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MICROLOCAL ANALYTIC AND GEVREY SINGULARITIES FOR SECOND ORDER BOUNDARY VALUE PROBLEMS

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Transversal reflection of singularities and microlocal analytic and Gevrey (non) regularity near hyperbolic and elliptic points for certain second order boundary value problems are obtained.

1. Introduction. The microlocal analytic and Gevrey singularities of the solutions of second order boundary value problems are studied in this paper. Asymptotic solutions in the space of the formal analytic symbols are constructed, thus allowing as in the C^∞ case the use of microlocal parametrices for the examination of boundary value problems. An existence result with a priori estimates for the G^σ approximative solution is also proposed. Some of the assertions in the paper were stated in [16].

Let us remind that if $X\subset \mathbb{R}^n$ is an open domain we denote by $G^\sigma(X)$, $\sigma\geqq 1$ the class of the Gevrey functions of order σ, i. e. $f(x)\in G^\sigma(X)\Leftrightarrow f(x)\in C^\infty(X)$ and for every compact $K\subset\subset X$, there exists positive constant C_k satisfying

$$|\partial_x^\alpha f(x)|\leqq C_k^{|\alpha|+1}(\alpha\,!)^\sigma \quad x\in K \tag{1.1}$$

$$\alpha=(\alpha_1,\ldots,\alpha_n)\in \mathbb{Z}_+^n \quad \alpha\,!=\alpha_1!\ldots\alpha_n!$$

In particular $G^1(X)=A(X)$ is the space of all real analytic functions on X. We point out that for $\sigma>1$ $G^\sigma(X)$ contains non-zero compactly supported functions (i. e. $G_0^\sigma(X)=G^\sigma(X)\cap C_0^\infty(X)\neq\{0\}$) while $G_0^1(X)=\{0\}$ [10, 12].

The Gevrey pseudodifferential operators were initially studied by L. Boutet de Monvel and P. Kree [3], later L. Hörmander set out the basic notions of the G^σ-microlocal analysis [10]. The analytic case was considered in [2, 18, 19] and recently J. Sjostrand investigated the analytic singularities of second order boundary value problems near diffractive points [20, 21]. The stationary phase method in the classes of G^σ-symbols, $\sigma>1$ was examined by the author [7, 8] using the relation $G_0^\sigma\neq\{0\}$. It enables us to proceed similarly to the C^∞ microlocal analysis, when $\sigma>1$.

Let us recall the definitions of the Gevrey wave front sets $WF_G\sigma u$, $\sigma\geqq 1$ [10].

Suppose that $\sigma>1$ and $\rho^0=(x^0,\xi^0)\in T^*X\setminus 0$, $u\in \mathcal{D}'(X)$. We say that ρ^α does not belong to the G^σ wave front set of the distribution u and write $\rho^0\notin WF_G\sigma u$ if there exist $\varphi(x)\in G_0^\sigma(X)$, $\varphi(x^0)\neq 0$, open cone $T\ni\xi^0$ in $\mathbb{R}_\xi^n\setminus 0$ and positive constant C such that

$$\begin{aligned} &|\widehat{\varphi u}(\xi)|\leqq C^{N+1}(N!)^\sigma|\xi|^{-N} \quad \xi\in T\cap\{|\xi|\geqq 1\}\\ &N=0,1,\ldots \qquad \widehat{v}(\xi)=\int e^{-ix\xi}v(x)dx. \end{aligned} \tag{1.2}$$

In the analytic case $\sigma=1$ the lack of non-zero compactly supported functions prompted the use of special sequences in G_0^∞ [1, 10]. More precisely for any two open bounded sets $U_0, U_1, \bar{U}_0 \subset U_1$ one can find sequence $h_N(x) \in C_0^\infty(U_1)$. $h_N|_{U_0} \equiv 1$ $N=0, 1, \ldots$ and positive number C with the property

$$(1.3) \qquad |\partial_x^\alpha h_N(x)| \leq C^{|\alpha|} N^{|\alpha|} \quad |\alpha| \leq N \quad x \in \mathbf{R}^n$$
$$N=0, 1, \ldots \quad \alpha \in \mathbf{Z}^n_+.$$

Let now $u \in \mathscr{D}'(X)$ and $\rho^0=(x^0, \xi^0) \in T^x X \setminus 0$. The point ρ^0 is not in the analytic wave front set of u, $WF_a u = WF_G$, u if there exist two open nighbourhoods of x^0, U_0, U_1, $\bar{U}_0 \subset U_1$, open cone $T \ni \xi^0$ in $\mathbf{R}^n_\xi \setminus 0$ sequence $h_N(x)$, $N=0, 1, \ldots$ as above and constant $C>0$ verifying

$$(1.4) \qquad |\widehat{h_N u}(\xi)| \leq C^{N+1} N! |\xi|^{-N} \xi \in T \cap \{|\xi| \geq 1\}$$
$$N=0, 1, \ldots$$

Equivalent definitions of $WF_\sigma u$ could be found in [2, 20]. For simplicity's sake we will write further $WF_\sigma u$ instead of $WF_G \sigma u$, $\sigma>1$.

The main results on second order boundary value problems are stated in 2, while 3 deals with Cauchy problems in the space of the formal analytic symbols (f. a. s. -s) and the consequent construction of asymptotic solutions in the hyperbolic and elliptic regions. The proofs of the assertions in 2 are situated in 4. Theorem 2.3 was obtained in cooperation with P. Popivanov. During the work on this subject the author had very useful communications with J. Sjostrand.

2. Statements of the main results. Let M be real-analytic manifold of dimension $n+1$, $n \geq 3$ with boundary ∂M and let $P(x, D)$ be a second order differential operator with analytic coefficients and with real principal symbol p satisfying the condition

$$(2.1) \qquad d_{\text{fibre}} p(\rho) \neq 0 \quad \rho \in \Sigma_p = p^{-1}(0) \cap (T^*M \setminus 0)$$

$$(2.2) \qquad \Sigma_p \neq 0.$$

The boundary ∂M is supposed non-characteristic for P, i. e. $p(\rho) \neq 0$, $\forall \rho \in N^*\partial M \setminus 0$, where $N^*\partial M = \ker i^* \subset T^*M \setminus 0|_{\partial M}$, $i: \partial M \to M$ is the natural inclusion. Without loss of generality we assume

$$(2.3) \qquad p(\rho)>0 \quad \forall \rho \in N^*\partial M \setminus 0.$$

If the opposite sign holds, one studies $-p$ instead of p.

Consider the analytic boundary operator in the form

$$(2.4) \qquad k(m)\partial_\nu + \mathscr{L} \quad m \in \partial M,$$

where ∂_ν is the natural unit normal vector field [17], $\mathscr{L}$ stands for first order analytic differential operator on ∂M and $k \in A(\partial M)$ $k \not\equiv 0$. We require that $k(m)=0$, $m \in \partial M \Rightarrow \mathscr{L}_1|_m \neq 0$ ($\mathscr{L}_1$ — the principal symbol of $\mathscr{L}$). If $k(m) \neq 0$, $\forall m \in \partial M$, we regard $k \equiv 1$.

