in the following.

T h e o r e m 1. *Under assumptions 1)-4) the considered problem (1)-(2) has a unique solution belonging to the class* $C^{2,\alpha}(\bar{\Omega})$, $0<\alpha<1$.

The main tools used are Bernstein's method of a priori bounds, a result of L. Nirenberg for equations in two variables and the method of continuity.

2. A priori estimates. Further we shall often use the equivalent form of equation (1)

$$\sum_{i,j=1}^{2} a^{ij}(D^2u)u_{x_ix_j}+\sum_{i=1}^{2} b^i(x,\ u,\ Du)u_{x_i}+c(x,\ u)u+g(x,\ 0,\ 0)=0, \tag{6}$$

where

$$\text{(i)}\quad a^{ij}(D^2u)=\int_0^1 f_{ij}(tD^2u)dt,$$

$$\text{(7)}\qquad \text{(ii)}\quad b^i(x,\ u,\ Du)=\int_0^1 g_{p_i}(x,\ u,\ tDu)dt,$$

$$\text{(iii)}\quad c(x,\ u)=\int_0^1 g_z(x,\ tu,\ 0)dt.$$

L e m m a 1. *If* $u\in C^2(\bar{\Omega})$ *is a solution of* (1)-(2), *then*

$$\max_{\bar{\Omega}}|u|\leq M,$$

where $M=M(\theta,\ \Theta,\ \eta,\ \delta,\ \max_{\bar{\Omega}}|g(x,\ 0,\ 0)|)$.

P r o o f. Set $z_0(x)=u^2(x)-N$ and

$$Lv=\Sigma\, a^{ij}(D^2u)v_{x_ix_j}+\Sigma\, b^i(x,\ u,\ Du)v_{x_i}+c(x,\ u)v,$$

then on $\partial\Omega$ we have

$$\partial_\nu z_0+\sigma z_0=2u\partial_\nu u+\sigma(x)u^2-\sigma(x)N$$

$$=2u(\partial_\nu u+\sigma(x)u)-\sigma(x)u^2-\sigma(x)N\quad=0-\sigma(x)u^2-\sigma(x)N\geq 0+0+\delta N>1$$

for N large enough. This means that z_0 can have no positive maximum at any point $x\in\partial\Omega$.

To prove that z_0 has no positive maximum in Ω we shall show that

$$Lz_0>1$$

for N large enough. Simple computations give

$$Lz_0 = 2\Sigma\, a^{ij}(D^2u)u_{x_i}u_{x_j} + 2u\,\Sigma\, a^{ij}(D^2u)u_{x_ix_j} + 2u\,\Sigma\, b^i(x,\ u,\ Du)u_{x_i} + cu^2 - cN$$
$$= 2\,\Sigma\, a^{ij}(D^2u)u_{x_i}u_{x_j} + 2uLu - cu^2 - cN$$
$$\geqq 2\theta\,|\,Du\,|^2 + 2uLu - cu^2 - cN = 2\theta\,|\,Du\,|^2 - 2ug\,(x,0,0) - cu^2 - cN$$
$$\geqq 2\theta|Du|^2 - (\varepsilon|2)\,u^2 - \ cu^2\ \ - (2/\varepsilon)g^2(x,\ 0,\ 0) - cN$$
$$\geqq -(c+\varepsilon/2)u^2 - (2/\varepsilon)g^2(x,\ 0,\ 0) - cN \geqq -(2/\varepsilon)g^2(x,\ 0,\ 0) - cN > 1$$

for N large enough, i. e. $z_0 \leqq 0$ in $\bar{\Omega}$ and hence $u^2 \leqq N$ in $\bar{\Omega}$. For $M = N^{1/2}$ this inequality gives us

$$\max_{\bar{\Omega}} |\,u\,| \leqq M.$$

L e m m a 2. *Let $u \in C^3(\Omega)$ be a solution of (1)-(2). Then for any domain $\Omega' \subset\subset \Omega$ we have*

$$\max_{\bar{\Omega}'} |\,Du\,| \leqq M_1',$$

where $M_1' = M_1'(\theta,\ \Theta,\ \eta,\ \delta,\ M,\ G(M),\ \Omega')$.

