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ON THE CONDITIONING OF MATRICES BY SCALING

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We give another argument for the wide spread opinion that the numerical properties of matrices are improved by scalings which make the norms of the rows and columns of the matrix approximately equal. Namely, we prove that non-singular matrices whose rows and columns have equal l_2-norms are optimally scaled with respect to a complex condition measure.

The condition of a non-singular matrix A with respect to the linear problem $Ax=b$ is defined as the sensibility of the solution vector x to errors in the right hand side vector b. A widely used measure of this sensibility is the condition number $k(A)=\|A\|.\|A^{-1}\|$, which gives an upper bound for the propagation of the relative error of b in the solution x (cf. [2]). A more refined tool for investigating the condition of a matrix A is its singular spectrum $\sigma_A=(\sigma_1,\ldots,\sigma_n)$, $\sigma_1\geqq\cdots\geqq\sigma_n$, where σ_i are the square roots of the (always positive) eigenvalues of AA^T. σ_A has the following geometrical meaning: if $S_n=\{x\in\mathrm{R}^n \mid \|x\|_2=1\}$, then $\sigma_1,\ldots,\sigma_n$ are the (lengths of the) semiaxes of the ellipsoid $A(S_n)$.

Now let us have to solve $Ax=b$, $b\neq 0$, and let us have solved a perturbed system $Ax^*=b^*$ instead. If we know some upper bound for the relative error in b $\rho(b)=\|b^*-b\|_2/\|b\|_2$, we will be interested in the question how large the relative error in x $\rho(x)=\|x^*-x\|_2/\|x\|_2$ might be.

Let $x=\lambda . x_1$, $e=x^*-x=\mu . e_1$, $x_1, e_1\in S_n$

Obviously

$$\rho(x)=\mu/\lambda,\quad \rho(b)=\|Ae\|_2/\|Ax\|_2=\mu\|Ae_1\|_2/\lambda\|Ax_1\|_2 \tag{1}$$
$$\rho(x)/\rho(b)=\|Ax_1\|_2/\|Ae_1\|_2$$

Thus we see that the probability of having $\rho(x)/\rho(b)\geqq t$, $1\leqq t\leqq k(A)$ depends on the distorsion of the ellipsoid $A(S_n)$, i. e. on the "dispersion" of σ_A. (The worst case of (1) is $\|Ax_1\|_2=\sigma_1$, $\|Ae_1\|_2=\sigma_n$. Then $\rho(x)/\rho(b)=k(A)$).

Now we are going to introduce a measure of the distorsion of $A(S_n)$ and to prove that a scaling strategy is optimal with respect to this measure.

For $x=(x_1,\ldots,x_n)$, $x_i>0$ let us denote:

$$m_2(x)=\left(\frac{1}{2}\Sigma x_i^2\right)^{1|2},\quad m_1(x)=\frac{1}{n}\Sigma x_i,\quad m_0(x)=(\Pi x_i)^{1|n}$$
$$w(x)=m_2(x)/m_0(x)$$

Obviously $w(x)\geqq 1$, and greater values of $w(x)$ correspond to more "dispersed" x_i-s. (In terms of the mean quadratic deviation: $D(x)=\Sigma\frac{1}{n}(x_i-m_1(x))^2=m_2^2(x)-m_1^2(x)$ and thus $D(x)/m_1^2(x)\leqq w^2(x)-1$).

For a non-singular matrix A we define: $w(A)=w(\sigma_A)$, where σ_A denotes the singular spectrum of A.

To compute $w(A)$ one need not know the singular spectrum σ_A because of the following equalities:

$$m_2(\sigma_A)=\left(\frac{1}{n}\Sigma a_{ij}^2\right)^{1/2} \tag{2}$$

$$m_0(\sigma_A)=|\det(A)|^{1/n} \tag{3}$$

(2) and (3) follow from the fact that $\sigma_1^2, \ldots, \sigma_n^2$ are the eigenvalues of AA^T (cf. [4]).

