Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

ON CERTAIN SUBCLASS OF STARLIKE FUNCTIONS

SHIGEYOSHI OWA

There are many classes of starlike functions in the unit disk U. We consider a class $S_p(\alpha, \beta)$ of univalent and starlike functions in the unit disk U. The purpose of this paper is to show a representation formula, a distortion theorem, a sufficient condition and an argument theorem for the class $S_p(\alpha, \beta)$. Furthermore, we give the radius of convexity for functions in the class $S_p(\alpha, \beta)$.

1. Introduction. Let S denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n,$$

analytic in the unit disk $U=\{|z|<1\}$. Further, let $S(\alpha, \beta)$ denote the class of functions $f(z)\in S$ satisfying the condition

$$(1)\qquad \left|\frac{\dfrac{zf'(z)}{f(z)}-1}{\alpha\dfrac{zf'(z)}{f(z)}+1}\right|<\beta,\quad z\in U$$

for some $\alpha(0\leq\alpha\leq1)$ and $\beta(0<\beta\leq1)$. For this class T. V. Lakshminarasimhan [2] established some results. Moreover, M. L. Mogar [3] and K. S. Padmanabhan [5] studies the class $S(1, \beta)$ of functions $f(z)\in S$ satisfying the condition (1), where $\alpha=1$, $0<\beta\leq1$, and R. Singh [7] studied the class $S(0, 1)$ of functions $f(z)\in S$ satisfying the condition (1) for $\alpha=0$ and $\beta=1$.

In this paper, we consider the function

$$f(z)=z+\sum_{n=1}^{\infty} a_{p+n} z^{p+n},\quad p\in N$$

analytic in the unit disk U and satisfying the condition (1) for some $\alpha(0\leq\alpha\leq1)$ and $\beta(0<\beta\leq1)$. We denote the class of all such functions $f(z)$ by $S_p(\alpha, \beta)$. The class $S_p(1, \beta)$ is precisely the class of functions studied by M. L. Mogra [4].

2. A representation formula. Here, we obtain a representation formula for the functions of the class $S_p(\alpha, \beta)$. In the first place, we require the following lemma.

Lemma 1. *Let the function* $H(z)=1+b_p z^p+b_{p+1} z^{p+1}+\cdots$, $(p\in N)$ *be analytic in the unit disk* U. *Then* $H(z)$ *is analytic and satisfies the condition* $\left|\frac{1-H(z)}{1+\alpha H(z)}\right|<\beta$, *for some* $\alpha(0\leq\alpha\leq1)$ *and* $\beta(0<\beta\leq1)$ *if, and only if, there exists an analytic function* $\Phi(z)$ *in the unit disk* U *such that* $|\Phi(z)|\leq\beta$ *for* $z\in U$ *and*

$$H(z)=\frac{1-z^p\,\Phi(z)}{1+\alpha z^p\,\Phi(z)}.$$

P r o o f. We employ the technique used by K. S. P a d m a n a b h a n [5] Let the function $H(z)=1+b_p z^p+b_{p+1}z^{p+1}+\cdots$, satisfies the condition

$$\left|\frac{1-H(z)}{1+\alpha H(z)}\right|<\beta$$

for some $\alpha\,(0\leqq\alpha\leqq 1)$ and $\beta\,(0<\beta\leqq 1)$. Setting $z^{p-1}h(z)=\frac{1-H(z)}{1+\alpha H(z)}$, we note that the function $h(z)$ is analytic in the unit disk U, satisfies $|h(z)|<\beta$ for $z\in U$ and $h(0)=0$. Consequently, using Schwarz's lemma, we have $h(z)=z\Phi(z)$, where $\Phi(z)$ is analytic in the unit disk U and satisfies $|\Phi(z)|\leqq\beta$ for $z\in U$. Thus we obtain $H(z)=\frac{1-z^{p-1}h(z)}{1+\alpha z^{p-1}h(z)}=\frac{1-z^p\Phi(z)}{1+\alpha z^p\Phi(z)}$. On the other hand, if $H(z)=\frac{1-z^p\,\Phi(z)}{1+\alpha z^p\,\Phi(z)}$ and $|\Phi(z)|\leqq\beta$ for $z\in U$, then the function $H(z)$ is analytic in the unit disk U. Furthermore, since $|z^p\,\Phi(z)|\leqq\beta|z|^p<\beta$ for $z\in U$, we obtain

$$\left|\frac{1-H(z)}{1+\alpha H(z)}\right|=|z^p\,\Phi(z)|<\beta,$$

for $z\in U$. This completes the proof of the lemma.