P r o o f. We use the auxiliary function

$$w(x) = \zeta^2\,|\,Du\,|^2 + N(u+M)^2 + N_1|x|^2,$$

where $\zeta \in C_0^\infty(\Omega)$. For large enough constants N, N_1 w has no maximum at any inner point; as $\zeta = 0$ on the boundary this yields an estimate for $|Du|$ in any domain $\Omega' \subset\subset \Omega$. The computations differ only slightly from those in [3]. The terms containing derivatives of ζ contain lower order derivatives of u than the others and thus mean no additional problem.

L e m m a 3. *If $u \in C^4(\bar{\Omega})$ is a solution of (1)-(2), then there exists a constant*

$$C = C(\theta,\ \Theta,\ \eta,\ \delta,\ M,\ \max_{\bar{\Omega}} |\,Du\,|)$$

and such that

$$\max_{\bar{\Omega}} |\,D^2u\,| \leqq C + \max_{\partial\Omega} |\,D^2u\,|.$$

P r o o f. The proof is contained in [3].

To achieve estimates for $\max_{\bar{\Omega}} |\,Du\,|$ and $\max_{\partial\Omega} |\,D^2u\,|$ we shall transform the domain Ω onto a disc. If Ω is simply connected, according to Riemann's theorem, it is conformly equivalent to a disc. To estimate the derivatives of the solution close to the boundary we shall consider the transformed equation

$$f(D^2u,\ Du,\ y) + g(\ y,\ u,\ Du) = 0 \tag{8}$$

in a crown close to the boundary S of the disc. (All functions are considered in the new variables but denoted by the same letters as before.) Without loss of generality we can choose $y_1 = \varphi$ to be the angular variable and $y_2 = \rho$ — the radial one. The boundary condition is thus transformed in

$$(u_\rho - \sigma(\ y)u)|_S = 0,\quad \sigma(\ y) \leqq -\delta < 0.$$

If Ω is not simply connected, all components of the boundary are treated similarly, i. e. we transform them into boundaries of discs and consider crowns close to them. For the inner components of the boundary the boundary condition is transformed in

$$(u_\rho+\sigma(y)u)|_{S_i}=0, \quad \sigma(y)\leq-\delta<0.$$

The transformed function $f(r, q, y)$ has the following properties:

(9)
(i) $f(0, 0, y)=0$;
(ii) the derivatives f_{kl}, $k, l=1, 2$, are bounded;
(iii) the derivatives f_{qk}, $k=1, 2$, are bounded;
(iv) the derivatives f_φ and f_ρ have linear growth with respect to the second derivatives of the solution.

We are now ready to proceed with the proofs of the bounds for $\max_{\bar\Omega}|Du|$ and $\max_{\partial\Omega}|D^2u|$. We shall consider a simply connected domain Ω. According to the above arguments this means no loss of generality. Let $B=\{0\leq\rho\leq R\}$ be the disc, conformly equivalent to Ω and $S=\partial B=\{\rho=R\}$.

Lemma 4. *Let $u\in C^3(\Omega)$ be a solution of (1)-(2). Then for N, N_0, N_1 large enough the function*

$$z_1(y)=|Du|^2-2\sigma(y)uu_\rho+Nu^2+N_0\rho^2-N_1$$

has no positive maximum in the set

$$V=\{R-\varepsilon<\rho\leq R\}.$$

Proof. If z_1 achieves a maximum at an inner point y_0 of V, then

$$\sum_{i,j=1}^{2} f_{ij}z_{1y_iy_j}(y_0)<0$$

because of the ellipticity.

Simple computations using the ellipticity give

$$\sum_{i,j} f_{ij}z_{1y_iy_j}\geq 2\theta\sum_{i,k}|u_{y_iy_k}|^2+2N\theta\sum_i|u_{y_i}|^2+2Nu\sum_{ij}f_{ij}u_{y_iy_j}+2\sum_{i,j,k}f_{ij}u_{y_k}u_{y_iy_jy_k}$$
$$-2\sigma u\sum_{i,j}f_{ij}u_{y_iy_j\rho}-2uu_\rho\sum_{i,j}f_{ij}\sigma_{y_iy_j}-4u_\rho\sum_{i,j}f_{ij}\sigma_{y_i}u_{y_j}-4u\sum_{i,j}f_{ij}\sigma_{y_i}u_{y_j\rho}$$
$$-2\sigma u_\rho\sum_{i,j}f_{ij}u_{y_iy_j}-4\sigma\sum_{i,j}f_{ij}u_{y_i}u_{y_j\rho}+2N_0f_{22}.$$

