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HOW SMOOTH IS THE SHADOW OF A SMOOTH CONVEX BODY?

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Kiom glata estas la ombro de konveksa korpo glata? Estu A barita korpo glata kaj konveksa en $\mathbf{R}^3$, kaj π lineara surĵeto de $\mathbf{R}^3$ sur $\mathbf{R}^2$. La artikolo pritraktas la glatecon de la bildo (ombro) $\pi(A)$. Se la rando de A estas dufoje derivebla kun Lipschitz-kontinuaj duaj derivaĵoj, tiam tiu de $\pi(A)$ estas dufoje derivebla — sed ne nepre dufoje kontinue derivebla, eĉ se A estas nefinie glata. Tamen, se la rando de A estas analitika, tiam $\pi(A)$ havas dufoje kontinue deriveblan randon.

1. Introduction. What degree of smoothness does a two-dimensional projection of a three-dimensional smooth convex compact set possess? It turns out that even if the set has a C^∞ boundary, its image need not be of class C^2. However, the curvature of the boundary always exists. In other words, the boundary of the shadow is described by a twice differentiable function whose second derivative may be discontinuous.

More precisely, we prove that if A is a bounded convex subset of $\mathbf{R}^3$ whose boundary is of class C^2 with Lipschitz continuous second derivatives, then its image under the map $(x, y, z) \mapsto (x, y)$ has a twice differentiable boundary. If the boundary of A is real analytic, then the boundary of the shadow is of Hölder class $C^{2+\varepsilon}$ for some $\varepsilon>0$, i. e., the second derivative of the function describing the boundary is Hölder continuous. These results are proved in Section 2, and in Section 3 we show that they cannot be much improved. In the real-analytic case the boundary of the projection may be exactly of class $C^{2+2/q}$ for any odd integer $q \geqq 3$; see Example 3.2. That the boundary of the shadow in the C^∞ case need not be of class C^2 is finally shown in Example 3.3 and Theorem 3.4.

The origin of the present paper lies in the following question, posed to me by Christer Borell: If A and B are smooth convex sets, does their vector sum $A+B$ have a smooth boundary?

Now, if A and B are smooth subsets of $\mathbf{R}^2$, then $A+B$ may be locally described as the projection of a smooth set in $\mathbf{R}^3$, so we are led to studying the shadow in $\mathbf{R}^2$ of a body in $\mathbf{R}^3$. One of the results is that if A and B are convex and have real-analytic boundaries, then the boundary of $A+B$ is of Hölder class $C^{20/3}$ but no better in general. This was the initial result in this investigation, but it seems that the methods needed to study the regularity of $A+B$ are by necessity different from those used to study the more general shadow problem, and no clarity will be gained in combining the two. For this reason we do not pay special attention to the degree of smoothness of $A+B$ here: although the positive results apply, they are far from being best possible in this case.

For projections $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ many problems seem to remain open.

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2. Smoothness results for the projection of a smooth convex body in $\mathbf{R}^3$

Theorem 2.1. *Let A be a convex compact subset of $\mathbf{R}^3$ and π a linear surjection of $\mathbf{R}^3$ onto $\mathbf{R}^2$. If A has a C^1 boundary, then so has $\pi(A)$. If A has a boundary* *of class C^2 with Lipschitz continuous second derivatives, then the boundary of $\pi(A)$ is twice differentiable. If A has a real-analytic boundary, then the boundary of $\pi(A)$ is of class $C^{2+\varepsilon}$ for some $\varepsilon>0$.*

To prove these results, we shall reformulate them in terms of functions. Let f be a real-valued function of two real variables and $h(x)=\inf_y f(x,y)$ its infimum for a fixed x. We take f convex and such that $f(x,y)\to+\infty$ as $|y|\to+\infty$. Then the epigraph of f, defined by $f(x,y)\leq z$, projects under π: $(x,y,z)\mapsto(x,z)$ to the epigraph of h, defined by $h(x)\leq z$. The points (x,y) where the infimum is attained form a set T which we call the **terminator**; when f is of class C^1 we thus have $T=\{(x,y);\, f_y(x,y)=0\}$. When f is of class C^2 we can also define the **strict terminator** S:

$$S=\{(x,y)\in T;\quad f_{xx}(x,y)=\inf_{t\in T_x} f_{xx}(x,t)\},$$

where T_x denotes the set of all y such that $(x,y)\in T$; we define S_x similarly. The sets S_x and T_x are non-empty compact intervals.

