# Serdica

Bulgariacae mathematicae publicationes

# Сердика

Българско математическо списание

# HOLOMORPHIC SUBMANIFOLDS OF GENERALIZED *B*-MANIFOLDS

GEORGI DIMITROV DJELEPOV

Let $\tilde{M}$ be a generalized $B$-manifold, a normal generalized $B$-manifold, an almost $B$-manifold or a $B$-manifold. It is proved that if $M$ is a holomorphic submanifold of $\tilde{M}$, then $M$ has the same property. In the case when $M$ is a holomorphic hypersubmanifold of a generalized $B$-manifold $\tilde{M}$ some properties are found. If $\tilde{M}$ and M are almost $B$-manifolds an identity for the second fundamental form is obtained, as well as for the second fundamental tensor. In this case some relations among special sectional curvatures of $\tilde{M}$ and $M$ are found.

**1.** Let $\tilde{M}$ be a pseudo-Riemannian manifold with a metric $g$ and let $M$ be a submanifold of $\tilde{M}$. We also use the symbol $g$ for the restriction on $M$ of the metric of $\tilde{M}$. Let $\nabla$ and $\tilde{\nabla}$ be the corresponding Levi-Civita connections. Then the Gauss formula and the Weingarten formula are as follows:

$$\tilde{\nabla}_x y = \nabla_x y + \sigma(x, y), \quad x, y \in \mathfrak{X}(M), \tag{1.1}$$

$$\tilde{\nabla}_x \xi = -A_\xi x + D_x \xi, \quad x, y \in \mathfrak{X}(M), \tag{1.2}$$

where: $\sigma$ is the second fundamental form, $A$ is the second fundamental tensor, D is the connection on $T(M)^\perp$. It is known that

$$g(\sigma(x, y), \xi) = g(A_\xi x, y) = g(x, A_\xi y), \quad x, y \in \mathfrak{X}(M), \ \xi \in \mathfrak{X}(M)^\perp. \tag{1.3}$$

We note by $\xi$, $\eta$ vector fields in $\mathfrak{X}(M)^\perp$ and by $x, y, z, w$ vector fields in $\mathfrak{X}(M)$. If $x, y, z, w$ are in $\mathfrak{X}(\tilde{M})$, we note this explicitly. According to [1], we have

$$(\nabla_x A)_\xi y = \nabla_x A_\xi y - A_\xi \nabla_x y - A_{D_x \xi} y, \tag{1.4}$$

$$(\nabla_x \sigma)(y, z) = D_x \sigma(y, z) - \sigma(\nabla_x y, z) - \sigma(y, \nabla_x z). \tag{1.5}$$

Let $\tilde{R}$ (resp. $R$) be the curvature tensor field on $\tilde{M}$ (resp. on $M$), and $R^\perp$ be the curvature tensor field on $T(M)^\perp$, i. e.

$$R^\perp(x, y)\xi = D_x D_y \xi - D_y D_x \xi - D_{[x, y]}\xi. \tag{1.6}$$

Then the equations of Gauss, Koddaci and Ricci are as follows:

$$\tilde{R}(x, y, z, w) = R(x, y, z, w) - g(\sigma(x, w), (\sigma(y, z)) + g(\sigma(y, w), \sigma(x, z)), \tag{1.7}$$

$$\tilde{R}(x, y)z^\perp = (\nabla_x \sigma)(y, z) - (\nabla_y \sigma)(x, z), \tag{1.8}$$

$$\tilde{R}(x, y, \xi, \eta) = R^\perp(x, y, \xi, \eta) - g([A_\xi, A_\eta J]x, y). \tag{1.9}$$

A $2n$-dimensional pseudo-Riemannian manifold $\tilde{M}$ is in the class $\mathcal{GB}$ of the generalized $B$-manifold [3] if $\tilde{M}$ admits an almost complex structure $J$ and a $B$-metric $g$, i. e.