Let $\bar{m} \in \partial M$. In view of (2.3) one can introduce local coordinates $x=(x', x_n)$, $x(\bar{m})=0$, $x'=(x_0, \ldots, x_{n-1})$ in a neighbourhood $\mathscr{U} \ni \bar{m}$ such that

$$(2.5) \qquad M \cap \mathscr{U} \cong \{(x', x_n): 0 \leq x_n < \delta, \ |x'| < \delta_0\}, \ \delta_0, \ \delta>0$$

$$(2.6) \qquad p(x, \xi) = \xi_n^2 + r(x, \xi').$$

In these variables $\partial_\nu = \partial_{x_n}$. Let us recall the division of $T^*\partial M \setminus 0$ into three separate classes: $H = \{\rho \in T^*\partial M \setminus 0$: $i^{*-1}(\rho) \cap \Sigma_p$ consists of two points$\}$, $G = \{\rho \in T^*\partial M \setminus 0$;

$i^{*-1}(\rho)\cap\Sigma_p$ contains one point}, $E=\{\rho\in T^*\partial M\setminus 0;\ i^{*-1}(\rho)\cap\Sigma_p=\emptyset\}$ called respectively the set of hyperbolic, glancing and elliptic points. In the coordinates (2.5, 2.6) H, E, G are given respectively by the inequalities $r_0(x', \xi')<0$, $r_0(x', \xi')>0$, $r_0(x', \xi')=0$, $r_0=r|_{x_n=0}$.

Consider the Hamiltonian vector field H_p of p. Its integral curves $\gamma(t, \rho)\subset T^*M\setminus 0$, $\rho\in\Sigma_p$ are called zero bicharacteristics of P and if $\rho\in H$ one has $i^{*-1}(\rho)\cap\Sigma_p=\{\rho^+, \rho^-\}$, $\gamma(t, \rho^\pm)\subset\mathring{M}$ for $\pm t>0$, $\mathring{M}=M\setminus\partial M$. Put $\gamma^\pm_\rho(t)=\gamma(t, \rho^\pm)$, $\rho\in H$.

We shall study the next boundary value problem

$$
(2.7)\qquad \begin{aligned} &P(x, D)u=f\in G^\sigma(M)\quad \sigma\geqq 1\\ &Bu|_{\partial M}=g\in\varepsilon'(\partial M). \end{aligned}
$$

Here B is either (2.4) or $B=1$ ((i. e. Dirichlet boundary condition)

D e f i n i t i o n 2.1. *The boundary value problem (2.7) verifies Agmon's condition at the point* $\rho\in T^*_m\partial M\setminus 0$ *if* $B=1$ or

$$
(2.8)\qquad k^2(m)\, i^*(p)(\rho)+(\mathscr{L}_1(\rho))^2\neq 0
$$

in the case B given by (2.4).

The first result of the paper is analogue in G^σ, $\sigma>1$ to the well-known theorems for transversal reflection of singularities in C^∞ category (by L a x and N i r e n b e r g [15]) and in analytic category (by S h a p i r a [19]).

T h e o r e m 2.1 *Let* $\sigma>1$, *the distribution* $u\in\mathscr{D}'(\mathring{M})$ *satisfy* (2.7) *and* $\rho^0\in H\setminus WF_\sigma g$. *Suppose in addition that* $\gamma^-_{\rho^0}(t)\notin WF_\sigma u$, $t<0$ $(\gamma^+_{\rho^0}(t)\notin WF_\sigma u$, $t>0)$. *Then we get* $\gamma^+_{\rho^0}(t)\notin WF_\sigma u$, $t>0$ $(\gamma^-_{\rho^0}(t)\notin WF_\sigma u$, $t<0$ *and*

$$
(2.9)\qquad \rho^0\in WF_\sigma(\partial^j_\nu u|_{\partial M})\quad j=0, 1, \ldots
$$

providing Agmon's condition is valid at ρ^0.

Next we shall study the analytic singularities (2.7) $(\sigma=1)$.

D e f i n i t i o n 2.2. (Sjostrand). Assume that the distribution $u\in\mathscr{D}'(\mathring{M})$ satisfies $Pu\in A(M)$. Then its wave front up to the boundary $WF_{ba}u$ is defined as follows

$$
(2.10)\qquad WF_{ba}u=WF_a(u)\cup WF_a(u|_{\partial M})\cup WF_a(\partial_\nu u|_{\partial M})\subset BM.
$$

Here $BM\sim T^*\mathring{M}\setminus 0\cup T^*\partial M\setminus 0$ is the well-known homogeneous topological space and the microlocal Holmgren theorem shows that $WF_{ba}u$ is closed conic set in BM [20].

Denote by $V_a\subset T^*\partial M\setminus 0$ the closed set, consisting of all points violating Agmon's condition for (2.7). As usually $\{f, g\}$ stands for the Poisson bracket.

T h e o r e m 2.2 *In the case* $k\equiv 1$ *one has*

i) if the point $\rho^0\in V_a\cap H_{\bar m}$, $(\bar m\in\partial M)$ *is such that*

$$
(2.11)\qquad \{Im\,\mathscr{L}_1,\ i^*p\}(\rho^0)\neq 2Re\,\mathscr{L}_1(\rho^0)\{I_m\mathscr{L}_1,\ Re\,\mathscr{L}_1\}(\rho^0)
$$

one can find open neighbourhood $W\subset M$ *of* $\bar m$ *and* $u_0\in\mathscr{D}'(W\cap\mathring{M})$ *satisfying*

$$
(2.12)\qquad \begin{aligned} &P(x, D)\,u_0\in A(W)\\ &\rho^0\in WF_{ba}u_0\setminus WF_a(Bu_0|_{W\cap\partial M}) \end{aligned}
$$

and either $\rho^+_{\rho^0}(t) \notin WF_a u_0$, $t > 0$, $\gamma^-_{\rho^0}(t) \notin WF_a u_0$, $t < 0$ *or* $\gamma^+_{\rho^0}(t) \notin WF_a u_0$ $t < 0$, $\gamma^-_{\rho^0}(t) \in WF_a u_0$, $t<0$.

ii) for any point $\rho^0 \in V_a \cap E_{\bar{m}}$ *having the property*

$$\{Re\,\mathscr{L}_1,\ i^*p\}(\rho^0) < 2Im\,\mathscr{L}_1\{Im\,\mathscr{L}_1,\ Re\mathscr{L}_1\}(\rho^0) \tag{2.13}$$

there exist neighbourhood $W \ni \bar{m}$ *and* $u_0 \in \mathscr{D}'(\mathring{W})$ *verifying* (2.12).

Remark 2.1. The assertion above is analytic microlocal analogue to Theorem 2 in [17]. The lack of global results is due to the fact that the analytic functions are not always extendible and $G^1_0 = \{0\}$.

Furher we require $k(m) = 0$ for some m and for simplicity's sake we assume k real-valued and $Re\mathscr{L}_1$, $Im\,\mathscr{L}_1$ in involution, i. e.

$$\{Re\,\mathscr{L}_1,\ Im\,\mathscr{L}_1\} \equiv 0. \tag{2.14}$$

Let $k(\bar{m}) = 0$, $m \in \partial M$ and χ be arbitrary tangential analytic vector field near $\bar{m}$. Put $N(\bar{m}, \chi)$ to be the miniml integer j such that $(\chi^j k)(\bar{m}) \neq 0$. In view of $k \not\equiv 0$, $N(\bar{m}, \chi) < \infty$.