Theorem 2.2. *Let f be a convex function of two real variables such that $f(x,y)\to+\infty$ as $|y|\to+\infty$, and define $h(x)=\inf_y f(x,y)$, $x\in\mathbf{R}$. If f is of class C^1, then so is h, and for any $(x,y)\in T$ we have $h'(x)=f_x(x,y)$. If the second derivatives of f exist and are Lipschitz continuous, then h is twice differentiable; for a point $(x,y)\in T$ such that $f_{yy}(x,y)>0$ we have (even if f is only of class C^2),*

$$h''(x)=f_{xx}(x,y)-\frac{f_{xy}(x,y)^2}{f_{yy}(x,y)}; \tag{2.1}$$

for an x such that $f_{yy}(x,y)=0$ for some (equivalently all) $y\in T_x$ we have

$$h''(x)=\inf_{t\in T_x} f_{xx}(x,t)=f_{xx}(x,s),\quad s\in S_x. \tag{2.2}$$

Finally if f is real analytic near T, then h is locally of class $C^{2+\varepsilon}$ for some $\varepsilon>0$.

The hypothesis that $f(x,y)\to+\infty$ with $|y|$ is essential, as witnessed by the real-analytic convex function $f(x,y)=(x^2+e^{2y})^{1/2}$ whose infimum, $h(x)=|x|$, is not even differentiable.

In the proof of Theorem 2.2 we shall need to know that the accumulation points of the terminator belong to the strict terminator.

Lemma 2.3. *Let $f\in C^2(\mathbf{R}^2)$ be convex and satisfy $f(x,y)\to+\infty$ as $|y|\to+\infty$. Let $(x_j,y_j)\in T$ be points converging to (a,b) and such that $x_j\neq a$ for all j. Then $(a,b)\in S$.*

Proof. Since T is closed we have $(a,b)\in T$. If $f_{yy}(a,b)>0$, then T_a is just the point $\{b\}$, so S_a must also be this point. To prove that $b\in S_a$ in general, we consider the Taylor expansion in the x direction at a point $(a,c)\in T$:

$$f(x,c)=f(a,c)+f_x(a,c)(x-a)+f_{xx}(a,c)(x-a)^2/2+o((x-a)^2),\quad x\to a,$$

and we shall use it for several choices of $c\in T_a$. (The full Taylor expansion is less useful due to the presence of a remainder term $o((x-a)^2+(y-c)^2)$.) First of all we note that $f(a,c)=h(a)$ and $f_x(a,c)=h'(a)$ are constant when $c\in T_a$, so we may assume that they are zero simply by subtracting the affine function $h(a)+h'(a)(x-a)$ from f, thus

$$f(x,c)=f_{xx}(a,c)(x-a)^2/2+o((x-a)^2),\quad x\to a, \tag{2.3}$$

provided $c\in T_a$. Now assume that $b\notin S_a$ and let s be any point in S_a. Thus $f_{xx}(a,s)<f_{xx}(a,c)$ for all $c\in T_a\setminus S_a$. On comparing

$$f(x,b)=f_{xx}(a,b)(x-a)^2/2+o(x-a)^2),\quad \text{and}$$

$$h(x)\leqq f(x, s)=f_{xx}(a, s)(x-a)^2/2+o((x-a)^2),$$

we see that we must have $f(x, b)>f(x, s)\geqq h(x)$ for all x close to a but different from a, in particular

$$f(x_j, b)>f(x_j, s)\geqq h(x_j)=f(x_j, y_j)$$

for j large. Since $y\mapsto f(x_j, y)$ is convex, this is only possible if y_j is between b and s for large j, and since T_a is an interval containing both b and s, we must also have $y_j\in T_a$ for these j. We may therefore take $c=y_j$ in (2.3):

$$f(x, y_j)=f_{xx}(a, y_j)(x-a)^2/2+o((x-a)^2),$$

to be compared with

$$f(x_j, y_j)=h(x_j)\leqq f(x_j, s)=f_{xx}(a, s)(x_j-a)^2/2+o((x_j-a)^2).$$

Since $f_{xx}(a, y_j)\to f_{xx}(a, b)>f_{xx}(a, s)$ we do get a contradiction which proves the lemma.