T h e o r e m 1. *Let $f(z)=z+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ be a function, analytic in the unit disk U. Then the function $f(z)$ is in the class $S_p(\alpha,\beta)$ if, and only if.*

$$f(z)=z\exp\{-(1+\alpha)\int_0^z\frac{t^{p-1}\,\Phi(z)}{1+\alpha\,t^p\,\Phi(z)}\,dt\},\tag{2}$$

where $\Phi(z)$ is an analytic function in the unit disk U and satisfies $|\Phi(z)|\leqq\beta$ for $z\in U$.

P r o o f. Let $f(z)$ be in the class $S_p(\alpha,\beta)$. Then, since $f(z)$ satisfies the condition (1), we can write $\frac{zf'(z)}{f(z)}=\frac{1-z^p\,\Phi(z)}{1+\alpha z^p\,\Phi(z)}$ by using Lemma 1. Hence we have $\frac{f'(z)}{f(z)}-\frac{1}{z}=\frac{-(1+\alpha)\,z^{p-1}\,\Phi(z)}{1+\alpha z^p\,\Phi(z)}$ which at once gives (2) on integration from 0 to z.

Conversely, if $f(z)$ has the representation (2), it follows that $\frac{zf'(z)}{f(z)}=\frac{1-z^p\Phi(z)}{1+\alpha z^p\Phi(z)}$ holds with $\Phi(z)$ as in Lemma 1. Accordingly we obtain that $f(z)\in S_p(\alpha,\beta)$ with the aid of Lemma 1.

3. A distortion theorem.

L e m m a 2. *Let $f(z)=z+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ be in the class $S_p(\alpha,\beta)$. Then we have*

$$\frac{1-\beta|z|^p}{1+\alpha\beta|z|^p}\leqq\mathrm{Re}\,\{\frac{zf'(z)}{f(z)}\}\leqq\frac{1+\beta|z|^p}{1-\alpha\beta|z|^p}$$

for $z\in U$.

P r o o f. Let $f(z)$ be in the class $S_p(\alpha,\beta)$. Then, by Lemma 1, we have $w(z)=\frac{zf'(z)}{f(z)}=\frac{1-Z}{1+\alpha Z}$ with $|Z|<\beta|z|^p$ for $z\in U$, so that

$$\frac{1-\beta|z|^p}{1+\alpha\beta|z|^p}\leqq\mathrm{Re}\,(w)=\mathrm{Re}\,\{\frac{1-Z}{1+\alpha Z}\}\leqq\frac{1+\beta|z|^p}{1-\alpha\beta|z|^p}.$$

This proves Lemma 2.

Theorem 2. *Let $f(z)=z+\Sigma_{n=1}^{\infty} a_{p+n} z^{p+n}$ be analytic in the unit disk U and suppose $f(z)\in S_p(\alpha,\beta)$. Then we have*

$$\frac{|z|}{(1+\alpha\beta|z|^p)^{(1+\alpha)/\alpha p}}\leq|f(z)|\leq\frac{|z|}{(1-\alpha\beta|z|^p)^{(1+\alpha)/\alpha p}}$$

for $0<\alpha\leq 1$, $0<\beta\leq 1$ and $z\in U$. Furthermore, we have

$$|z|\exp\left(-\frac{\beta}{p}|z|^p\right)\leq|f(z)|\leq|z|\exp\left(\frac{\beta}{p}|z|^p\right)$$

for $\alpha=0$, $0<\beta\leq 1$ and $z\in U$.

Proof. Since the function $f(z)$ is in the class $S_p(\alpha,\beta)$, we have $\frac{zf'(z)}{f(z)}=\frac{1-z^p\Phi(z)}{1+\alpha z^p\Phi(z)}$, where $\Phi(z)$ is an analytic function in the unit disk U and $|\Phi(z)|\leq\beta$ for $z\in U$. Hence we obtain

$$\frac{f'(z)}{f(z)}-\frac{1}{z}=\frac{-(1+\alpha)z^{p-1}\Phi(z)}{1+\alpha z^p\Phi(z)}. \tag{3}$$