Theorem 2.3. *Let* $\bar{m} \in \partial M$ *and* $k(\bar{m}) = 0$.

iii) if $N_r = N(\bar{m}.\ Im\,\mathscr{L}_1)$ *is even we claim that assumptions:* $\rho^0 \in V_a \cap H_{\bar{m}}$, $u \in D'(\mathring{M})$ *solves* (2.7), $\rho^0 \notin WF_a(g)$ *and either* $\gamma^+_{\rho^0}(t) \notin WF_a u$, $t>0$ *or* $\gamma^-_{\rho^0}(t) \notin WF_a u$, $t<0$ *imply*

$$\rho^0 \notin WF_{ba} u. \tag{2.15}$$

The relations N_1 odd, $H_{\bar{m}} \cap V_a \neq \emptyset$ lead to the existence of a point $\rho^0 \in H_{\bar{m}} \cap V_a$ verifying the conclusions of i) Theorem 2.2.

iv) if $N_2 = N(\bar{m},\ Re\mathscr{L}_1)$ is even we state that for every $\rho^0 \in E_{\bar{m}} \cap V_a$ (2.15) is valid, providing $u \in \mathscr{D}'(\mathring{M})$ is a solution to (2. 7), $\rho^0 \notin WF_a(g)$. The assertion above remains true when N_2 is odd and $(Re\mathscr{L}_1)^{N_2} k(\bar{m}) > 0$. If N_2 is odd and $(Re\,\mathscr{L}_1)^{N_2} k(\bar{m}) < 0$ one can find a point $\rho^0 \in E \cap V_a$ for which the conclusions of ii) Theorem 2.2 are true.

Let now $Q \subset \mathscr{R}^n_{x_1, \ldots, x_n}$ be an open domain with analytic boundary ∂Q and let $M = \mathsf{R}_{x_0} \times \bar{Q}$. The second order differential operator $P(x, D)$ is supposed strictly hyperbolic with respect to x_0 in R^{n+1} and:

$$\inf_{\substack{x \in \mathsf{R}^{n+1} \\ (\xi_1, \ldots, \xi_n) \in S^n}} |\lambda_1(x, \xi_1, \ldots, \xi_n) - \lambda_2(x, \xi_1, \ldots, \xi_n)| > 0 \tag{2.16}$$

where $\lambda_{1,2}$ are the two disjoint real roots of the equation with respect to $\xi^0 : p(x, \xi_0, \xi_1, \ldots, \xi_n) = 0$. One demands in addition that no zero bicharacteristic of P hits twice ∂M.

Under these conditions we have

Theorem 2.4. *Let* $\sigma > 1$. *Then for every* $g \in \varepsilon'(\partial M)$, $WF_\sigma g \subset H$ *there exists* $u_g \in \mathscr{D}'(\mathring{M})$ *approximative solution to the mixed problem*

$$\begin{aligned} &P(x, D)u_g = f(x) \in G^\sigma(M) \\ &u_g|_{\partial M} = g + h, \quad h \in G^\sigma_0(\partial M) \\ &u_g|_{x_0 \ll 0} = 0. \end{aligned} \tag{2.17}$$

Moreover, for any compactly based conic neighbourhood $\Gamma\subset T^*\partial M\setminus 0$, $\bar{\Gamma}\subset H$ one can find $t\in\mathbb{R}$ and $C>0$ with the next property: if $l\in\mathbb{Z}_+$, $g\in\mathcal{E}'(\partial M)\cap H^1(\partial M)$, $WF_\sigma g\subset\Gamma$ there exists u_g satisfying (2.17) and

$$\|u_g\|_{H^t(M_l)}\leqq C^{l+1}\sum_{j=0}^{l}((l-j)!)^\sigma\|g\|_j\,. \tag{2.18}$$

Here $\|g\|_j$ means the norm in $H^j(\partial M)$, $M_t=M\cap\{x_0\leqq t\}$.

3. The construction of asymptotic solutions. Let $\Gamma\subset\mathbb{R}^n_x\times(\mathbb{R}^d_\theta\setminus 0)$ be open compactly based cone. The formal sum $\sum_{k=0}^\infty p_{m-k}(x,\theta)$ is called formal analytic symbol (f. a. s.) of order m in Γ if $p_{m-k}(x,\theta)\in A(\Gamma)$, $\operatorname{ord}_\theta p_{m-k}=m-k$, $k=0,1,\ldots$, and there is a positive constant C such that

$$|\partial_x^\alpha\partial_\theta^\beta p_{m-k}(x,\theta)|\leqq C^{k+|\alpha|+|\beta|+1}\,k!\,\alpha!\,\beta!\,|\theta|^{m-k}\quad(x,\theta)\in\bar{\Gamma}. \tag{3.1}$$
$$\alpha\in\mathbb{Z}^n_+,\quad\beta\in\mathbb{Z}^d_+\quad k=0,1,\ldots$$

It is well known [3], that (3.1) is equivalent to the fact that $p_{m-k}(x,\theta)$, $k=0,1\ldots$ are analytic in $\tilde{\Gamma}\subset\mathbb{C}^n_x\times(\mathbb{C}^d_\theta\setminus 0)$ complex conic neighbourhood of $\bar{\Gamma}$ and the inequalities (3.1) hold for $(x,\theta)\in\tilde{\Gamma}$ (with another $C>0$). Further we will assume Γ complex cone.

Denote by $S_a(m,\Gamma)$ the set of all f. a. s.-s of order m in Γ. We introduce in $S_a(m,\Gamma)$ norms, similar to those proposed in [3]. More precisely let $E_m(T,\Gamma)=\{p=\sum_{k=0}^\infty p_{m-k}\in S_a(m,\Gamma):\|p\|_T<\infty\}$, where

$$\|p\|_T=\sum_{k,\alpha,\beta}M^T_{\alpha,\beta,k}(p)\quad T>0 \tag{3.2}$$

$$M^T_{\alpha,\beta,k}(p)=\frac{\varepsilon_k\,k!\,T^{2k+|\alpha|+|\beta|}}{(k+|\alpha|)!\,(k+|\beta|)!}\sup_{\substack{(x,\theta)\in\Gamma\\|\theta|=1}}|\partial_x^\alpha\partial_\theta^\beta p_{m-k}(x,\theta)|$$

and $\varepsilon_k\searrow 0$ is fixed sequence.

The set $E_m(T,\Gamma)$ is Banach space and evidently the next properties hold: a) $T_1<T_2\Rightarrow E_m(T_1,\Gamma)\subset E_m(T_2,\Gamma)$; b) $\bigcup_{T>0}E_m(T,\Gamma)=S_a(m,\Gamma)$.

Let $\mathcal{L}$ be the analytic differential operator

$$\mathcal{L}=\sum_{j=1}^n\gamma_j(x,\theta)\partial_{x_j}+\sum_{i=1}^d\delta_i(x,\theta)\partial_{\theta_i} \tag{3.3}$$

$$\gamma_j,\ \delta_i\in A(\bar{\Gamma}),\ \operatorname{ord}_\theta\gamma_j=0,\ \operatorname{ord}_\theta\delta_i=1,\ j=1,\ldots,n,\ i=1,\ldots,d.$$

We demand that

$$\mathcal{L}\text{ and }\sum_{i=1}^n\theta_i\partial_{\theta_i}\text{ are linearly independent at every point }(x,\theta)\in\Gamma. \tag{3.4}$$

Choose and fix a conic complex analytic hyperplane $S\subset\Gamma$ transversal to $\mathcal{L}$. It means that for each $(x^0,\theta^0)\in S$ there exists analytic in conic neighbourhood of (x^0,θ^0) function $f(x,\theta)$ such that near (x^0,θ^0) $S=\{f=0\}$, $(\mathcal{L}f)|_{f=0}\neq 0$.