Definition: *A real n by n matrix A is said to be l_2-balanced if the l_2-norms of its rows and columns are equal to 1. A is row-balanced (column-balanced) if its rows (columns) have equal l_2-norms.*

Lemma 1.: *Let A be a non-singular matrix and $D=diag(d_1, \ldots, d_n)$, $d_i \neq 0$. Then*

(i) *if A is row-balanced, then $w(DA)=w(D)w(A)$.*

(ii) *if A is column-balanced, then $w(AD)=w(A)w(D)$.*

Proof: We prove only (i); the proof of (ii) is analogous. Let $r=\| A_{i*}\|_2=(\Sigma a_{ij}^2)^{1/2}$ for $i=1, \ldots, n$ Then by (2):

$$m_2(\sigma_A)=\left(\frac{1}{n}\Sigma \| A_{i*}\|_2^2\right)^{1/2}=r$$

$$m_2(\sigma_{DA})=\left(\frac{1}{n}\Sigma \| d_i . A_{i*}\|_2^2\right)^{1/2}=\left(\frac{1}{n}\Sigma d_i^2 . r^2\right)^{1/2}=r . m_2(\sigma_D)$$

and thus $m_2(\sigma_{DA})=m_2(\sigma_D) . m_2(\sigma_A)$. By (3):

$$m_0(\sigma_{DA})=|\det(DA)|^{1/n}=|\det(D)|^{1/n} . |\det(A)|^{1/n}=m_0(D) . m_0(A) \text{ q. e. d.}$$

Theorem 1.: *Let A be a non-singular matrix, $r_i=\| A_{i*}\|_2$, $c_j=\| A_{*j}\|_2$ and*

$$A^c=A . \mathrm{diag}(c_1^{-1}, \ldots, c_n^{-1}), \quad A^r=\mathrm{diag}(r_1^{-1}, \ldots, r_n^{-1}) . A$$

i. e. A^c and A^r are the column- and row-balanced matrices obtained from A by right and left scaling respectively. Then:

(i) $w(A^c)=w(A)/w(c_1, \ldots, c_n)$

(ii) $w(A^r)=w(A)/w(r_1, \ldots, r_n)$.

Proof: Apply Lemma 1 to

$$A=A^c \mathrm{diag}(c_1, \ldots, c_n) \text{ and } A=\mathrm{diag}(r_1, \ldots, r_n)A^r \ \square$$

Let us consider the sequence obtained from a non-singular matrix A by consecutive left and right scalings:

$$A_0 = A \quad A_{n+1} = \begin{cases} D_{r,n} \cdot A_n & \text{for even } n \\ A_n \cdot D_{c,n} & \text{for odd } n \end{cases} \tag{4}$$

where $D_{r,n}$ (or $D_{c,n}$) is a diagonal matrix chosen to make the l_2-norms of the rows (or columns) of A_{n+1} equal to one.

Theorem 1 implies that $w(A_0) \geqq w(A_1) \geqq \cdots \geqq 1$ and thus $w(A_n)$ converges to a minimal value $w_{\min}(A)$.

Moreover, the matrix sequence $A_0, A_1, \ldots,$ converges to an l_2-balanced matrix A^*. In fact, this claim is equivalent to the assertion that the matrix sequence $A'_0, A'_1, \ldots$ where $(A'_n)_{ij} = (A_n)^2_{ij}$ converges to a doubly stochastic matrix, what follows from the well-known result of Sinkhorn and Knopp [3]. (Since A_n is non-singular, A'_n always possesses a positive diagonal, what is the necessary and sufficient condition for the convergence.). From the theorem of Sinkhorn and Knopp it follows, further, that the necessary and sufficient condition for the existence of two diagonal matrices D_c and D_r such that $A^* = D_r A D_c$ is that A be fully indecomposable.

Now we are going to show that the scaling strategy (4) is the best what can be done to improve $w(A)$.

