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VARIETIES OF PAIRS OF ALGEBRAS WITH A DISTRIBUTIVE LATTICE OF SUBVARIETIES

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This paper deals with varieties of pairs of algebras over a field of characteristic O. One of the algebras is Lie and the other is its associative enveloping algebra. Necessary and sufficient conditions for the distributivity of the lattice of subvarieties are found.

Introduction. All algebras will be over a fixed field K of characteristic O. Let $A=A(X)=K\langle X\rangle=K\langle x_1, x_2, \dots\rangle$ be the free associative algebra with free generators $x_1, x_2, \dots, A_m$ — the subalgebra of rank m generated by $x_1, \dots, x_m$. Let $L=L(X)$ be the Lie subalgebra of A generated by $x_1, x_2, \dots$ with respect to the new multiplication $[u, v]=u(\mathrm{ad}\ v)=uv-vu$. It is known that L is a free Lie algebra. We denote by $\mathrm{Sym}(n)$ and GL_m the symmetric group and the general linear group, acting on the set of symbols $\{1, 2, \dots, n\}$ and on an m-dimensional vector space, respectively.

Let G be a Lie algebra and let R be its associative enveloping algebra. The polynomial $f(x_1, \dots x_n)$ from $K\langle X\rangle$ is a weak identity for the pair (R, G) if $f(g_1, \dots, g_n)=0$ for any $g_1, \dots, g_n \in G$. The set T of all weak identities for (R, G) is a weak T-ideal in $K\langle X\rangle$ definied by (R, G). If I is a weak T-ideal, then $f(u_1, \dots, u_n)$ belongs to I for any $f(x_1, \dots, x_n) \in I$ and $u_1, \dots, u_n \in L(X)$. The class of all pairs (R, G) satisfying a given set of weak identities forms a variety of pairs of algebras.

Many properties of varieties of algebras can be transferred verbatim to varieties of pairs. For example all subvarietes of a given variety of pairs form a lattice with respect to intersection and union. The weak identities are introduced by R a z m y s l o v [5] in his studying of the 2×2 matrix algebra. They can be applied to other Lie and associative algebras as well [3].

Let I be a weak T-ideal in $A(X)$ and let $\mathfrak{M}$ be the variety of pairs corresponding to I. Then $(A/I, L/(L\cap I))$ is called a relatively free pair of $\mathfrak{M}$. We denote A/I by $F(\mathfrak{M})$ and $A_m/(A_m\cap I)$ by $F_m(\mathfrak{M})$. Since the characteristic of K is O, any weak T-ideal in $A(X)$ can be generated by its multilinear polynomials. We denote by $P_n(\mathfrak{M})$ the set of all multilinear polynomials from $F_n(\mathfrak{M})$ of degree n. The space $P_n(\mathfrak{M})$ has a structure of a left $\mathrm{Sym}(n)$-module. The action of the symmetric group is inherited from that on $P_n\subset A_n$ and is defined by the equality $\sigma(x_{i_1}\dots x_{i_n})=x_{\sigma(i_1)}\dots x_{\sigma(i_n)}$, $\sigma \in \mathrm{Sym}(n)$, $x_{i_1}\dots x_{i_n} \in P_n(\mathfrak{M})$. The algebra A_m is isomorphic to the tensor algebra of a vector space of dimension m. Thus $F_m(\mathfrak{M})$ is a left GL_m-module with the action $g(x_{i_1}\dots x_{i_n})=g(x_{i_1})\dots g(x_{i_n})$, $g \in GL_m$, $x_{i_1}\dots x_{i_n} \in F_m(\mathfrak{M})$.

The irreducible $\mathrm{Sym}(n)$- and GL_m-modules are described by Young diagrams. For any partition $\lambda=(\lambda_1, \dots, \lambda_r)$ of n we shall denote $M(\lambda)$ and $N_m(\lambda)$ the $\mathrm{Sym}(n)$- and GL_m-modules corresponding to λ. It is known [4], that the homogeneous component $F_m^{(n)}(\mathfrak{M})$ of $F_m(\mathfrak{M})$ and $P_n(\mathfrak{M})$ have the same module structures. It means that if $P_n(\mathfrak{M})=\Sigma k(\lambda)M(\lambda)$, then $F_m^{(n)}(\mathfrak{M})=\Sigma k(\lambda)N_m(\lambda)$. The necessary information about the represen-

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tations of the symmetric and general linear groups and their application to polynomial identities are given in [2, 1, 4].