The proof of Theorem 2.2 will depend on an estimate of the mixed second derivative f_{xy}.

Lemma 2.4. *Let f be a convex function of two real variables, defined and of class C^2 in the rectangle spanned by four points (a, b), (A, b), (A, B) and (a, B). Assume that $f_y(a, b)=f_y(A, B)$. Then*

$$\Big(\int_b^B |f_{xy}(A, t)|dt\Big)^2\leqq\int_b^B f_{xx}(A, t)dt\,(f_y(A, B)-f_y(A, b))=-\int_b^B f_{xx}(A, t)\,dt\int_a^A f_{xy}(s, b)\,ds,$$

so that $|\int_b^B |f_{xy}(A, t)|\,dt|>\leqq|A-a|^{1/2}|B-b|^{1/2}(\sup_V f_{xx})^{1/2}(\sup_H|f_{xy}|)^{1/2}$, *where V denotes the side of the rectangle containing (A, b) and (A, B), H that containing (a, b) and (A, b).*

Proof. The convexity implies that $f_{xy}^2\leqq f_{xx}f_{yy}$ everywhere, so Hölder's inequality gives

$$\Big(\int_b^B |f_{xy}(A, t)\,dt\Big)^2\leqq\int_b^B f_{xx}(A, t)\,dt\int_b^B f_{yy}(A, t)dt=\int_b^B f_{xx}(A, t)\,dt\,[f_y(A, B)-f_y(A, b)]$$

$$=\int_b^B f_{xx}(A, t)\,dt\,[f_y(a, b)-f_y(A, b)]=\int_b^B f_{xx}(A, t)dt\int_A^a f_{yx}(s, b)\,ds,$$

where we have used the special assumption $f_y(A, B)=f_y(a, b)$.

Proof of Theorem 2.2. If h is not of class C^1, it admits a minorant of the form $h(a)+c(x-a)+\varepsilon|x-a|$ for some $a\in\mathbb{R}$, $c\in\mathbb{R}$ and $\varepsilon>0$. Then f has the same minorant, which implies that it cannot be differentiable at a point (a, b) such that $f(a, b)=h(a)$. Second, if $f\in C^2$ and $f_{yy}(a, b)>0$ at a point $(a, b)\in T$, the result is also elementary, for then the solution to $f_y(x, y)=0$ is unique when x is close to a, and the implicit theorem tells us that $y=\varphi(x)$ for a C^1 function φ. We have $h(x)=f(x, \varphi))$ and $h'(x)=f_x(x, \varphi(x))$ from which (2.1) follows by differentiation:

$$h''(x)=f_{xx}(x, \varphi(x))+f_{xy}(x, \varphi(x))\,\varphi'(x),$$

φ' being determined from the defining equation $f_y(x, \varphi(x))=0$: $f_{yx}(x, \varphi(x))+f_{yy}(x, \varphi(x))\,\varphi'(x)=0$.

Consider now a point $(a, b)\in T$ such that $f_{yy}(a, b)=0$. We have $h'(x)=f_x(x, y)$ for all $(x, y)\in T$, so that, for all (x, y), $(a, b)\in T$ with $x\neq a$,

$$\frac{h'(x)-h'(a)}{x-a}=\frac{f_x(x, y)-f_x(a, b)}{x-a}=\frac{f_x(x, y)-f_x(x, b)}{x-a}+\frac{f_x(x, b)-f_x(a, b)}{x-a}.$$