If F is linear continuous operator in $E_l(T,\Gamma)$, $T\in(0,T_0)$, T_0, $l\in\mathcal{R}$ fixed, we study the following Cauchy problem

$$\begin{aligned}&\mathcal{L}\Big(\sum_{k=0}^\infty p_{l-k}\Big)+F\Big(\sum_{k=0}^\infty p_{l-k}\Big)=\sum_{k=0}^\infty f_{l-k}=f\in E_l(T,\Gamma)\\&\sum_{k=0}^\infty p_{l-k}|_S=\sum_{k=0}^\infty b_{l-k}=b\in E_l(T,S).\end{aligned} \tag{3.5}$$

Theorem 3.1. *Let $z^0=(x^0, \theta^0)\in S$ be an arbitrary point. Under the assumption* (3.4) *one can find conic neighbourhood $\tilde{\Gamma}^0\subset\Gamma$ of z^0, positive number T_1 with the property: for every $b\in E_l(T, \tilde{S}^0)$, $\tilde{S}^0=S\cap\Gamma^0$ $f\in E_l(T, \Gamma^0)$ there exists unique $p\in E_l(T, \Gamma^0)$ solution to* (3.5), *providing $T\in(0, T_1]$. Moreover, the linear operator transforming the data (b, f) into the unique solution p is continuous from $E_l(T, \tilde{S}^0)\times E_l(T, \tilde{\Gamma}^0)$ to $E_l(T, \tilde{\Gamma}^0)$.*

Proof: In view of (3.4) and the transversality demand on we may think without loss of generality (after appropriate change of the variables near z^0) that in conic neighbourhood $\Gamma^0\ni z^0$ we have $z^0=(0, \theta^0)$, $S=\{x_n=0\}$, $\mathscr{L}=\partial_{x_n}$. In these coordinates the Cauchy problem is written as integral equation in the Banach space $E_l(T, \Gamma^0)$

$$p(x, \theta)+\int_0^{x_n} F(p)(x', \tau, \theta)d\tau=b(x, \theta')+\int_0^{x_n} f(x', \tau, \theta)d\tau. \tag{3.6}$$

We used the natural inclusion $E_l(T, S^0)\frown E_l(T, \Gamma^0)$, $S^0=S\cap\Gamma^0$. There exist two small positive numbers δ, T_1 such that

$$\|\int_0^{x_n} F(p)(x', \tau, \theta)d\tau\|_T\leq 2^{-1}\|p\|_T, \quad \forall p\in E_l(T, \Gamma^0_\delta) \tag{3.7}$$
$$\forall T\in(0, T_1], \quad \Gamma^0_\delta=\Gamma^0\cap\{|x_n|<\delta\}.$$

Indeed, in view of the inequalities

$$M^T_{\alpha,\beta,k}(\int_0^{x_n} F(p))\leq\begin{cases}\delta M^T_{\alpha,\beta,k}(p) & \text{if } \delta_n=0\\ TM^T_{(\alpha',\alpha_n-1),\beta,k}(p) & \text{if } \alpha_n\geq 1\end{cases} \tag{3.8}$$

one obtains (3.7) taking T_1 and δ small enough. The estimate (3.7) imply that $(Id-F)^{-1}$ is continuous which proves Theorem 3.1.

Suppose now $\Gamma=X\times\Omega$, where $X\subset\mathbf{C}^n_x$ is open domain, $\Omega\subset\mathbf{C}^n_\xi\setminus 0$ is open cone and let $p(x, D)=\Sigma^\infty_{k=0}p_{m-k}(x, D)$ be a formal analytic p. d. o. in Γ (i. e. $p(x, \xi$ $\in S_a(0, \Gamma))$. Consider the cone $F\subset\mathbf{C}^n_x\times(\mathbf{C}^d_\xi\setminus 0)$ verifying the inclusion $\{x\in\mathbf{C}^n;\ \exists\theta\in\mathbf{C}^d\setminus 0$ $(x, \theta)\in F\}\subset X$.

Let $\varphi(x, \theta)$ be analytic function in $\bar{F}$, $\operatorname{ord}_\theta\varphi=1$ and

$$\varphi_x(x, \theta)\in\Omega \quad (x, \theta)\in\bar{F}. \tag{3.9}$$

If we put $\varphi_1(x, z, \theta)=-\varphi(x, \theta)+\varphi(z, \theta)+(z-x)\varphi_x(x, \theta)$ then, according to [10],

$$D_z(e^{i\varphi_1(x,z,\theta)})|_{z=x}=\sum_{0\leq j\leq\frac{|\alpha|}{2}} C^\alpha_j(x, \theta),\ \operatorname{ord}_\theta C^\alpha_j=j \quad \alpha\in\mathbf{Z}^n_+ \tag{3.10}$$

and there exists $C_1>0$ such that

$$|\partial^\beta_x\partial^\gamma_\theta C^\alpha_j(x, \theta)|\leq C_1^{|\alpha|+|\beta|+|\gamma|-j+1}\frac{|\alpha|!}{j!}\beta!\,\gamma!\,|\theta|^{j-|\gamma|} \tag{3.11}$$
$$\alpha, \beta\in\mathbf{Z}^n_+, \gamma\in\mathbf{Z}^d_+, 0\leq j\leq\frac{|\alpha|}{2}, (x, \theta)\in\bar{F}.$$

Consider the formal action of $p(x, D)$ on $e^{i\varphi}a$, $a=\Sigma^\infty_{k=0}a_{l-k}\in S_a(l, \Gamma)$. We have

$$v(x, D)(e^{i\varphi(\cdot,\theta)}a(\cdot, \theta))=e^{i\varphi(x,\theta)}\sum_{j=0}^\infty\Lambda_{m+l-j}(x, \theta), \tag{3.12}$$

where

(3.13) $$\Lambda_{m+l-j}(x,\theta)=\sum_{\substack{-\nu+t+|\gamma+\delta|+s=j\\ 0\le\nu\le\frac{\delta}{2}}}\partial_\xi^{\gamma+\delta}p_{m-t}(x,\varphi_x)\frac{C_\nu^\gamma(x,\theta)}{\delta!}\frac{D_x^\alpha a_{l-s}(x,\varphi_x)}{\gamma!}.$$

We will study the linear operator $I_{p,\varphi}(a)=\Sigma_{j=0}^\infty\Lambda_{m+l-j}(x,\theta)$ providing p, φ fixed. Further we suppose that

(3.14) $$\varepsilon_k=2^{-k}(n+d)^{-k}\quad k=0,1,\dots$$

Theorem 3.2. *The linear operator $I_{p,\varphi}$ acts from $S_a(l,F)$ to $S_a(l+m,F)$. Moreover, there exist positive numbers T_0 and C_0 such that $I_{p,\varphi}$ is continuous from $E_l(T,F)$ to $E_{l+m}(T,F)$ and its $\|I_{p,\varphi}\|_T$ is bounded by C_0, $\forall T\in(0,T_0]$, $\forall l\in\mathbf{R}$.*

Proof: Put

(3.15) $$b_{m-r}^{(\gamma)}(x,\theta)=\sum_{\substack{t+|\delta|-\nu=r\\ 0\le\nu\le r}}\partial_\xi^{\gamma+\delta}p_{m-t}(x,\varphi_x)\frac{c_\nu^\delta(x,\theta)}{\delta!}.$$

Clearly $\operatorname{ord}_\theta b_{m-r}^{(\gamma)}=m-r-|\gamma|$. Thus one may write

(3.16) $$\Lambda_{m+l-j}(x,\theta)=\sum_{s+r+|\gamma|=j}\frac{1}{\gamma!}b_{m-r}^{(\gamma)}(x,\theta)D_x^\gamma a_{l-s}(x,\theta),$$

which resembles the composition of two p. d. o.-s. In view of (3.9) one has

(3.17) $$c_1\le|\theta|^{-1}|\varphi_x(x,\theta)|\le c_2,\quad (x,\theta)\in\overline{F},\ c_1,c_2>0.$$

Taking into account (3.11) and (3.17), we deduce that

(3.18) $$|\partial_x^\alpha\partial_\theta^\beta b_{m-r}^{(\gamma)}(x,\theta)|\le C^{r+|\gamma|+|\alpha|+|\beta|+1}\alpha!\,\beta!\,\gamma!\,|r||\theta|^{m-r-|\gamma|}$$
$$\alpha,\gamma\in\mathbf{Z}_+^n,\ \beta\in\mathbf{Z}_+^d,\ r=0,1,\dots\ C=C(\varphi,p,F)>0.$$

The estimates (3.18) and the relation (3.16) lead to the continuity of $I_{p,\varphi}$ by repeating the arguments in the proof of Lemma 1.2 in [3].