The main result. We shall prove the following main theorem in this paper.

Theorem. *Let $\mathfrak{M}$ be a variety of pairs over a field of characteristic O. The lattice of subvarieties of $\mathfrak{M}$ is distributive if and only if the pairs from $\mathfrak{M}$ satisfy the weak identity*

$$(*)\qquad \alpha[x, y]y+\beta y[x, y]=0$$

for suitable α, β *from* K, *such that* $(\alpha, \beta)\neq(0, 0)$.

Remark. This result is an analog of Anan'in and Kemer's result [1] for varieties of associative algebras. But if one compares the description of $P_n(\mathfrak{M})$ in both cases, one can see that the lattice of subvarieties in the case of pairs is more complicated than in the associative case.

It is known that condition for distribution of the lattice of subvarieties is equivalent to the condition for distributivity of the lattice of Sym (n)-submodules in $P_n(\mathfrak{M})$. Therefore our task is to find necessary and sufficient conditions for $P_n(\mathfrak{M})$ to be a sum of non-isomorphic irreducible Sym (n)-submodules for every $n\geqq 1$. The Sym (n)-module P_n of the multilinear polynomials in the free associative algebra is isomorphic to the group algebra $K\,\mathrm{Sym}\,(n)$ and $P_n=\Sigma\,(\dim M(\lambda))\,M(\lambda)$. The least n with $\dim M(\lambda)>1$ for a given λ is $n=3$, when $\dim M(2, 1)=2$. Hence a necessary condition for the distributivity of the lattice is the existence of an identity, which "glues" both isomorphic modules $M(2, 1)$. Such an identity is $(*)$. In order to prove the theorem, it suffices to establish that the identity $(*)$ implies the condition $P_n(\mathfrak{M})\subset\Sigma M(\lambda)$ for any $n\geqq 3$.

Denote by $\mathfrak{M}$ the variety of pairs determined by the weak identity $(*)$. We shall examine four different cases, as is done in [1]:

(i) $\alpha\neq 0$, $\beta\neq 0$, $\alpha-\beta\neq 0$, $\alpha+\beta\neq 0$;
(ii) $\alpha=0$, $\beta\neq 0$ (the case $\alpha\neq 0$, $\beta=0$ is similar);
(iii) $\alpha-\beta=0$, $\alpha\neq 0$;
(iv) $\alpha+\beta=0$, $\alpha\neq 0$.

Proposition 1. *Let* $\alpha\neq 0$, $\beta\neq 0$, $\alpha-\beta\neq 0$, $\alpha+\beta\neq 0$ *in the identity* $(*)$. *Then* $P_n(\mathfrak{M})\subset M(n)+M(n-1, 1)+M(n-2, 1^2)$. (*Actually one can prove that the module* $M(n-2, 1^2)$ *does not enter into* $P_n(\mathfrak{M})$.)

Proof. We shall divide the proof in several steps.

1) We linearize $(*)$ and write it in the form

$$a([x, y, z]+[x, z, y])=[x, yz+zy],\quad a=(\beta-a)(\alpha+\beta)^{-1}.$$

Then we substitute for x the Lie element $[x, t]$ and using the obvious identity $[xt, y, z]=x[t, y, z]+[x, z][t, y]+[x, y][t, z]+[x, y, z]t$ we get $[[x, y], [z, t]]+[[x, t], [z, y]]=0$.
For $z=t$ we obtain that $[[x, y], [z, x]]=0$ and, hence

$$\Sigma(-1)^{\sigma}[[x_{\sigma(1)}, x_{\sigma(2)}], [x_{\sigma(3)}, x_1]]=0,\quad \sigma\in\mathrm{Sym}\,(3).$$

2) Rewriting the identity $(*)$ in the form $[x, y]y=by[x, y]$, $b=-\beta/\alpha$ and multiplying by x from the right-hand side, we obtain $[x, y]yx=by[x, y]x=b^2yx[x, y]$. Permuting x and y gives $[y, x]xy=b^2xy[y, x]$. Adding the last two identities $(b^2-1)[x, y]^2=0$, and keeping in mind that $\alpha\pm\beta\neq 0$ we establish $[x, y]^2=0$.