As $x\to a$, the second term here tends to $f_{xx}(a,b)$ by definition. We shall prove that the first tends to zero under the additional assumption that $(a,b)\in S$. By hypothesis $f_y(a,b)=0=f_y(x,y)$, so Lemma 2.4 yields the estimate

$$\left|\frac{f_x(x,y)-f_x(x,b)}{x-a}\right|=\frac{1}{|x-a|}\left|\int_b^y f_{xy}(x,t)\,dt\right|\leq|x-a|^{-1/2}|y-b|^{1/2}(\sup_V f_{xx})^{1/2}(\sup_H |f_{xy}|)^{1/2},$$

where H is the segment $[a,x]\times\{b\}$ or $[x,a]\times\{b\}$. By our assumption in this degenerate case $f_{yy}(a,b)$ vanishes, and this implies that also $f_{xy}(a,b)=0$, for $f_{xy}^2\leq f_{xx}f_{yy}$ everywhere. Moreover, f_{xy} is Lipschitz continuous, so that $|f_{xy}(s,b)|\leq M|s-a|$ for s near a and a suitable constant M, and $\sup_H|f_{xy}|^{1/2}\leq M^{1/2}|x-a|^{1/2}$. We take M so large that it majorizes also f_{xx} in a neighborhood of (a,b) and conclude that

$$\left|\frac{f_x(x,y)-f_x(x,b)}{x-a}\right|\leq M|y-b|^{1/2}.$$

This shows that if x_j tends to a, $x_j\neq a$, and if we can choose $y_j\in T_{x_j}$ so that y_j converges to some point b, then

$$\frac{h'(x_j)-h'(a)}{x_j-a}\to f_{xx}(a,b). \tag{2.4}$$

We now claim that

$$\frac{h'(x)-h'(a)}{x-a}\to f_{xx}(a,b)\quad\text{as}\quad x\to a,\ x\neq a, \tag{2.5}$$

for any $b\in S_a$. In fact, if this were not true, there would exist a sequence (x_j) tending to a, $x_j\neq a$, such that

$$\left|\frac{h'(x_j)-h'(a)}{x_j-a}-f_{xx}(a,b)\right|\geq\varepsilon>0. \tag{2.6}$$

Pick a point y_j in T_{x_j}; by passing to a subsequence we may assume that (y_j) converges to some point, say c. But by what we just proved, see (2.4), we then know that

$$\frac{h'(x_j)-h'(a)}{x_j-a}\to f_{xx}(a,c). \tag{2.7}$$

Since b is in S_a (by assumption) and c is in S_a (by Lemma 2.3), we must have $f_{xx}(a,b)=f_{xx}(a,c)$, for $f_{xx}(a,y)$ is constant in S_a. Now we see that (2.6) and (2.7) are incompatible, which means that (2.5) must hold: $h''(a)$ exists and is equal to $f_{xx}(a,b)$, $b\in S_a$, i. e., we have proved (2.2).

Now assume, finally, that f is real analytic near T. For a fixed x, the function $y\mapsto f_y(x,y)$ is a strictly increasing real-analytic function in a neighborhood of its zero set, so is has only one zero, possible multiple. Denote this by $\varphi(x)$. We shall study the behaviour of φ and h near a point which we take to be $x=0$; we may choose coordinates so that $\varphi(0)=0$. Let f have the power series expansion $f(x,y)=\Sigma a_{jk}x^jy^k$ at the origin; we thus have $a_{01}=0$. It is known that the solution to the equation $f_y(x,y)=0$ admits a fractional power series expansion, a Puiseux series, to the right and to the left of the origin:

$$\varphi(x)=\sum_{m=1}^{\infty}b_m x^{m/q},\ 0\leq x\leq\delta,$$

for some integer q and some $\delta>0$; a similar series exists for $-\delta\leq x\leq 0$. If f_y is a polynomial, this result is due to Puiseux [1850]; in the general case we can first use the Weierstrass preparation theorem to find a factorization $f_y = gw$ where g is analytic with $g(0,0) \neq 0$ and $w(x, y)$ is a polynomial in y with coefficients which are analytic in x (for a proof see e. g. Hörmander [1973, Cor. 6. 1. 2] or [1983, Th. 7. 5. 1]). That the zeros of $w(x, y) = 0$ are analytic functions of $x^{1/q}$ for some q is proved e. g. in van der Waerden [1939, § 14].