Integrating both sides of (3) from 0 to z and taking the real parts on both sides of the resulting equation,

$$\log\left|\frac{f(z)}{z}\right|=\operatorname{Re}\{\log\left(\frac{f(z)}{z}\right)\}=\operatorname{Re}\int_0^z\left\{\frac{f'(t)}{f(t)}-\frac{1}{t}\right\}dt=\operatorname{Re}\int_0^z\frac{-(1+\alpha)t^{p-1}\Phi(t)}{1+\alpha t^p\Phi(t)}dt$$

$$\leq\int_0^{|z|}\frac{(1+\alpha)|\Phi(te^{i\theta})|t^{p-1}}{|1+\alpha t^p e^{ip\theta}\Phi(te^{i\theta})|}dt.$$

Hence,

$$\log\left|\frac{f(z)}{z}\right|\leq\int_0^{|z|}\frac{(1+\alpha)\beta t^{p-1}}{1-\alpha\beta t^p}dt=-\log(1-\alpha\beta|z|^p)^{(1+\alpha)/\alpha p}$$

for $0<\alpha\leq 1$ and

$$\log\left|\frac{f(z)}{z}\right|\leq\int_0^{|z|}\beta t^{p-1}dt=\frac{\beta}{p}|z|^p$$

for $\alpha=0$. Consequently we have $|f(z)|\leq\frac{|z|}{(1-\alpha\beta|z|^p)^{(1+\alpha)/\alpha p}}$ for $0<\alpha\leq 1$ and $|f(z)|\leq|z|\exp\left(\frac{\beta}{p}|z|^p\right)$ for $\alpha=0$.

On the other hand, by Lemma 2,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\geq\frac{1-\beta|z|^p}{1+\alpha\beta|z|^p}$$

for $0\leq\alpha\leq 1$, $0<\beta\leq 1$ and $z\in U$. This gives

$$r\operatorname{Re}\left\{\frac{\partial}{\partial r}\left(\log\frac{f(z)}{z}\right)\right\}=\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-1\right\}\geq\frac{1-\beta r^p}{1+\alpha\beta r^p}-1=\frac{-(1+\alpha)\beta r^p}{1+\alpha\beta r^p}$$

for $|z|=r$. Thus we obtain

$$\log\left|\frac{f(z)}{z}\right|=\operatorname{Re}\left\{\log\frac{f(z)}{z}\right\}\geq\int_0^r\frac{-(1+\alpha)\beta t^{p-1}}{1+\alpha\beta t^p}dt.$$

Hence, $\log|\frac{f(z)}{z}|\geqq -\log(1+\alpha\beta r^p)^{(1+\alpha)/\alpha p}$ for $0<\alpha\leqq 1$ and $\log|\frac{f(z)}{z}|\geqq -\frac{\beta}{p}r^p$ for $\alpha=0$. Consequently we have

$$|f(z)|\geqq \frac{|z|}{(1+\alpha\beta|z|^p)^{(1+\alpha)/\alpha p}}$$

for $0<\alpha\leqq 1$ and

$$|f(z)|\geqq |z|\exp(-\frac{\beta}{p}|z|^p)$$

for $\alpha=0$. The equality is attained for $f(z)=\frac{z}{(1-\alpha\beta z^p)^{(1+\alpha)/\alpha p}}$, when $0<\alpha\leqq 1$ and $0<\beta\leqq 1$ and for $f(z)=z\exp(\frac{\beta}{p}z^p)$, when $\alpha=0$ and $0<\beta\leqq 1$.

4. A sufficient condition for the class $S_p(\alpha, \beta)$.

Theorem 3. *Let $f(z)=z+\Sigma_{n=1}^{\infty} a_{p+n}z^{p+n}$ be analytic in the unit disk U. If we have*

$$\sum_{n=1}^{\infty}\{(p+n-1)+\beta(1+\alpha p+\alpha n)\}|a_{p+n}|\leqq\beta(1+\alpha) \tag{4}$$

for some α $(0\leqq\alpha\leqq 1)$ and β $(0<\beta\leqq 1)$, then the function $f(z)$ belongs to the class $S_p(\alpha, \beta)$.