3) We substitute $[z, t]$ for z in the linearization of $(*)$: $[y, x][z, t]+y[z, t]x-[z, t]xy=b(xy[z, t]-x[z, t]y+[z, t][y, x])$. Now the summation over all permutations of y, x, z, t with an alternating change of signs gives $2(1-b)S_4=0$. Therefore the standard polynomial $S_4(x, y, z, t)$ belongs to the weak T-ideal of the variety $\mathfrak{M}$.

4) The linear space Γ_n of proper (or commutator) multilinear polynomials in P_n is a Sym(n)-submodule of P_n. It is well known that $\Gamma_4=M(3,1)+M(2^2)+M(2,1^2)+M(1^4)$. The submodule generated by $[x_1, x_2][x_3, x_4]$ is a sum of $M(2^2)$, $M(2,1^2)$ and $M(1^4)$. On the other side, $M(2^2)$, $M(2,1^2)$ and $M(1^4)$ are generated by the linearizations of the elements $[x,y]^2$, $\Sigma(-1)^\sigma[[x_{\sigma(1)}, x_{\sigma(2)}], [x_{\sigma(3)}, x_1]]$, $S_4(x,y,z,t)$, respectively. We have seen that these elements are from the weak T-ideal of the variety. Hence we obtain the new identity $[x,y][z,t]=0$.

5) We substitute z with $[z,t]$ in $(*)$ and use the obvious identity $[xt,y,z]=x[t,y,z]+[x,y,z]t+[x,z][t,y]+[x,y][t,z]$. Some calculations give $[x,y[z,t]+[z,t]y]=a([[x,y],[z,t]]+[x,[z,t],y])=a([x,y,z]t+[x,z,y]t)+a(z[x,y,t]+z[x,t,y])+a([z,y][x,t]+[x,z][t,y]-[t,y][x,z]-[x,t][z,y])-a(t[x,y,z]+t[x,z,y])-a([x,y,t]z+[x,t,y]z)=[x,yz+zy]t+z[x,yt+ty]-t[x,yz+zy]-[x,yt+ty]z$. Hence $[x,yz+zy]t+z[x,yt+ty]=[x,y(zt)+(zt)y]-2z[x,y]t=[x,y(zt)+(zt)y]-2z[x,y]t-[x,y(tz)+(tz)y]+2t[x,y]z=[x,y[z,t]+[z,t]y]-2(z[x,y]t-t[x,y]z)$ and we establish the identity $z[x,y]t=t[x,y]z$.

6) Using similar transformations we obtain $x[y,t]z=bxz[y,t]$ from the linearization of the identity $z[y,x]x=bzx[y,x]$. Analogously, the identity $[y,t]zx=bz[y,t]x$ follows from the linearization of $[y,x]xz=bx[y,x]z$.

7) We rewrite this identity in the form $t[x,y,z]=(b-1)tz[x,y]$.

8) The free associative algebra A_m is a universal enveloping algebra of the free Lie algebra L_m. Let $u_1<u_2<\cdots$ be an ordered basis of L_m consisting of commutators $u_i=[x_{i_1},\ldots,x_{i_k}]$, such that $\deg u_1\leq\deg u_2\leq\cdots$. By the Poincaré-Birkhoff-Witt theorem, the products $u_{i_1}\ldots u_{i_r}$, $i_1\leq\cdots\leq i_r$ form a basis of A_m. By the universal property of the free associative algebra, there is a homomorphism $A_m\to F_m(\mathfrak{M})$ which extends the map $x_i\to x_i$, $i=1,\ldots,m$. Therefore $F_m(\mathfrak{M})$ is spanned by the polynomials $x_{i_1}\cdots x_{i_{n-s}}[x_{k_1},\ldots]\cdots[\ldots,x_{k_s}]$, $i_1\leq\cdots\leq i_{n-s}$, $s=0,2,3,\ldots$.