The further arguments are similar to those used in [3] to establish the bounds for the first derivatives. We get

$$\sum_{i,j} f_{ij}z_{1y_iy_j}\geq 2\theta\sum_{i,j}|u_{y_iy_j}|^2+2N\theta\sum_i|u_{y_i}|^2+2N_0f_{22}+2Nu\sum_{i,j}f_{ij}u_{y_iy_j}+2\sum_{i,j,k}f_{ij}u_{y_k}u_{y_iy_jy_k}$$
$$-2\sigma u\sum_{i,j}f_{ij}u_{y_iy_j\rho}-C'|u|^2-C''|Du|^2-(1/\mu)C'''|Du|^2-\mu C^{IV}\sum_{i,j}|u_{y_iy_j}|^2.$$

The differentiated equation gives

$$\sum_{i,j}f_{ij}u_{y_iy_j\rho}+\sum_i f_{q_i}u_{y_i\rho}+f_\rho+\sum_i g_{p_i}u_{y_i\rho}+g_uu_\rho+g_\rho=0$$

and consequently

$$|\sum_{i,j} f_{ij} u_\rho u_{y_i y_j \rho}| \leq C|Du||D^2u| + \bar{C}|Du|^2$$

as f_ρ has linear growth with respect to the second derivatives. After the polar change of variables the function f may not be convex any longer so after choosing N large enough we now have to use the inequality

$$|2Nu \sum_{i,j} f_{ij} u_{y_i y_j}| \leq (N^2u^2)/\mu' + \mu' |\sum_{i,j} f_{ij} u_{y_i y_j}|^2$$

with small enough μ'.

It is now obvious that suitable choice of constants provides

$$\sum_{i,j} f_{ij} z_{1 y_i y_j} > 0.$$

What remains is to prove that z_1 has no positive maximum at any point belonging to $\{\rho = R\}$.

On S we have $u_\rho = \sigma u$ and consequently

$$-\partial_\rho z_1 + \sigma z_1 = -2u_\rho u_{\rho\rho} - 2u_\varphi u_{\varphi\rho} + 2\sigma_\rho u u_\rho$$

$$+2\sigma u_\rho^2 + 2\sigma u u_{\rho\rho} - 2Nuu_\rho + \sigma u_\rho^2 + \sigma u_\varphi^2 - 2\sigma^2 u u_\rho + \sigma N u^2 - \sigma N_1 - 2N_0\rho + \sigma N_0 \rho^2$$

$$= -2u_\varphi(\sigma u)_\varphi + 2\sigma_\rho u(\sigma u) + 2\sigma(\sigma u)^2 - 2Nu(\sigma u)$$

$$+\sigma(\sigma u)^2 + \sigma u_\varphi^2 - 2\sigma^2 u(\sigma u) + \sigma N u^2 - \sigma N_1 - 2N_0 R + \sigma N_0 R^2$$

$$= -2\sigma_\varphi u u_\varphi - \sigma u_\varphi^2 + 2\sigma\sigma_\rho u^2 - \sigma N u^2 + \sigma^3 u^2 - \sigma N_1 - 2N_0 R + \sigma N_0 R^2$$

$$\geq -|\sigma_\varphi| u^2/\mu'' - |\sigma_\varphi| \mu'' u_\varphi^2 - \sigma u_\varphi^2 + 2\sigma\sigma_\rho u^2 - \sigma N u^2 + \sigma^3 u^2 - \sigma N_1 - 2N_0 R + \sigma N_0 R^2.$$

The constant N_0 has already been determined. We can now choose the remaining constants so that

$$-\partial_\rho z_1 + \sigma z_1 \geq 1 \text{ on } S$$

and consequently z_1 has no positive maximum on S which completes the proof.

To find an estimate for $\max_{\partial\Omega} |D^2u|$ we shall use a modification of a procedure described in [5].

L e m m a 5. *Let $u \in C^4(\bar{\Omega})$ be a solution of (1)-(2). Then*

$$\max_{\partial\Omega} |D^2u| \leq M_2',$$

where $M_2' = M_2'(\theta, \Theta, \eta, \delta, M, G(M), M_1)$, $M_1 = \max_{\bar{\Omega}} |Du|$.