Proof. Assume that the condition (4) holds. Then we obtain

$$|zf'(z)-f(z)|-\beta|\alpha zf'(z)+f(z)|=|\sum_{n=1}^{\infty}(p+n-1)a_{p+n}z^{p+n}|$$

$$-\beta|(1+\alpha)z+\sum_{n=1}^{\infty}(1+\alpha p+\alpha n)a_{p+n}z^{p+n}|\leqq\sum_{n=1}^{\infty}(p+n-1)|a_{p+n}||z|^{p+n}$$

$$-\beta(1+\alpha)|z|+\sum_{n=1}^{\infty}\beta(1+\alpha p+\alpha n)|a_{p+n}||z|^{p+n}$$

$$<[\sum_{n=1}^{\infty}\{(p+n-1)+\beta(1+\alpha p+\alpha n)\}|a_{p+n}|-\beta(1+\alpha)]|z|\leqq 0$$

for $z\in U$. Hence, by the maximum modulus theorem, the function $f(z)$ belongs to the class $S_p(\alpha, \beta)$.

5. An argument theorem.

Theorem 4. *Let $f(z)=z+\Sigma_{n=1}^{\infty}a_{p+n}z^{p+n}$* be in the class $S_p(\alpha, \beta)$. Then we have

$$|\arg\frac{zf'(z)}{f(z)}|\leqq\sin^{-1}(\frac{\beta(1+\alpha)|z|^p}{1+\alpha\beta^2|z|^{2p}}).$$

Proof. We use a method of R. M. Goel and N. S. Sohi [1]. Since $f(z)$ is in the class $S_p(\alpha, \beta)$, by Lemma 1, we obtain

$$\frac{zf'(z)}{f(z)}=\frac{1-z^p\Phi(z)}{1+\alpha z^p\Phi(z)},$$

where $\Phi(z)$ is a function analytic in the unit disk U and $|\Phi(z)|\leqq\beta$ for $z\in U$. After a simple calculation, we have

$$\left|\frac{zf'(z)}{f(z)}-\frac{1+\alpha\beta^2|z|^{2p}}{1-\alpha^2\beta^2|z|^{2p}}\right|\leqq\frac{\beta(1+\alpha)|z|^p}{1-\alpha^2\beta^2|z|^{2p}}.$$

Consequently,

$$\left|\arg\frac{zf'(z)}{f(z)}\right|\leqq\sin^{-1}\left(\frac{\beta(1+\alpha)|z|^p}{1+\alpha\beta^2|z|^{2p}}\right).$$

Corollary I. Let $f(z)=z+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ be a starlike function of order $\gamma\,(0\leqq\gamma<1)$ in the class $S_p(\alpha,\beta)$. Then we have

$$|\arg\{f'(z)\}|\leqq 2(1-\gamma)\sin^{-1}|z|+\sin^{-1}\left(\frac{\beta(1+\alpha)|z|^p}{1+\alpha\beta^2|z|^{2p}}\right).$$

Proof. Since $f(z)$ is a starlike function of order $\gamma\,(0\leqq\gamma<1)$, using a result of B. Pinchuk [6], we obtain

$$\left|\arg\frac{f(z)}{z}\right|\leqq 2(1-\gamma)\sin^{-1}|z|.$$

Hence we have the corollary with the aid of Theorem 4.

6. The radius of convexity for functions of the class $S_p(\alpha,\beta)$.

Theorem 5. *Let $f(z)=z+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ be in the class $S_p(\alpha,\beta)$ with $0<\alpha\leqq 1$ and $0<\beta\leqq 1$. Then $f(z)$ map the disk*

$$|z|<\left(\frac{2+(1+\alpha)p-\sqrt{(1+\alpha)p\{(1+\alpha)p+4\}}}{2\beta}\right)^{1/p}$$

onto a convex domain if $C(\alpha,p)\leqq\beta\leqq 1$, where $C(\alpha,p)$ is the value of β which satisfies the equation

$$2\beta a_1^{p+2}-2\beta a_1^p-\{2+(1+\alpha)p\}a_1^2-2(1+\alpha)a_1+2+(1+\alpha)p=0$$

and

$$a_1=\left(\frac{2+(1+\alpha)p-\sqrt{(1+\alpha)p\{(1+\alpha)p+4\}}}{2\beta}\right)^{1/p}.$$