Now we shall construct asymptotic solutions for the boundary value problem (2.7).

Let $\rho^0\in H$ and $\rho^0=(0,\xi^{0\prime})$ in the local coordinates (2.5), (2.6). There are two formal asymptotic solutions near ρ^0 in the form

$$e^{i\varphi_\pm(x,\xi')}\sum_{k=0}^\infty a_{-k}^\pm(x,\xi')$$

(3.19) $$|x_n|<\delta,\ (x',\xi')\in\Gamma=X\times T_{\delta_2}$$
$$X=\{x',\ |x'|<\delta_1\},\quad T_{\delta_2}=\{\xi'\in\mathbf{R}^n\setminus 0,\ \left|\frac{\xi'}{|\xi'|}-\frac{\xi^{0\prime}}{|\xi^{0\prime}|}\right|<\delta_2\},$$
$$\delta_1,\ \delta_2>0$$

where the phase function $\varphi_\pm(x,\xi')$ satisfies the eikonal equation

(3.20) $$\varphi_{x_n}^\pm\mp\sqrt{-r(x,\varphi_{x'}^\pm)}=0\quad\sqrt{1}=1$$
$$\varphi^\pm|_{x_n=0}=x'\cdot\xi'$$

while $a_{-k}^\pm$, $k=0,1,\dots$ are solving the transport equations

$$\mathcal{L}^\pm a_{-k}^\pm+\frac{1}{|\xi'|}\Big(\sum_{l,s=0}^n\frac{\partial^2 p}{\partial\xi_l\partial\xi_s}(x,\varphi_x^\pm)\varphi_{x_lx_s}^\pm+p_1(x,\varphi_x^\pm)\Big)a_{-k}^\pm=\frac{i}{|\xi'|}P(x,D)a_{-k+1}^\pm$$

(3.21) $a^{\pm}_{-k}|_{x_n=0}=\delta_{0k}.\ \delta_{00}=1, \quad \delta_{0k}=0 \quad k>0, \quad k_1=0$

$$\mathscr{L}^{\pm}=\frac{1}{|\xi'|}\sum_{j=0}^{n}\partial_{\xi_j}p(x,\ \varphi^{\pm}_x)\partial_{x_j}.$$

Using Theorem 3.1 and Theorem 3.2, one easily obtains that $\Sigma_{k=0}^{\infty}a^{\pm}_{-k}$ are f. a. s.-s. In fact $\varphi^{\pm}(x,\ \xi')$, $a^{\pm}_{-k}(x,\ \xi')$ are defined and analytic in $\tilde{\Gamma}_\delta=\{(x,\ \xi')\in \mathbf{C}^{n+1}\times(\mathbf{C}^n\setminus 0)$ $(x',\ \xi')\in\tilde{\Gamma},\ |x_n|<\delta\}$, $\tilde{\Gamma}$ complex conic neighbourhood of Γ, $\Sigma_{k=0}^{\infty}a^{\pm}_{-k}(x,\ \xi')\in S_a(0,\ \tilde{\Gamma}_\delta)$.

If $\rho^0=(0,\ \xi^{0\prime})\in E$ one constructs asymptotic solution for (2.7) near ρ^0 in the form

$$e^{i\varphi(x,\ \xi')}\sum_{k=0}^{\infty}a_{-k}(x,\ \xi'), \tag{3.22}$$

where $(x,\ \xi')$ belong to complex conic neighbourhood $\tilde{\Gamma}_1\subset\mathbf{C}_x^{n+1}\times(\mathbf{C}^n_{\xi'}\setminus 0)$ of $\Gamma_1=X\times(-\delta,\ \delta)\times T_1$, $T_1\ni\xi^{0\prime}$, $\varphi(x,\ \xi')$ is the unique solution of

$$\begin{aligned}&\varphi_{x_n}-i\sqrt{r(x,\ \varphi_{x'})}=0 \quad \sqrt{1}=1\\ &\varphi|_{x_n=0}=x',\ \xi'\end{aligned} \tag{3.23}$$

In particular $Im\,\varphi\geq 0$ for $x_n\geq 0$, $(x',\ \xi')$ real. The functions a_{-k} are obtained from the corresponding transport equations and $\Sigma_{k=0}^{\infty}a_{-k}\in S_a\ (0,\ \tilde{\Gamma}_1)$.

4. Proof of the results, stated in 2. Let $\sigma>1$ and choose $\varkappa\in G_0^\sigma\ (\mathbf{R}^n)$, $\varkappa(\xi')=1$, $|\xi'|\leq 1$. Put $a^{\pm}(x,\ \xi')=\Sigma_{k=0}^{\infty}a^{\pm}_{-k}(x,\ \xi')(1-\varkappa(\frac{\xi'}{Rk}))$ $R\gg 1$. Then $a^{\pm}(x,\ \xi')$ is complete G^σ realization of $\Sigma_{k=0}^{\infty}a^{\pm}_k(x,\ \xi')$, i. e. for some $C>0$

$$\begin{aligned}&|\partial_x^\alpha\partial_{\xi'}^\beta(a^{\pm}-\sum_{k=0}^{N}a^{\pm}_{-k})(x,\ \xi')|\leq C^{|\alpha|+|\beta|+N+1}(\alpha!\ \beta!\ N!)^\sigma\langle\xi'\rangle^{-N-|\beta|-1}\\ &\langle\xi^0\rangle=(1+|\xi'|^2)^{1/2},\ \alpha\in\mathbf{Z}_+^{n+1},\ \beta\in\mathbf{Z}_+^n,\ N=0,\ 1,\ldots\end{aligned} \tag{4.1}$$

and we write briefly $a^{\pm}\overset{\sigma}{\simeq}\Sigma_{k=0}^{\infty}a^{\pm}_{-k}$ [3, 7, 8]. Moreover, $a^{\pm}$ are unique modulo symbol from the space $C_\sigma^{-\infty}=\{b(x,\ \xi')\ \exists C>0:\ |\partial_x^\alpha b(x,\xi')|\leq C^{|\alpha|+N+1}(\alpha!\ N!)^\sigma\langle\xi'\rangle^{-N},\ \alpha\in\mathbf{Z}_+^n,$ $N\in\mathbf{Z}_+\}$ and $\alpha^{\pm}$ verify (with another constant $C>0$)

$$|\partial_x^\alpha\partial_{\xi'}^\beta a^{\pm}(x,\ \xi')|\leq C^{|\alpha|+|\beta|+1}(\alpha!\ \beta!)^\sigma\langle\xi'\rangle^{-|\beta|}. \tag{4.2}$$

Let $h(\xi)\in G^\sigma$, $h(\xi')=0$ for $\xi'\notin T_{\delta_0}\cap\{|\xi'|\geq\frac{1}{2}\}$, $h=1$ on $T_{\frac{\delta_0}{2}}\cap\{|\xi'|\geq 1\}$, $0<\delta_0<\delta_2$.