9) We substitute $[x,u]$ for x in the identity $[x,y][z,t]=0$. It follows that $[x,u]y[z,t]=0$. Applying induction we obtain the identity $[x_1,x_2]x_3,\ldots,x_{k-2}[x_{k-1},x_k]=0$. Hence we can exchange the variables placed on the left of the commutator. We conclude from this fact and the identities $[x,y][z,t]=0$, $[y,t]zx=bz[y,t]x=b^2xz[y,t]$, that $F_m(\mathfrak{M})$ is spanned by the elements $x_{i_1}\ldots x_{i_k}$, $x_{i_1}\ldots x_{i_k}[x_{j_1},x_{j_2}]$, $i_1\leq\cdots\leq i_k$.

10) The commutators $[x_i,x_j]$ generate in A_m an irreducible GL_m-submodule isomorphic to $N_m(1^2)$. Now we use an idea from [6]. We consider the map φ from $K[x_1,\ldots,x_m]\otimes N_m(1^2)$ into $F_m(\mathfrak{M})$ defined by $x_{i_1}\ldots x_{i_{n-2}}\otimes[x_{i_{n-1}},x_{i_n}]\to x_{i_1}\ldots x_{i_{n-2}}[x_{i_{n-1}},x_{i_n}]$. In virtue of the identity $[x_1,x_2]x_3\ldots x_{k-2}[x_{k-1},x_k]=0$, this map is a homomorphism of GL_m-modules. Moreover, $F_m(\mathfrak{M})=Im\,\varphi+K[x_1,\ldots,x_m]$ and $K[x_1,\ldots,x_m]=\Sigma N_m(n)$. Using the Littlewood-Richardson rule for the tensor product of GL_m-modules we obtain that $K[x_1,\ldots,x_m]\otimes N_m(1^2)=\Sigma N_m(n-1,1)+\Sigma N_m(n-2,1^2)$. Therefore the irreducible submodules of $P_n(\mathfrak{M})$ are among $M(n)$, $M(n-1,1)$ and $M(n-2,1^2)$.

Proposition 2. *Let $\beta=0$ in the identity $(*)$. Then*

$$P_n(\mathfrak{M})\subset M(n)+M(n-1,1)+M(n-2,1^2).$$

Proof. First, one can see that the proof of the identity $[x,y][z,t]=0$ holds for $\beta=0$ as well. Now the identity $(*)$ is of the form $[x,y]y=0$. We substitute y with $[z,t]+y$ and obtain $x[z,t]y-[z,t]xy=0$. Then we deduce, as in Step 6 of Proposition 1, that $x[y,z]t=0$. Obviously, it follows that $[x,y]zt=0$. As in Steps 8 and 9 of Proposition 1 we obtain that $F_m(\mathfrak{M})$ is spanned modulo the last two identities by $x_{i_1}\ldots x_{i_k}$, $x_{i_1}\ldots x_{i_k}[x_{j_1},x_{j_2}]$, $i_1\leq\cdots\leq i_k$. Now, the proposition follows as in case (i).

We shall consider the third case. Let $\Xi=K[\xi_{ij}^{(k)}]$ be the polynomial algebra in a countable set of indeterminates $\xi_{ij}^{(k)}$, $1\leq i,j\leq n$, $\kappa=1,2,\ldots$. The matrices $\xi^{(k)}=(\xi_{ij}^{(k)})$

$\in M_n(\Xi)$, $k=1, 2, \ldots$, generate the algebra $K\langle\xi\rangle$ of $n\times n$ generic matrices [7]. This algebra is isomorphic to the relatively free algebra of the variety generated by $M_n(K)$.

Similarly, for $\Theta=K[\theta_{ij}^{(k)} \mid 1\leq i, j\leq n, k=1, 2, \ldots, \Sigma_{i=1}^{n} \theta_{ii}^{(k)}=0]$ we introduce traceless generic matrices $\theta^{(k)}=(\theta_{ij}^{(k)})$. Repeating the arguments for $K\langle\xi\rangle\cong F$ (var $M_n(K)$) we establish that the subalgebra of $M_n(\Xi)$ generated by $\theta^{(k)}$, $k=1, 2, \ldots$, is isomorphic to the relatively free algebra $F(\mathrm{var}(M_n(K), sl_n(K)))$.

P r o p o s i t i o n 3. *Let $\alpha-\beta=0$ in the identity* (*). *Then*

$$P_n(\mathfrak{M})=\Sigma M(\lambda),$$

where the summation ranges over all $\lambda=(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1+\lambda_2+\lambda_3=n$.