Proof. Since $f(z)$ is in the class $S_p(\alpha,\beta)$, by Theorem 1, we have

$$\frac{zf'(z)}{f(z)}=\frac{1-z^p\Phi(z)}{1+\alpha z^p\Phi(z)},$$

where $\Phi(z)$ is an analytic function in the unit disk U and satisfies $|\Phi(z)|\leqq\beta$ for $z\in U$. Differentiating logarithmically the both sides of the above equality with respect to z, we obtain

$$1+\frac{zf''(z)}{f'(z)}=\frac{1-z^p\Phi(z)}{1+\alpha z^p\Phi(z)}-\frac{(1+\alpha)(pz^p\Phi(z)+z^{p+1}\Phi'(z))}{(1+\alpha z^p\Phi(z))(1-z^p\Phi(z))}.$$

Moreover, we have

(5) $$\left|\frac{\Phi'(z)}{\beta}\right|\leqq\frac{1-|\Phi(z)/\beta|^2}{1-|z|^2}$$

for the analytic function $\Phi(z)$ in the unit disk U. Since

$$\mathrm{Re}\left\{\frac{1-z^p\Phi(z)}{1+\alpha z^p\Phi(z)}\right\}=\frac{1-\alpha|z^p\Phi(z)|^2-(1-\alpha)\mathrm{Re}(z^p\Phi(z))}{|1+\alpha z^p\Phi(z)|^2}$$

$$\geqq \frac{(1+\alpha\,|\,z^p\,\Phi(z)\,|)\,(1-|\,z^p\,\Phi(z)\,|)}{|\,1+\alpha z^p\,\Phi(z)\,|^2} \geqq \frac{1-|\,z^p\,\Phi(z)\,|}{1+\alpha\,|\,z^p\,\Phi(z)\,|}$$

and

$$\mathrm{Re}\,\{\frac{pz^p\,\Phi(z)+z^{p+1}\,\Phi'(z)}{(1+\alpha z^p\,\Phi(z))\,(1-z^p\,\Phi(z))}\} \leqq \frac{p\,|\,z^p\,\Phi(z)\,|+|\,z^{p+1}\,\Phi'(z)\,|}{(1-\alpha\,|\,z^p\,\Phi(z)\,|)\,(1-|\,z^p\,\Phi(z)\,|)}$$

$$\leqq \frac{p\,|\,z^p\,\Phi(z)\,|+|\,z^{p+1}\,\Phi(z)\,|}{(1-\alpha\,|\,z^p\,\Phi(z)\,|)^2},$$

we get

$$\mathrm{Re}\,\{1+\frac{zf''(z)}{f'(z)}\} \geqq \frac{1-|\,z^p\,\Phi(z)\,|}{1+\alpha\,|\,z^p\,\Phi(z)\,|} - \frac{(1+\alpha)\,(\,p\,|\,z^p\,\Phi(z)\,|+|\,z^{p+1}\,\Phi'(z)\,|)}{(1-\alpha\,|\,z^p\,\Phi(z)\,|\,)^2}.$$

Assume that

(6) $$1+|\,z^p\,\Phi(z)\,|^2-\{2+(1+\alpha)\,p\}\,|\,z^p\,\Phi(z)\,|-(1+\alpha)\,|\,z^{p+1}\,\Phi'(z)\,|>0$$

Then we obtain

$$\mathrm{Re}\,\{1+\frac{zf''(z)}{f'(z)}\}>0.$$

Now, in virtue of (5), the condition (6) will be satisfied

$$1+|\,z^p\,\Phi(z)\,|^2-\{2+(1+\alpha)\,p\}\,|\,z^p\,\Phi(z)\,|-(1+\alpha)\,|\,z\,|^{p+1}\,\frac{\beta-|\,\Phi(z)\,|^2/\beta}{1-|\,z\,|^2}>0.$$

Writing $a=|z|$ and $t=|\,z^p\,\Phi(z)\,|$, the above condition can be re-written as

$$(1-a^2)\,[1+t^2-\{2+(1+\alpha)\,p\}\,t]-(1+\alpha)\,(\beta a^{p+1}-\frac{t^2}{\beta a^{p-1}})>0,$$

that is,

(7) $$t^2\,\{(1-a^2)+\frac{1+\alpha}{\beta a^{p-1}}\}-t\,(1-a^2)\,\{2+(1+\alpha)\,p\}+1-a^2-(1+\alpha)\,\beta a^{p+1}>0,$$

where $0<a<1$ and $0\leqq t\leqq a^p\,\beta$. Let $G(t)$ denote the left hand member of (7). Then we see that