Consider the following F. I. O.

$$\begin{aligned}&J^{\pm}v(x)=\int e^{i\varphi^{\pm}(x,\xi')}\alpha^{\pm}(x,\ \xi')h(\xi')\hat{v}(\xi')\,d\xi'\\ &v\in\varepsilon'_{\Gamma^0}(\sigma)=\{v\in\varepsilon'(X_0):\ WF_\sigma v\in\Gamma^0\}\\ &\Gamma^0=X_0\times T_{\frac{\delta_0}{2}}\ X_0=\{|x|<\delta_0\}.\end{aligned} \tag{4.3}$$

As in the C^∞ category $J^{\pm}$ posses the properties

$$P(x,\ D)J^{\pm}v\in G^\sigma(X\times(-\delta,\ \delta)); \tag{4.4}$$

$$J^{\pm}v|_{x_n=0}-v\in G^\sigma(X); \tag{4.5}$$

(4.6) $$WF_\sigma(J^\pm v)=\gamma^\pm(WF_\sigma v)=\{\gamma^\pm_\rho(t),\ \rho\in WF_\sigma v\},\ v\in \varepsilon'_{\Gamma^0}(\sigma).$$

The relation (4.4) follows from (4.1) and the fact that $e^{i\varphi^\pm}\Sigma_{k=0}^\infty a^\pm_{-k}$ is asymptotic solution of P, while the initial data for $a^\pm_{-k}$ lead to (4.5). The inclusion $WF_\sigma(J^\pm v)\subset\gamma^\pm(WF_\sigma v)$ is obtained with integration by parts for G^σ symbols [7, 8, 13] and the equality (4.6) is deduced from (4.5) and the propagation of G^σ singularities for differential operators of real principal type [10].

Proof of Theorem 2.1: Let the distribution u satisfy (2.7) and let $\rho^0=(0,\xi^{0\prime})\in H\setminus WF_\sigma(Bu|_{\partial M})$, $\gamma^-_{\rho^0}(t)\notin WF_\sigma u$, $t<0$.

Having in mind the considerations in [3, 10], we decompose $p(x, D)$ microlocally near ρ^0

(4.7) $$P(x, D)=(D_{x_n}-\lambda^-(x, D'))\circ(D_{x_n}-\lambda^+(x, D'))+R$$
$$D'=(D_{x_0},\dots,D_{x_{n-1}}),$$

where $\lambda^\pm(x,\xi')\sigma\,\Sigma_{k=0}^\infty\lambda^\pm_{1-j}(x,\xi')$, $\lambda^\pm_1(x,\xi')=\pm\sqrt{-r(x,\xi')}$; $Rw\in G^\sigma$ if $WF_\sigma w$ is contained in a conic neighbourhood of $\{\rho^{0+},\rho^{0-}\}$.

In the case of Dirichlet boundary condition the proof follows the steps of Nirenberg [15] in the C^∞ category. More precisely the ellipticity of the p. d. o. $D_{x_n}-\lambda^-(x, D')$ near $\gamma^-_{\rho^0}(t)$ implies that u verifies microlocally near $\rho^{0+}=(0, 0, \xi^{0\prime}, \sqrt{-r_0(0,\xi^{0\prime})})$

(4.8) $$(D_{x_n}-\lambda^+(x, D'))u\in G^\sigma\ (x_n\geqq 0)$$
$$\rho^0\notin WF_\sigma(u|_{x_n=0})\qquad .$$

Applying the microlocal parametrix J^+ (the construction of $e^{i\varphi^+}\Sigma_{k=0}^\infty a^+_{-k}$ shows that it is asymptotic solution of $D_{x_n}-\lambda^+(x, D')$) we get $\gamma^+_{\rho^0}(t)\notin WF_\sigma u$, $t>0$, $\rho^0\notin WF_\sigma(\partial_{x_n}u|_{x_n=0})$. One deals similarly with the Dirichlet problem when $\gamma^+_{\rho^0}(t)\in WF_\sigma u$, $t>0$ holds.

If B is the first order boundary operator the conclusions of Theorem 2.1 are equivalent with G^σ hypoellipticity near ρ^0 for

(4.9) $$b^\pm(x', D')v=BJ^\pm v|_{x_n=0}=g.$$

The fulfillment of Agmon's condition at ρ^0 means that $b^\pm(x, D')$ are elliptic at ρ^0 which proves the theorem.

Let $\sigma=1$. The considerations in the analytic category are rather different from the C^∞ and G^σ, $\sigma>1$ cases, due to $G^1_0=\{0\}$.

According to [20, 21, 24], we can find functions (denoting them again $a^\pm(x,\xi')$ and $a(x,\xi')$) analytic realisations of $\Sigma_{k=0}^\infty a^\pm_{-k}$, $\Sigma_{k=0}^\infty a_{-k}$, respectively, i. e. there exists $C>0$ such that

(4.10) $$|\partial_x^\alpha(a^\pm(x,\xi')-\sum_{k=0}^N a^\pm_{-k}(x,\xi'))|\leq C^{|\alpha|+N+1}\alpha!\,N!\,\langle\xi'\rangle^{-N-1};$$

(4.11) $$|\partial_x^\alpha(a^\pm(x,\xi')-\sum_{k=0}^N a_{-k}(x,\xi'))|\leq C^{|\alpha|+N+1}\alpha!\,N!\,\langle\xi'\rangle^{-N-1}.$$
$$\alpha\in Z_+^{n+1},\quad N=0, 1,\dots$$

Moreover, for some $\varepsilon_0>0$

(4.12) $$|\partial_{\bar{\xi}_j} a^{\pm}(x, \xi')| \leq \frac{1}{\varepsilon_0} e^{-\varepsilon_0|\xi'|}$$

(4.13) $$|\partial_{\bar{\xi}_j} a(x, \xi')| \leq \frac{1}{\varepsilon_0} e^{-\varepsilon_0|\xi'|};$$

$$j=0, 1, \dots, n-1, \ \partial_{\xi_j}=\frac{1}{2}(\partial_{Re\xi_j}+i\partial_{Im\rho_j}).$$

The variables (x, ξ') in these inequalities belong to $\tilde{\Gamma}_1$.

According to Lemma 1.4, chap. V, [24], if T, T^* are open cones in $\mathbb{R}^n\setminus 0$, $\bar{T}\subset T^*$ there exists $M>0$ with the following properties: for every $\varepsilon>0$ there is a function $g_\varepsilon(\zeta')\in C^\infty(\mathbb{C}^n - i\mathbb{R}^n)$ verifying

(4.14) $$\partial \leq g_\varepsilon(\xi') \leq 1 \quad \xi'\in\mathbb{R}^n;$$

(4.15) $$g_\varepsilon(\zeta')=\begin{cases} 0 & \xi'\notin T^* \\ 1 & \xi'\in T \end{cases} \quad \zeta'=\xi'+i\eta', \quad \xi', \eta'\in\mathbb{R}^n;$$

(4.16) $$|g_\varepsilon(\zeta')| \leq Me^{\varepsilon M|\eta'|};$$

(4.17) $$|\partial_{\bar{\zeta}} g_\varepsilon(\zeta)| \leq Me^{(M\varepsilon|\eta'|-\varepsilon|\xi'|)}.$$

Let $T=T_{\delta_0/3}$, $T^*=T_{\delta_0/4}\cap\{|\xi'|\geq 1\}$. Consider the next analytic F. I. O.-s

(4.18) $$J_\varepsilon^{\pm}v(x)=\int e^{i\varphi^{\pm}(x,\xi')}a^{\pm}(x, \xi')\, g_\varepsilon(\xi')\,\hat{v}(\xi')\, d\xi';$$

(4.19) $$J_\varepsilon v(x)=\int e^{i\varphi(x,\xi')}\, a^{\pm}(x, \xi') g_\varepsilon(\xi')\hat{v}(\xi')\, d\xi', \quad v\in\varepsilon'.$$

The number ε will be fixed later.