By substituting the series for φ into the series for f we obtain $h(x) = f(x, \varphi(x)) = \sum_{j,k=0}^{\infty} a_{jk} x^j (\sum_{m=1}^{\infty} b_m x^{m/q})^k = \sum_{j=0}^{\infty} c_j x^{j/q}$, $0 \leq x \leq \delta$. Here all series converge uniformly for these x; as a matter of fact we are calculating with analytic functions of a variable $t = x^{1/q}$ (but possibly different functions to the left and right). We can now read off the differentiablity properties of h from this series: which is the smallest noninteger exponent? We know from the result already proved that $h'(0)$ and $h''(0)$ exist; hence there can be no exponent smaller than 2, except 0 and 1. Moreover, the coefficients c_0, c_q and c_{2q}, i. e. the coefficients corresponding to the exponents 0, 1 and 2, must be the same from the left as from the right. Therefore h must have the form $h(x) = h(0) + h'(0)x + h''(0)x^2/2 + \sum_{j=2q+1}^{\infty} c_j x^{j/q}$, $0 \leq x \leq \delta$, and similarly for $-\delta \leq x \leq 0$. Hence h'' is Hölder continuous with some positive exponent. Theorem 2.2 and, consequently, Theorem 2.1 are now proved. We shall see that they are essentially sharp.

3. A smooth convex compact set whose shadow is not of class C^2. In this section we present examples which show that the projection of a compact convex set in $\mathbb{R}^3$ with C^∞ boundary need not have a C^2 boundary. We shall also see that the degree of smoothness given by Theorem 2.2 in the real-analytic case cannot be improved.

The study of $h(x) = \inf_y f(x, y)$ leads us to the implicit equation $f_y(x, y) = 0$ for the terminator, i. e. the points where the infimum is attained. To produce examples it is useful to have a class of convex functions for which this equation can be solved with some success. We shall first define a class of function such that the solution $y = \varphi(x)$ to $f_y(x, y) = 0$ has an explicitly defined inverse $x = \varphi^{-1}(y)$ (Examples 3.2 and 3.3). Then we find a somewhat larger class for which $x = \varphi^{-1}(y)$ admits a good approximation (Theorem 3.5) and in which we obtain a necessary and sufficient condition for h'' to be continuous.

The following lemma defines a class of convex functions whose terminators are easy to describe.

Lemma 3.1. *Put $f(x, y) = x^2/v(y) + u(y)$ where u is convex and v is concave and strictly positive on some interval I. Then f is convex on $\mathbb{R} \times I$.*

Proof. It is convenient to represent f as $f(x, y) = \frac{x^2}{v(y)} + u(y) = \sup_\xi (x\xi - \frac{v(y)}{4}\xi^2 + u(y))$. Now $-v(y)\xi^2/4$ is a convex function of y, so f is a supremum of convex functions of (x, y), thus itself convex in $\mathbb{R} \times I$.

Our standard choice for v will be $v(y) = \dfrac{1}{4 - y + \frac{1}{2}y^2}$, which is concave when $|y| \leq 1/2$. Thus if u is convex for these y, the function

$$f(x, y) = x^2(4 - y + \tfrac{1}{2}y^2) + u(y) \tag{3.1}$$

is convex as a function of (x, y) in the strip $|y| \leq 1/2$, and its terminator is explicitly given by $x^2 = u'(y)/(1-y)$.

Example 3.2. Let q be an odd natural number. The polynomial

$$f(x, y) = x^2(4 - y + \tfrac{1}{2}y^2) + \frac{1}{q+1} y^{q+1} - \frac{1}{q+2} y^{q+2}$$

is convex in the strip $|y| < 1/2$ and $h(x) = \inf_{|y|<1/2} f(x, y) = 4x^2 - \frac{1}{q+1}|x|^{2+2/q} + \frac{q}{2q+4}|x|^{2+4/q}$ for x sufficiently small. Therefore h is of Hölder class $C^{2+2/q}$ but no better if $q \geq 3$.