*SERDICA Bulgaricae mathematicae publicationes. Vol. 12, 1986, p. 283—287.*

$$J^2=-id,\ g(Jx, Jy)=-g(x,y),\ x,y\in\mathfrak{X}(\tilde{M}). \tag{1.10}$$

If $\tilde{M}\in\mathcal{GB}$ and the Nijenhuis tensor field $\tilde{N}$ of $J$ vanishes identically, then $\tilde{M}$ is in the class $\mathcal{NGB}$ of the normal generalized $B$-manifolds. If $\tilde{M}\in\mathcal{GB}$ and

$$g((\tilde{\nabla}_x J)y,z)+g((\tilde{\nabla}_y J)z,x)+g((\tilde{\nabla}_z J)x,y)=0,\quad x,y,z\in\mathfrak{X}(M), \tag{1.11}$$

then $\tilde{M}$ is in the class $\mathcal{AB}$ of the almost $B$-manifolds [3]. In fact $\mathcal{NGB}\cap\mathcal{AB}=\mathcal{B}$, where $\mathcal{B}$ is the class of the well-known $B$-manifolds [6].

**2.** Let $\tilde{M}\in\mathcal{GB}$. A submanifold $M$ of $\tilde{M}$ is holomorphic if $Jx\in\mathfrak{X}(M)$ for every $x\in\mathfrak{X}(M)$. We note that the holomorphic submanifolds of $B$-manifolds are studied in [7]. Here we consider some properties of holomorphic (non-isotropic) submanifolds in $\mathcal{GB}$. At first we prove the following assertion.

Lemma 1. *Let $\tilde{M}$ be in $\mathcal{GB}$, $\mathcal{NGB}$, $\mathcal{AB}$ or $\mathcal{B}$. If $M$ is a holomorphic submanifold of $\tilde{M}$, then $M$ has the same property.*

Proof. Let $\tilde{M}\in\mathcal{GB}$, $M$ be a holomorphic submanifod of $\tilde{M}$. If $J$ is the restriction on $M$ of the almost complex structure of $\tilde{M}$ and $x,y\in\mathfrak{X}M$, then (1.10) is also satisfied on $M$. Consequently $M\in\mathcal{GB}$. If $\tilde{M}\in\mathcal{NGB}$, then $\tilde{N}=0$. Since the restriction $N$ of $\tilde{N}$ on $M$ is a Nijenhuis tensor field (the proof of this is similar to the proof of the corresponding assertion in the almost Hermitian geometry — see [2]) we have $N=0$, i. e. $M\in\mathcal{NGB}$. Now let $\tilde{M}\in\mathcal{AB}$. From (1.1) it follows

$$(\tilde{\nabla}_x J)y=(\nabla_x J)y+\sigma(x,Jy)-J\sigma(x,y). \tag{2.1}$$

Using (1.11) and (2.1), we get $g((\nabla_x J)y,z)+g((\nabla_y J)z,x)+g((\nabla_z J)x,y)=0$, i. e. $M\in\mathcal{AB}$. Since $\mathcal{B}=\mathcal{NGB}\cap\mathcal{AB}$ it is clear that if $\tilde{M}\in\mathcal{B}$, then $M\in\mathcal{B}$. Hence the lemma is proved.

The assertion of the lemma in the case of $B$-manifolds is proved directly in [7].

Let $M$ be a holomorphic hypersubmanifold of $\tilde{M}\in\mathcal{GB}$. Then there exists an orthonormal $J$-basis [3] in $\mathfrak{X}(M)^\perp$, i. e. a pair of vector fields $\xi$, $J\xi\in\mathfrak{X}(M)^\perp$ such that

$$g(\xi,\xi)=1,\ g(\xi,J\xi)=0. \tag{2.2}$$

Lemma 2. *Let $\tilde{M}(2n\geq 4)$ be in $\mathcal{GB}$. If $M$ is a holomorphic hypersubmanifold of $\tilde{M}$ and $\{\xi, J\xi\}$ is an orthonormal $J$-basis in $\mathfrak{X}(M)^\perp$, then*

$$D_x\xi=0,\ D_xJ\xi=0 \quad \textit{for every } x\in\mathfrak{X}(M). \tag{2.3}$$