The operators $J_\varepsilon^{\pm}$, J have the properties

(4.20) $$P(x, D)Jv\in A(X\times(-\delta, \delta));$$

(4.21) $$WF_a(Jv|_{x_n=0}-v)\cap(X_0\times T_{\delta_0/4})=\emptyset$$

for $J=J_\varepsilon^{\pm}$, J_ε and

(4.22) $$WF_a(J_\varepsilon^{\pm}v)\cap\gamma^{\mp}(t, \rho^0)=\emptyset.$$

The relation (4.20) is obtained in the same way as (4.4) (taking into account (4.10, 4.11)) while (4.21) is a consequence from the equality $Jv|_{x_n=0}-v=\int e^{ix'\xi'}(1-g_\varepsilon(\xi'))\hat{v}(\xi')\, d_{\xi'}$ (mod $A(X)$) and $1-g_\varepsilon(\xi')=0$, $(\xi')=0, \xi'\in T_{\delta_0/4}$ [24].

The validity of (4.22) will be established following the ideas in [20, 21] for changing the integration's contour.

Consider $J_\varepsilon^{+}v$. Further we will shrink Γ and $\tilde{\Gamma}$ several times without explicitly mentioning it.

In view of (2.1) we may assume that P is strictly hyperbolic with respect to x_0 in Γ_δ and after appropriate change in x'-variables we can write $r_0(x, \xi')=-\xi_0^2+\mu(x', \xi'')$, $\mu>0$, $0<c_1<-\xi_0|\xi'|^{-1}<c_2$, $\xi'\in T_{\delta_0}$. So $dx_0/dt|_{x_n=0}=-2\xi_0>0$ and, replacing t with x_0 as parameter in the zero bicharacteristics of p, we deduce that $\pm dx_n/dx_0>0$ on $\gamma_\rho^{\pm}(x_0)\rho\in H$. The eikonal equation (3.20) and $\partial_{\xi_0}r_0>0$ imply

$$\varphi^+_{\xi_0} = x_0 + x_n\varphi_1(x,\xi') : \varphi_1(x,\xi') < 0 \quad \varphi_1|_{x_n=0} = -\partial_{\xi_0}\sqrt{-r_0(x',\xi')} > 0. \tag{4.23}$$

Let $x^+_{\rho^0}(x_0) = \pi_x(\gamma^+_{\rho^0}(x^0))$ (π_x the projection on the base $X\times(-\delta,\delta)$) and let $\bar{x}\in {}^+\rho^0(x^0)$. Such point $\bar{x}$ exists in view of $\pm dx_n/dx_0 > 0$ on $\gamma^+_{\rho^0}(x^0)$. Evidently (4.23) yield the existence of positive nimber d satisfying

$$\varphi^+_{\xi_0}(\bar{x}.\ \xi') - y_0 < -2d \quad \xi'\in T_{\delta_0},\ y'\in X_0. \tag{4.24}$$

Because φ^+ is analytic function we can find open neighbourhood W of $\bar{x}$, $x_n \geq \delta' > 0$ when $x\in \bar{W}$, open complex cone $\tilde{T}_0 \supset \bar{T}_{\delta_0}$, positive number d_1, such that

$$Re(\varphi^+(x,\zeta') - y'.\zeta') < -d,\ x_0 - y_0 < -d_1,\ (x,y',\zeta')\in W\times X_0\times\tilde{T} = \Omega. \tag{4.25}$$

The Cauchy-Riemann equations with respect to ζ_0 and (4.25) give

$$\begin{aligned}&\partial_{\eta_0} Im\,\varphi^+(x,\ \xi_0 + i\eta_0,\ \xi'' + i\eta'') - y_0 < -d\\ &(x,\ y',\ \zeta')\in\Omega,\ \zeta' = \xi' + i\eta'\ \xi',\ \eta'\in \mathrm{R}^n.\end{aligned} \tag{4.26}$$

Put $\tilde{\varphi}^+(x,y',\zeta') = \varphi^+(x,\zeta') - y'.\zeta'$ and denote by $C(t)$, $t\in[0,1]$ the following contour of integration

$$\mathrm{R}^n \ni \left(\frac{\xi_0}{\xi''}\right) \to C(t) = \begin{cases}\zeta_0 = \xi_0 - iht\,\xi_0\,\tilde{\varphi}^+_{\xi_0}(x,\ y',\ \xi').\\ \xi''\end{cases} \tag{4.27}$$

Here $h>0$ is small, satisfying $\{C(t),\ t\in[0,1]\}\subset \tilde{T}_0$ for $(x,y',\xi')\in W\times X_0\times T_{\delta_0}$ and

$$-\frac{\xi_0 th\tilde{\varphi}^+_{\xi_0}(x,\ y',\ \xi')}{|\xi'|} \geq \frac{1}{2M} \quad (x,\ y',\ \xi')\in W\times X_0\times T_{\delta_0}, \tag{4.28}$$

where M is the constant in (4.16, 4.17).

The inequality (4.26) shows that $Im\,\varphi^+(x,\ \xi_0 + i\eta_0,\ \xi'')\ \eta_0 \leq 0$ and in particular (having in mind the restriction $0 < c_1 < -\xi_0|\xi'|^{-1} < c_2$ $\xi'\in T_{\delta_0}$) one gets

$$\begin{aligned}&Im\,(\tilde{\varphi}^+(x,\ y',\ \zeta_0,\ \xi'')) \geq \frac{1}{2}(-\xi_0)thd^2 \geq \frac{1}{2}thc_1d^2|\xi'|\\ &(x,\ y',\ \zeta_0,\ \xi'')\in W\times X_0\times\{C(t),\ t\in[0,1]\}.\end{aligned} \tag{4.29}$$

Choose now ε to be given by

$$\varepsilon = \min\left(\frac{\varepsilon_0}{2},\ \frac{1}{4}hc_1d^2\right), \tag{4.30}$$

where ε_0 is the constant in (4.12, 4.13).

Then we can write for $x\in W$ (using Stoke's formula)

$$\begin{aligned}J^+_\varepsilon v(x) = &\iint_{X_0\times C(1)} e^{i\tilde{\varphi}^+(x,y',\zeta_0,\xi'')} a^+ . g_\varepsilon(\zeta_0,\ \xi'')v(y')dy'd\xi'\\ &+ \iint_{X_0\times\{C(t),\ t\in[0,1]\}} e^{i\varphi^+(x,y',\xi_0+i\eta_0,\xi'')}[\partial_{\bar{\zeta}_0} a^+ . g_\varepsilon + a^+ . \partial_{\bar{\zeta}_0} g]v(y')dy'\,d\xi'd\eta_0.\end{aligned} \tag{4.31}$$

The choice of ε (4.30) gives us that the term in the first integral is estimated by $C\exp(-\frac{1}{4}hc_1d^2|\xi'|)$ while that in the second integral is bounded by $C\exp((-\varepsilon_0/2)|\xi'|)$ i. e. $J^+_\varepsilon v(x)$ is analytic function in W.

Remark 4.1. In fact we proved $J_\varepsilon^+ v$ is analytic in $\{x_0 < \inf\{y_0 \ni y''.y' \in \operatorname{supp} v\}\}$. But it will not be applied in this paper.

Proof of Theorem 2.2 and Theorem 2.3. Using the parametrices $J_\varepsilon^\pm$, J_ε and microlocal Holmgren's theorem we reduce our investigations to the study of boundary pseudo-differential equations near ρ^0

$$b_\varepsilon^\pm(x', D')v = BJ_\varepsilon^\pm v|_{x_n=0} = g \tag{4.32}$$

when $\rho^0 \in H$ and

$$b_\varepsilon(x', D')v = BJ_\varepsilon v|_{x_n=0} = g \tag{4.33}$$

in the case $\rho^0 \in E$.