P r o o f. Differentiate the equation with respect to ρ and set $v = u_\rho - \sigma u$. Then $v|S = 0$ and

(10) $$F_{u_{\varphi\varphi}} v_{\varphi\varphi} + 2F_{u_{\varphi\rho}} v_{\varphi\rho} + F_{u_{\rho\rho}} v_{\rho\rho} + G(u_{\varphi\varphi}, u_{\varphi\rho}, u_{\rho\rho}, u_\varphi, u_\rho, u, \varphi, \rho) = 0,$$

where

$$G(u_{\varphi\varphi}, u_{\varphi\rho}, u_{\rho\rho}, u_\varphi, u_\rho, u, \varphi, \rho) = F_{u_\varphi} v_\varphi + F_{u_\rho} v_\rho + F_u v$$

$$+F_{u_{\varphi\varphi}}(\sigma u)_{\varphi\varphi} + 2F_{u_{\varphi\rho}}(\sigma u)_{\varphi\rho} + F_{u_{\rho\rho}}(\sigma u)_{\rho\rho} + F_{u_\varphi}(\sigma u)_\varphi + F_{u_\rho}(\sigma u)_\rho + F_u(\sigma u) + F_\rho.$$

We set $v=\varkappa(w)$. We then have

$$v_\varphi=\varkappa' w_\varphi, \quad v_\rho=\varkappa' w_\rho,$$

$$v_{\varphi\rho}=\varkappa' w_{\varphi\rho}+\varkappa'' w_\varphi w_\rho.$$

The function w satisfies the following equation

$$F_{u_{\varphi\varphi}} w_{\varphi\varphi}+2F_{u_{\varphi\rho}} w_{\varphi\rho}+F_{u_{\rho\rho}} w_{\rho\rho}+(\varkappa''/\varkappa')[F_{u_{\rho\rho}} w_\rho^2+2F_{u_{\varphi\rho}} w_\varphi w_\rho+F_{u_{\varphi\varphi}} w_\varphi^2]+G/\varkappa'=0.$$

Choosing $\varkappa$ so that $\varkappa'>0$, $\varkappa''<0$ we can derive the inequality

$$F_{u_{\varphi\varphi}} w_{\varphi\varphi}+2F_{u_{\varphi\rho}} w_{\varphi\rho}+F_{u_{\rho\rho}} w_{\rho\rho}\geqq-(\varkappa''/\varkappa')\theta\,|Dw|^2-G/\varkappa'.$$

We shall now estimate the growth of G. We know that the derivatives of f (and hence of F) with respect to y_i have linear growth with respect to the second derivatives of u; the derivatives with respect to u_φ, u_ρ, $u_{\varphi\varphi}$, $u_{\varphi\rho}$, $u_{\rho\rho}$, are bounded.
From the definition of v it follows that $|u_{\varphi\rho}|$, $|u_{\rho\rho}|\leqq C\,|Dv|+C$, hence

$$|G|\leqq C\,|Dv|+C\,|u_{\varphi\varphi}|+C.$$

We can write the equation in the equivalent form $\Sigma\, a^{ij} u_{y_i y_j}+\cdots=0$, as a^{ij} are bounded we get $|u_{\varphi\varphi}|\leqq C\,|u_{\varphi\rho}|+C\,|u_{\rho\rho}|+C$, i. e.

$$|G|\leqq C\,|Dv|+C\leqq C\varkappa'^2\,|Dw|^2+C.$$

We then have

$$F_{u_{\varphi\varphi}} w_{\varphi\varphi}+2F_{u_{\varphi\rho}} w_{\varphi\rho}+F_{u_{\rho\rho}} w_{\rho\rho}\geqq-(\varkappa''/\varkappa')\theta\,|Dw|^2-\varkappa' C\,|Dw|^2-C/\varkappa'.$$

If we now in addition choose $\varkappa$ so that

$$-(\varkappa''/\varkappa')\theta-C\varkappa'\geqq 0, \quad \varkappa(0)=0, \text{ e. g. } \varkappa(\cdot)=(\theta/C)\ln(1+\cdot),$$

we get

$$v=(\theta/C)\ln(1+w), \quad w=-1+e^{(c/\theta)v}, \quad w\,|\,S=0,$$

$$F_{u_{\varphi\varphi}} w_{\varphi\varphi}+2F_{u_{\varphi\rho}} w_{\varphi\rho}+F_{u_{\rho\rho}} w_{\rho\rho}\geqq-(C/\varkappa')\geqq-\tilde{C}.$$