P r o o f. It is known from the paper of R a z m y s l o v [5], that the weak identity $[xy+yx, z]=0$ generates the weak T-ideal of the pair $(M_2(K), sl_2(K))$. Clearly, this identity and $[x, y]y+y[x, y]=[x, y^2]=0$ are equivalent.

Let $g(x_1, \ldots, x_n)$ be from P_n and generate an irreducible Sym(n)-module corresponding to the partition $\lambda=(\lambda_1, \ldots, \lambda_r)$. As it is known [4], the polynomial $g(x_1, \ldots, x_n)$ is equivalent to a polynomial $f(x_1, \ldots, x_r)=\Sigma a_\tau f_\tau(x_1, \ldots, x_r)$, $a_\tau\in K$, $\tau\in$ Sym(n), which generates in a standard way an irreducible GL_m-module for $m\geq r$. The variables of $f_\tau(x_1, \ldots, x_r)$ are distributed in groups of $m_1, \ldots, m_s$, where m_j is the length of the j-th column of the Young diagram $[\lambda]$ related to the partition λ. Any group consists of $x_1, \ldots, x_{m_j}$ and $f_\tau(x_1, \ldots, x_r)$ is a skew-symmetric sum over any such group.

Since $[xy+yx, z]$ generates the weak T-ideal of the pair $(M_2(K), sl_2(K))$, a multilinear polynomial $g(x_1, \ldots, x_n)$ belongs to $T(\mathfrak{M})$ if and only if $g(x_1, \ldots, x_n)$ vanishes upon substitutions with arbitrary elements from $sl_2(K)$ for the indeterminates x_i. Thus, it is enough to substitute the elements e_{12}, e_{21}, $e_{11}-e_{22}$, which form a basis of $sl_2(K)$.

We use arguments from [4]. Assume that $r\geq 4$. Since $\dim sl_2(K)=3$, and $f_\tau(x_1, \ldots, x_r)$ is skew-symmetric in the group of variables $x_1, \ldots, x_r$, we obtain that $f_\tau(x_1 \ldots, x_r)=0$ in $F(\mathfrak{M})$. This means that the Young diagrams $[\lambda]$ of the irreducible submodules $M(\lambda)$ of $P_n(\mathfrak{M})$ contain no more than three rows.

Without loss of generality, we shall consider identities in three variables. A polynomial $f(x_1, x_2, x_3)\in A_3$ belongs to the weak T-ideal of the pair $(M_2(K), sl_2(K))$ if and only if $f(\xi, \eta, \zeta)=0$, where

$$\xi=\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & -\xi_{11} \end{pmatrix}, \quad \eta=\begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & -\eta_{11} \end{pmatrix}, \quad \zeta=\begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & -\zeta_{11} \end{pmatrix}$$

are 2×2 generic matrices with zero traces. Applying a suitable diagonalization, as in [4], we can assume that

$$\xi=a(e_{11}-e_{22}), \quad \eta=b_1(e_{11}-e_{22})+b(e_{12}+e_{21}), \quad \zeta=c_1(e_{11}-e_{22})+c_2(e_{12}+e_{21})+c(e_{12}-e_{21}),$$

where a, b_1, b, c_1, c_2, c are algebraically independent indeterminates. Let $\lambda=(\lambda_1, \lambda_2, \lambda_3)$ and let $f(x_1, x_2, x_3)=\Sigma a_\tau f_\tau(x_1, x_2, x_3)$ generate $N_3(\lambda)$. The variables in the monomials of $f_\tau(x_1, x_2, x_3)$ are grouped in λ_3 triples, $\lambda_2-\lambda_3$ pairs and $\lambda_1-\lambda_2$ single variables x_1, such that x_1, x_2, x_3 are skew-symmetric in the triples, x_1, x_2 are skew-symmetric in the pairs. Therefore

$$f_\tau(\xi, \eta, \zeta)=f_\tau(a(e_{11}-e_{22}), b(e_{12}+e_{21}), c(e_{12}-e_{21}))=a^{\lambda_1}b^{\lambda_2}c^{\lambda_3}\varepsilon_\tau(e_{11}-e_{22})^{\delta_1}(e_{12}+e_{21})^{\delta_2}(e_{12}-e_{21})^{\delta_3},$$

where $\varepsilon_\tau\in K$, $\delta_i=0, 1$, $\delta_i\equiv\lambda_i \pmod 2$, [4]. Thus we obtain that $\varepsilon_{\tau_2}f_{\tau_1}-\varepsilon_{\tau_1}f_{\tau_2}=0$ is a weak identity for any $\tau_1, \tau_2\in$ Sym(n). This means that every two isomorphic irreducible modules glue together, i. e. the multiplicity of the irreducible modules in $P_n(\mathfrak{M})$ is not more than 1.