T h e o r e m 2: *Let A be a non-singular l_2-balanced matrix, and $B = DAE$, where $D = \operatorname{diag}(d_1, \ldots, d_n)$ and $E = \operatorname{diag}(e_1, \ldots, e_n)$ are two non-singular diagonal matrices. Then*

$$w(A) \leqq w(B),$$

whereby the equality is reached if and only if $w(D) = w(E) = 1$, i. e. iff $|d_1| = \cdots = |d_n|$ *and* $|e_1| = \cdots = |e_n|$.

P r o o f: Let $E' = (e'_1, \ldots, e'_n)$, where e'_i is equal to the l_2-norm of the i-th column of DA, and let $E'' = E'E$.

Evidently $B = DA(E')^{-1}E''$ and, by Lemma 1 and Theorem 1:

$$w(B) = w(DA(E')^{-1}) \,.\, w(E'') = w(DA) \,.\, w^{-1}(E') \,.\, w(E'') = w(D) \,.\, w(A) \,.\, w^{-1}(E') \,.\, w(E'')$$

Since $w(E'') \geqq 1$ it suffices to show that

$$w(D) \,.\, w(A) \,.\, w^{-1}(E') \geqq w(A), \text{ i. e. that } w(D) \geqq w(E').$$

$$w(D) = w(|d_1|, \ldots, |d_n|) = m_2(d)/m_0(d), \text{ where } d = (|d_1|, \ldots, |d_n|)$$

$$w(E') = w(e'_1, \ldots, e'_n) = m_2(e')/m_0(e'), \text{ where } e' = (e'_1, \ldots, e'_n)$$

Since A is row-balanced: $m_2(e') = \left(\frac{1}{n} \sum_{i,j} d_i^2 \,.\, a_{ij}^2\right)^{1/2} = \left(\frac{1}{n} \sum_i d_i^2\right)^{1/2} = m_2(d)$ and we must only show that $m_0(e') \geqq m_0(d)$.

The proof is based on the following generalization of the Bernoulli inequality:

$$\prod_{i=1}^{n} x_i^{\mu_i} \leqq \sum_{i=1}^{n} \mu_i x_i \quad \text{for} \quad x_i, \mu_i \geqq 0, \Sigma \mu_i = 1, \tag{5}$$

where the equality is reached iff $x_1 = \cdots = x_n$ (The case $n = 2$ is the well-known Bernoulli inequality, and the general case is easily proved by induction on n.)

Denoting $\delta_i = d_i^2$, $\alpha_{ij} = a_{ij}^2$ we have:

$$m_0(e') = \left(\Pi \left(\left(\sum_i d_i^2 \,.\, a_{ij}^2\right)^{1/2}\right)^{1/n}\right) = \left(\Pi \sum_i \delta_i \, \alpha_{ij}\right)^{1/2n}.$$

Since A is l_2-balanced we can apply (5) to each sum in this product thus obtaining: $m_0(e') \geqq (\Pi_j \Pi_i \delta_i^{\alpha_{ij}})^{1/2n} = (\Pi_i \delta_i^{\Sigma_j \alpha_{ij}})^{1/2n} = (\Pi_i \delta_i)^{1/2n} = m_0(d)$ q. e. d.

This theorem shows that a balanced matrix A can not be "re-balanced" to improve $w(A)$, and that the optimally scaled matrix A^*, obtained by (4) is unique up to scalings by matrices of the form $D = \text{diag}(\pm 1, \ldots, \pm 1)$.

Let us note in conclusion, that an l_2-balanced matrix A is not necessarily optimally scaled with respect to the classical condition number $k(A)$ (cf. [1]). However, trying to improve the condition number of such a matrix by scaling, one should remember that this will increase $w(A)$ and the probability of having large growth of errors in the solution. It should be mentioned also, that the necessary and sufficient conditions given in [1] are quite hard to be practically tested, and that no algorithm is known for optimally scaling a matrix with respect to $k(A)$.

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