$$G'(z)=2t\,\{(1-a^2)+\frac{1+\alpha}{\beta a^{p-1}}\}-(1-a^2)\,\{2+(1+\alpha)\,p\}=0$$

for $t=t_1=(1-a^2)\,\{2+(1+\alpha)\,p\}/2(1-a^2)+(1+\alpha)/\beta a^{p-1}\}$.
Further

$$G''(z)=2\,\{(1-a^2)+\frac{1+\alpha}{\beta a^{p-1}}\}>0,$$

because $0<a<1$. Now t_1 is positive and negative with $2\beta a^{p+2}-2\beta a^p-\{2+(1+\alpha)\,p\}a^2-2\,(1+\alpha)\,a+2+(1+\alpha)\,p$, respectively. Let

$$E(a)=2\beta a^{p+2}-2\beta a^p-\{2+(1+\alpha)\,p\}\,a^2-2\,(1+\alpha)\,a+2+(1+\alpha)\,p$$

and let a_0 be the positive root of $E(a)=0$ lying in the open interval (0, 1). Then $E(a)$ is positive for $0<a<a_0$ and so $t_1>a^p\beta$. Therefore $G'(t)$ is negative for $0\leqq t\leqq a^p\beta$, $G(a^p\beta)<G(t)$ and the condition (7) is satisfied if $G(a^p\beta)>0$. This is equivalent to

$$(1-a^2)\,[a^{2p}\beta^2-a^p\beta\{2+(1+\alpha)p\}+1]>0$$

which holds for $a<A^{1/p}$, where we have denoted

$$A^{1/p}=\Big(\frac{2+(1+\alpha)p-\sqrt{(1+\alpha)p\{(1+\alpha)p+4\}}}{2\beta}\Big)^{1/p}.$$

Furthermore, we can show that $a_0>A^{1/p}$ if $\frac{2+(1+\alpha)p-\sqrt{(1+\alpha)p\{(1+\alpha)p+4\}}}{2}\leq\beta\leq 1$. The condition for β implies $A<1$, and so $a_1=A^{1/p}<a_0$ if $E(a_1)$ is positive If $E(a_1)=0$, then $a_0=A^{1/p}$. This shows that

$$|z|<\Big(\frac{2+(1+\alpha)p-\sqrt{(1+\alpha)p\{(1+\alpha)p+4\}}}{2\beta}\Big)^{1/p}$$

is mapped onto a convex domain by $f(z)$ provided $C(\alpha,p)\leq\beta\leq 1$, where $C(\alpha,p)$ is the value of β which satisfies the equation $E(a_1)=0$.

Finally, the estimate is sharp. Choose the function $f(z)=\frac{z}{(1+\alpha\beta z^p)^{(1+\alpha)/\alpha p}}$ so that $f(z)\in S_p(\alpha,\beta)$ while $1+\frac{zf''(z)}{f'(z)}=0$, when $z=A^{1/p}$, $0<\alpha\leq 1$ and $0<\beta\leq 1$ so that $f(z)$ is not convex in any disk $|z|<R$ if R exceeds

$$\Big(\frac{2+(1+\alpha)p-\sqrt{(1+\alpha)p\{(1+\alpha)p+4\}}}{2\beta}\Big)^{1/p}.$$

This completes the proof of the theorem.

REFERENCES

1. R. M. Goel, N. S. Sohi. On certain classes of analytic functions. *Indian J. Pure App.. Math,,* **11**, 1980, 1308-1324.
2. T. V. Lakshminarasimhan. On subclasses of functions starlike in the unit disc. *J. Indian Math. Soc.,* **41**, 233-243.
3. M. L. Mogra. On a class of starlike functions in the unit disc I. *J. Indian Math. Soc.,* **40**, 1976, 159-161.
4. M. L. Mogra. On a class of starlike functions in the unit disc II. *Indian J. Pure Appl. Math.,* **8**, 1977, 157-165.
5. K. S. Padmanabhan. On certain classes of starlike functions in the unit disc. *J. Indian Math. Soc.,* **32**, 1968, 89-103.
6. B. Pinchuk. On starlike and convex functions of order α. *Duke Math. J.,* **35**, 1968, 721-734.
7. R. Singh. On a class of starlike functions. *J. Indian Math. Soc.,* **32**, 1968, 207-213.

Kinki University
Higashi-Osaka, Osaka 577
Japan

Received 10. 10. 1984