It is not hard to see that the multiplicities equal 1. The polynomial $S_3^p(x_1, x_2, x_3)$ $[x_1, x_2]^q x_1^r$, $3p+2q+r=n$, generates an irreducible GL_3-module corresponding to the partition $(p+q+r, p+q, p)$. If we substitute for x_1, x_2, x_3 the matrices

$$a=\frac{1}{2}(-e_{11}+e_{22})\sqrt{-1},\ b=\frac{1}{2}(e_{12}+e_{21})\sqrt{-1},\quad c=\frac{1}{2}(e_{12}-e_{21}),$$

we obtain $S_3(a, b, c)=a^2+b^2+c^2=-3e/4\neq 0$. Hence this polynomial is non-zero in $F(\mathfrak{M})$ and $P_n(\mathfrak{M})=\Sigma M(\lambda_1, \lambda_2, \lambda_3)$.

Now, let $\alpha+\beta=0$. In this case (*) has the form $[x, y, y]=0$. This identity is equivalent to $[x, y, z]=0$.

Proposition 4. *Let* $\mathfrak{M}$ *be the variety of pairs defined by the weak identity* $[x, y, z]=0$. *Then any irreducible* $Sym(n)$*-module* $M(\lambda)$ *has multiplicity 1 in* $P_n(\mathfrak{M})$, $n\geqq 1$.

Proof. The proof of this proposition is obtained by Volichenko [3]. We shall give an independent proof. The free associative algebra A_m is multigraded in a natural way, according to the degree in each variable $x_1, \dots, x_m$. Then $A_m\cap T(\mathfrak{M})$ is a graded subspace and this allows us to define the Hilbert series of $F_m(\mathfrak{M})=A_m/(A_m\cap T(\mathfrak{M}))$ $H(F_m(\mathfrak{M}), t_1, \dots, t_m)=\Sigma(\dim F_m(\mathfrak{M})^{(\lambda)})t_1^{\lambda_1}\dots t_m^{\lambda_m}$. The algebra $F_m(\mathfrak{M})$ is isomorphic to the universal enveloping algebra of $F_m(\mathfrak{N}_2)$, which is the relatively free algebra of rank m of the variety of Lie algebras determined by the identity $[x, y, z]=0$. By the Poincaré-Birkhoff-Witt theorem, $F_m(\mathfrak{M})$ has a basis $x_1^{a_1}x_2^{a_2}\dots x_m^{a_m}\Pi[x_i, x_j]^{b_{ij}}$, a_i, $b_{ij}\geqq 0$, $1\leqq i<j\leqq m$. We compute the Hilbert series $H(F_m(\mathfrak{M}), t_1, \dots, t_m)$, $=\Pi_{i=1}^m(\Sigma t_i^a)\Pi_{i<j}(\Sigma(t_it_j)^b)=\Pi_{i=1}^m(1-t_i)^{-1}\Pi_{i<j}(1-t_it_j)^{-1}$. The last product equals $\Sigma s_\lambda(t_1, \dots, t_m)$, where $S_\lambda(t_1, \dots, t_m)$ are the Schur functions [8]. Using that the following expression is unique $H(F_m(\mathfrak{M}), t_1, \dots, t_m)=\Sigma\mathfrak{X}_\lambda S_\lambda(t_1, \dots, t_m)$, where the summation runs over all partitions $\lambda=(\lambda_1, \dots, \lambda_m)$, and the coefficients $\mathfrak{X}_\lambda$ are equal exactly to the multiplicities of the irreducible $Sym(n)$-submodules of $P_n(\mathfrak{M})$, we obtain the proof of the proposition.

Propositions 1—4 give the proof of the main theorem.

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