Set $S_1=\{\rho=R-\varepsilon\}$, $S=\{\rho=R\}$, $\max\limits_{S_1}|w|=l$ and choose m such that

$$m>l/(e^R-e^{R-\varepsilon}).$$

Consider the function $z=w+me^\rho$. On S_k we have $w+me^{R-\varepsilon}\leqq l+me^{R-\varepsilon}<me^R-me^{R-\varepsilon}+me^{R-\varepsilon}=(w+me^\rho)|_S$. On the other hand,

$$F_{u_{\varphi\varphi}} z_{\varphi\varphi}+2F_{u_{\varphi\rho}} z_{\varphi\rho}+F_{u_{\rho\rho}} z_{\rho\rho}\geqq-C+mF_{\rho\rho}e^\rho>0$$

for m large enough. This means that $\partial_\rho z\geqq 0$ on S, i. e. $\partial_\rho w\,|_S+me^R\geqq 0$, which gives us one-sided estimate for $\partial_\rho w$ and hence for $\partial_\rho v$ on S, i. e. $\partial_\rho v\geqq-\bar{C}$.

To get an estimate of the form $\partial_\rho v\leqq\bar{C}$ we can use the auxiliary function $1-e^{-(C/\theta)v}-me^\rho$ and argue as above. Hence

$$\max_S|\partial v/\partial\rho|\leqq\tilde{M},$$

i. e. we achieved an estimate for $u_{\rho\rho}|_S$. The boundary condition and the equation supply estimates for the remaining second derivatives of $u: u_{\varphi\rho}|_S=(\sigma u)_\varphi|_S$.

Together with Lemma 3 this gives us $\max_{\overline{\Omega}}|D^2u|\leqq M_2$.

Lemma 6. *If* $u\in C^4(\overline{\Omega})$ *is a solution of (1)-(2), then* $|u|_{2,\alpha;\overline{\Omega}}\leqq C$, *where*

$$C=C(\theta,\ \Theta,\ \eta,\ \delta,\ M,\ G(M),\ M_1,\ M_2).$$

Proof. Since we consider the two-dimensional case only all we have to do is apply a result of Nirenberg (Theorem 11.4 and further, [4], p. 247) to the differentiated equation. The interior $C^{1,\alpha}$-estimates for any first derivative of the solution, i. e. the interior $C^{2,\alpha}$-estimates for the solution itself are a direct application of Theorem 11.4 [4] (see [3]). What remains is to establish bounds for the Hölder-coefficients of $u_{\varphi\varphi}$, $u_{\varphi\rho}$ and $u_{\rho\rho}$ near the boundary. To achieve this we consider the solution $v=u_\rho-\sigma u$ of the Dirichlet — problem for (10). The theorem and the following remark, concerning global estimates, yield a bound for $|v|_{1,\alpha}$ near the boundary. The Hölder coefficient of the remaining second derivative of the solution, $u_{\varphi\varphi}$, can be estimated using the implicit function theorem as in [3]. Namely, the uniform ellipticity permits us to express this derivative as a smooth function of the already estimated ones. Thus we finally get

$$|u|_{2,\alpha;\overline{\Omega}}\leqq C.$$

3. Existence of a solution. We can now establish the existence of a solution to our problem using the method of continuity. Consider the problem

$$\left|\begin{array}{l}\theta(1-\lambda)\Delta u+\lambda F[u]=0\\ (\partial_\nu u+\sigma u)|_{\partial\Omega}=0,\end{array}\right.\quad \text{in }\Omega,$$

where $\lambda\in[0,\ 1]$, $F[u]=f(D^2u)+g(x,\ u,\ Du)=F(x,\ u,\ Du,\ D^2u)$. We can now proceed exactly as in [3], substituting the space B used there with the space

$$B_\nu=\{u\in C^{2,\alpha}(\overline{\Omega})/(\partial_\nu u+\sigma u)|_{\partial\Omega}=0\}.$$

We shall go into no further details except remind that the proof of the fact that the set of all λ for which the above problem is solvable is relatively open in [0, 1] uses the implicit function theorem in Banach spaces and the solvability of the third boundary value problem for linear equations.

Uniqueness follows trivially from 4) (i), (ii).

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