Indeed, the convexity follows from Lemma 3.1 and the remarks concerning (3.1). We calculate f_y: $f_y(x, y)=(y^q-x^2)(1-y)$. The equation $f_y=0$ is satisfied by $y=|x|^{2/q}$. For x small enough the infimum is attained at this y, so that $h(x)=f(x, |x|^{2/q})$. We obtain an expression for h which in particular shows that Theorem 2.2 cannot be improved in the real-analytic case.

Example 3.3. Let u be a convex C^2 function defined for $|y|<1/2$ and satisfying $u'(0)=0$, $u'(y)>0$ when $0<y<1/2$. Define

$$f(x, y)=x^2(4-y+\frac{1}{2}y^2)+u(y), \tag{3.1}$$

and $h(x)=\inf_{|y|<1/2} f(x, y)$. Then we have for $x>0$ and small enough

$$h''(x)=f_{xx}(x, y)-\frac{f_{xy}(x, y)^2}{f_{yy}(x, y)}=8-2y+y^2-\frac{4(1-y)^2}{1+(1-y)u''(y)/u'(y)}, \tag{3.2}$$

where y is uniquely determined by x from the relation $x^2(1-y)=u'(y)$.

Indeed, $f_y(x, y)=x^2(-1+y)+u'(y)$ and $f_{yy}(x, y)=x^2+u''(y)$, so that the terminator is defined by $x^2=u'(y)/(1-y)$; that y is uniquely determined by $x>0$ follows from the fact that $f_{yy}(x, y)\geqq x^2>0$ then. We see that h'' is a continuous function of x for $0<x<\varepsilon$, in fact C^∞ there if u is C^∞. Moreover, $h''(x)\to 8$ as $x\to 0$, $x>0$, if and only if $u''(y)/u'(y)\to+\infty$ as $y\to 0$, $y>0$. Now there always exists a sequence of points $b_j\to 0$ such that $u''(b_j)/u'(b_j)\to+\infty$. To see this, let (c_j) be any sequence of positive numbers tending to zero. By the mean-value theorem, there exist points b_j, $0<b_j<c_j$, such that $0<u'(c_j)=u'(c_j)-u'(0)=u''(b_j)\,c_j$. Hence $u''(b_j)/u'(b_j)\geqq u''(b_j)/u'(c_j)=1/c_j\to+\infty$. If $a_j=(u'(b_j)/(1-b_j))^{1/2}$ are the corresponding points on the x-axis, we therefore see, in view of (3.2), that $h''(a_j)\to 8$.

However, we can find a function u such that $u''(y)/u'(y)$ does not tend to $+\infty$. In fact, u'' can be any smooth positive function which is not identically zero in any interval $[0, \varepsilon]$, $\varepsilon>0$, so it may well have infinitely many zeros $y_j\to 0$. An example is

$$u''(y)=\sin^2(-\frac{1}{y})\exp(-\frac{1}{y}),\ y>0. \tag{3.3}$$

With such a function u, and with the corresponding points x_j, we have since $u''(y_j)=0$:

$$h''(x_j)=8-2y_j+y_j^2-4(1-y_j)^2\to 4\neq 8.$$

Hence h'' is not continuous at the origin; this conclusion is independent of Theorem 2.2, i. e. we need not know that $h''(0)$ exists. (However, it is easy to see that $(h'(x)$

$$-h'(0))/x=2(4-y+\frac{1}{2}y^2)\to 8.)$$

Theorem 3.4. *There exists a convex compact set in* $\mathbb{R}^3$ *with* C^∞ *boundary such that the boundary of its projection in* $\mathbb{R}^2$ *is not of class* C^2.

Proof. Let f be defined by (3.1) in a neighbourhood of the origin with u even and defined by (3.3), and extend f in a suitable way to a convex C^∞ function in all of $\mathbb{R}^2$. Then its epigraph, defined by $f(x, y)\leqq z$, has all the required properties under the projection $(x, y, z)\mapsto(x, z)$, except of course that it is unbounded. A suitable compact subset of it will do.

What are the crucial properties of the functions (3.1)? It turns out that the conditions $f_{xy}(0, y)=0$, $f_{xxy}(0, 0)\neq 0$ determine a class which allows us to conclude very much like in Example 3.3. All functions of the form (3.1) as well as those in Lemma 3.1 satisfy $f_{xy}(0, y)=0$ for all y; those of the form (3.1) satisfy $f_{xxy}(0, 0)=-2$.