The inequality (2.11) ((2.13)) means that the famous (ψ)-condition for the subelliptic operators is violated for $b_\varepsilon^\pm(x', D')(b(x', D'))$. The results in [22] show $b_\varepsilon^\pm(x', D')(b_\varepsilon(x', D'))$ is not analytically hypoelliptic near ρ^0 which proves Theorem 2.2, because u_0 could be chosen as $u_0 = J_\varepsilon^\pm v_0$, v_0 verifying $\rho^0 \in WF_a v_0 \setminus WF_a b^\pm(x', D')v_0$ if (2.11) holds and $\rho^0 \in WFv_0 \setminus WF_a b_\varepsilon(x', D')v_0$ in the case (2.13) valid. Similarly we apply the positive results on the analytic subelliptic p. d. o.-s [22] for the demonstration of Theorem 2.3.

Proof of Theorem 2.4. The assumption that $P(x, D)$ is strictly hyperbolic with respect to x_0 allows us to think x_0 as parameter in the zero bicharacteristics of P. Thus we put $i^{*-1}(\rho) \cap \Sigma_p = \{\rho^+, \rho^-\}$, $\rho \in H$ and $\pm dx_n/dx_0 > 0$ on $\gamma_\rho^\pm(x_0)$.

Let Γ is compactly based conic neighbourhood in $T^*\partial M \setminus 0$, $\bar{\Gamma} \subset H$. It is enough to consider the case $\Gamma = X \times T_0 \subset T^*(\partial\mathcal{U}) \setminus 0$, where $\mathcal{U}$ is a coordinate neighbourhood of $\bar{x} \in \partial M$, verifying (2.5, 2.6).

Let $g \in \varepsilon'(\partial M)$, $WF_\sigma g \subset \Gamma$. In the variables (x', x_n) we write

$$J^+ g(x) = \int e^{i\varphi^+(x,\xi')} a^+(x, \xi') h(\xi') \hat{g}(\xi') \, d\xi'. \tag{4.34}$$

Shrinking eventually Γ, we can assure that the G^σ singular support of $J^+ g \in \mathcal{D}'(\mathcal{U}$ is contained in $\{0 \leq x_n < \delta/3\}$. Choose $\beta(x) \in G_0^\sigma(\mathcal{U})$ such that $\operatorname{supp} \beta \subset \{0 \leq x_n \leq \delta/2$ $x_0 \geq \inf\{y_0 : \exists y'', y' \in \operatorname{sing\,supp}_\sigma g\} - \varepsilon_1, 0 < \varepsilon_1 \ll 1\}$ and $\beta \equiv 1$ in a neighbourhood $\mathcal{U}_1$ of $\operatorname{sing\,supp}_\sigma J^+ g \cap \{x_0 \leq t_0\}$, $\bar{x}_0 < t_0 < \bar{x}_0 + \delta_0$. Put $t_1 = \sup\{x_0 \ni x'', (x', 0) \in \bar{\mathcal{U}}_1\}$. Evidently $w_g = \beta J^+ g$ becomes distribution on $\mathring{M}$, after natural extension as 0 in $M \setminus \mathcal{U}$, and $Pw_g = F$ with $\operatorname{supp} F \subset \mathcal{U}$, $\operatorname{sing\,supp}_\sigma F \subset \mathring{\mathcal{U}}$. So we can extend F as G^σ function in $\mathbb{R}^{n+1} \setminus M$ and, having in mind the definition of t_1, we regard w_g as distribution in $\{x_0 > t_1\}$ equal to 0 in $(M \setminus \mathbb{R}^{n+1}) \cap \{x_0 > t_1\}$.

Let $\alpha \in G^\sigma(\mathbb{R}^{n+1})$, $\alpha \equiv 1$ in $\mathbb{R}^{n+1} \setminus V_1$, $\alpha \equiv 0$ in V_2, where $V_2 \subset V_1$ are neighbourhoods of $\operatorname{sing\,supp}_\sigma F$, $\bar{V}_1 \subset \{x_0 > t_2\} \cap \mathring{\mathcal{U}}$, $t_2 > t_1$,

Consider the following Cauchy problem

$$\begin{aligned} &Pw_g^{(1)} = F \quad \text{in} \quad x_0 > t_3 \\ &w_g^{(1)}|_{x_0=t_3} = w_g|_{x_0=t_3} \\ &\partial_{x_0} w_g^{(1)}|_{x_0=t_3} = \partial_{x_0} w_g|_{x_0=t_3} \qquad t_3 = \frac{t_1+t_2}{2}. \end{aligned} \tag{4.35}$$

The results on strictly hyperbolic operators [3, 7] (taking into account (2.16)) yield $w_g^{(1)} = w_g$ in the strip $t_3 < x_0 < t_2$. Hormander's theorem on the propagation of G^σ

singularities for differential operators of type (2.1) [10] and the requirement that no zero bicharacteristics hits twice ∂M imply

(4.36) $$\operatorname{sing\,supp}_\sigma w_g^{(1)} \cap (\mathbf{R}^{n+1}\setminus M) \cap \{x_0 \geqq t_3\} = \emptyset.$$

Therefore the distribution

$$u_g = \begin{cases} w_g^{(1)} & x_0 > t_3 \\ w_g & x_0 < t_2 \end{cases}$$

verifies (2.17).

Let $t=t_2$. Because $\operatorname{supp} u_g \cap \{x_0 < t_2\} \subset \mathscr{U}$ we will estimate the H^j norms in the local coordinates (2.5, 2.6). Noting that $\beta(x)a^+(x, \xi')$ fulfils the inequalities (4.2), we write again $a^+(x, \xi')$ instead of $\beta(x)a^+(x, \xi')$. Then

(4.37) $$D_x^\alpha u_g(x) = \sum_{\beta\leqq\alpha} \binom{\alpha}{\beta} \int \partial_x^\beta (e^{i\varphi^+}) \partial_x^{\alpha-\beta} a^+ . h(\xi')\hat{g}(\xi')\, d\xi'$$

$$\sum_{\beta\leqq\alpha} \binom{\alpha}{\beta} \sum_{j=0}^{|\beta|} \int e^{i\varphi^+(x,\xi')} d_j^\beta(x, \xi') \partial_x^\alpha a^+(x, \xi') h(\xi') \hat{g}(\xi')\, d\xi',$$

where $\partial_x^\beta (e^{i\varphi^+(x,\xi')}) = e^{i\varphi^+(x,\xi')} \Sigma_{j=0}^{|\beta|} d_j^\beta(x, \xi')$, $\operatorname{ord}_\xi d_j^\beta = j$.

Similarly to (3.11) [10] one proves that for some $C>0$

(4.38) $$|d_j^\beta(x, \xi')| \leqq C^{|\beta|+1} \frac{|\beta|!}{j!} |\xi'|^j \quad (x, \xi') \in \operatorname{supp} a^+$$

$$\beta \in \mathbf{Z}_+^n \quad 0 \leqq j \leqq |\beta|.$$

So we have taking into account (4.2)

$$\| D_x^\alpha u_g \|_0 \leqq \sum_{\beta\leqq\alpha} \sum_{j=0}^{|\beta|} C^{|\alpha|-j+1} ((|\alpha|-j)!)^\sigma \| g \|_j$$

$$\leqq \tilde{C}^{|\alpha|+1} \sum_{j=0}^{|\alpha|} ((|\alpha|-j)!)^\sigma \| g \|_j, \quad \tilde{C} = \tilde{C}(P, t, \beta(x)) > 0$$

which proves the estimate (2.18) because $\| u \|_l^2 = \Sigma_{|\alpha|\leqq l} \| D_x u \|_0^2$, $l \in \mathbf{Z}_+$.

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