Theorem 3.5. *Let* $f \in C^4(\mathbb{R}^2)$ *be convex and satisfy* $f_y(0,0)=f_{yy}(0,0)=0$, $f_y(0,y)\neq 0$ *for* $y\neq 0$, $f_{xy}(0,y)=0$ *for all* y, *and* $f_{xxy}(0,0)<0$. *Then the second derivative of* $h(x)=\inf_y f(x,y)$ *is continuous to the right at the origin if and only if* $f_{yy}(0,y)/f_y(0,y)$ *tends to* $+\infty$ *as* $y\to 0, y>0$.

Proof. Consider the Taylor expansions of f_y, f_{yy} and f_{xy} in the x direction around a point $(0, y)$:

$$f_y(x,y)=f_y(0,y)+\frac{1}{2}x^2 f_{xxy}(0,y)+x^3 g_1(x,y), \tag{3.4}$$

$$f_{yy}(x,y)=f_{yy}(0,y)+x^2 g_2(x,y), \tag{3.5}$$

$$f_{xy}(x,y)=xf_{xxy}(0,y)+x^2 g_3(x,y), \tag{3.6}$$

where the g_k are some continuous functions. Here we have already used that $f_{xy}(0,y)=f_{xyy}(0,y)=0$. Since $f_{xxy}(0,0)<0$, we see from (3.6) that $f_{xy}(x,y)^2/x^2$ is bounded from above and below by positive constants in some neighbourhood of the origin. In particular, there is an $a>0$ such that $f_{xx}(x,y)f_{yy}(x,y)\geq f_{xy}(x,y)^2\geq ax^2$, which shows that $f_{yy}(x,y)>0$ for $x>0$ and sufficiently small. The solution $y=\varphi(x)$ to $f_y(x,y)=0$ must therefore be unique if $x>0$, and the implicit function theorem shows that φ is of class C^3 for $0<x<\varepsilon$. On differentiating $h(x)=f(x,\varphi(x))$ we obtain $h'(x)=f_x(x,\varphi(x))$ (which incidentally shows that h is of class C^4 for $0<x<\varepsilon$), and

$$h''(x)=f_{xx}(x,\varphi(x))+f_{xy}(x,\varphi(x))\varphi'(x)=f_{xx}(x,\varphi(x))-f_{xy}(x,\varphi(x))^2/f_{yy}(x,\varphi(x)), \tag{3.7}$$

all valid for $x>0$ and sufficiently small.

From (3.4) we see that when $y=\varphi(x)$ and x is small enough, then $bx^2\leq f_y(0,y)\leq Bx^2$ for some positive constants b and B. In particular, we see that $\varphi(x)\to 0$ as $x\to 0$, $x>0$, and we define $\varphi(0)=0$. By Theorem 2.2, $h''(0)=f_{xx}(0,0)$, for $T_0=\{0\}$. Therefore h is continuous to the right at the origin if and only if $\frac{f_{xy}(x,\varphi(x))^2}{f_{yy}(x,\varphi(x))}\to 0$ as $x\to 0$, $x>0$; i. e. if and only if

$$\frac{f_{yy}(x,\varphi(x))}{f_y(0,\varphi(x))}\to +\infty \quad \text{as} \quad x\to 0,\ x>0, \tag{3.8}$$

for $f_{xy}(x,\varphi(x))^2$, $f_y(0,\varphi(x))$ and x^2 are all comparable. Now, in view of (3.5) $\frac{f_{yy}(x,\varphi(x))}{f_y(0,\varphi(x))}=\frac{f_{yy}(0,\varphi(x))}{f_y(0,\varphi(x))}+\frac{x^2 g_2(x,\varphi(x))}{f_y(0,\varphi(x))}$, where the last term is bounded, so (3.8) is in turn equivalent to $f_{yy}(0,y)/f_y(0,y)\to+\infty$ as claimed. (If f is of class C^3 only, we need not have $f_{yy}(x,y)-f_{yy}(0,y)=O(x^2)$ so this last step in the proof will not work; all others will.)

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