Proof. Let $\tilde{M}\in\mathcal{GB}$, $2n\geq 4$, $\{\xi, J\xi\}$ be an orthonormal $J$-basis in $\mathfrak{X}(M)^\perp$, where $M$ is a hypersubmanifold of $\tilde{M}$. Then for every $x$ in $\mathfrak{X}(M)$ the following relations hold:

$$D_x\xi=a\xi+bJ\xi,\ D_xJ\xi=-b\xi+aJ\xi, \tag{2.4}$$

where $a, b$ are smooth functions. From (2.2) we obtain

$$xg(\xi,\xi)=0,\ xg(\xi,J\xi)=0. \tag{2.5}$$

Since $\tilde{\nabla}$ is the Levi-Civita connection (1.2) and (2.5) imply, respectively,

$$g(D_x\xi,\xi)=0,\ g(D_x\xi,J\xi)+g(D_xJ\xi,\xi)=0. \tag{2.6}$$

Using (2.2), (2.4) and (2.6), we get (2.3). Thus the lemma is proved.

Theorem 1. *Let $\widetilde{M}\in\mathcal{GB}$, $2n\geqq 4$, $M$ be a holomorphic hypersubmanifold of $\widetilde{M}$. If $\{\xi, J\xi\}$ is an orthonormal J-basis in $\mathfrak{X}(M)^{\perp}$, then the following assertions are true:*

$$\widetilde{\nabla}_x\xi=-A_\xi x\,; \tag{2.7}$$

$$R^{\perp}(x, y)\eta=0\,; \tag{2.8}$$

$$\widetilde{R}(x,y,\xi,\eta)=-g\,([A_\xi, A_\eta]x, y)\,; \tag{2.9}$$

$$\widetilde{R}(x,y)z^{\perp}=g((\nabla_x A)_\xi y-(\nabla_y A)_\xi x, z)\xi-g\,((\nabla_x A)_{J\xi}y-(\nabla_y A)_{J\xi}x, z)J\xi\,. \tag{2.10}$$

Proof. Equality (2.7) follows directly from (1.2) and (2.3). Let $\eta\in\mathfrak{X}(M)^{\perp}$, i. e $\eta=p\xi+qJ\xi$ for some smooth functions $p$ and $q$. By virtue of (1.6) we have $R^{\perp}(x,y)\xi=R^{\perp}(x,y)J\xi=0$ and consequently (2.8) is true. Then from (1.9) we find (2.9). Evidently $\sigma(x,y)=g(\sigma(x,y)\xi)\xi-g\,(\sigma(x,y), J\xi)J\xi$. Using (1.3) in the last identity, we get

$$\sigma(x, y)=g(A_\xi x, y)\xi-g(A_{J\xi}x,y)J\xi\,. \tag{2.11}$$

Taking into account (2.3) and (2.11), we obtain $D_z\sigma(x,y)=zg\,(A_\xi\, x,y)-zg\,(A_{J\xi}\,x,y)\,J\xi$ and consequently

$$D_z\sigma(x, y)=(g\,(\nabla_z A_\xi x, y)+g(A_\xi x, \nabla_z y))\xi-g(\nabla_z A_{J\xi}x, y)+g(A_{J\xi}x, \nabla_z y))J\xi\,. \tag{2.12}$$

Using (1.4), (1.5), (1.8) and (2.12), we find (2.10). Hence the theorem is proved.

3. Now we consider holomorphic submanifolds of manifolds in $\mathcal{AB}$. At first we prove the following theorem.

Theorem 2. *Let $M$ be a holomorphic submanifold of $\widetilde{M}\in\mathcal{AB}$. Then*

$$J\sigma(x,Jy)+J\sigma(y, Jx)=\sigma(Jx, Jy)-\sigma(x,y), \tag{3.1}$$

$$J\circ A_{J\xi}+A_{J\xi}\circ J=J\circ(A_\xi\circ J+J\circ A_\xi)\,. \tag{3.2}$$

Proof. Let $M$ be a holomorphic submanifold of $\widetilde{M}\in\mathcal{AB}$. Then (1.11) is valid where we put $y=Jx\in\mathfrak{X}(M)$ and $z=\xi\in\mathfrak{X}(M)^{\perp}$. Using the identily [3] $g((\widetilde{\nabla}_z J)y, Jy)=0$, $z,y\in\mathfrak{X}(\widetilde{M})$, we get

$$g((\widetilde{\nabla}_x J)Jx, \xi)+g((\widetilde{\nabla}_{Jx}J)\xi, x)=0\,. \tag{3.3}$$

Equality (1.2) implies

$$(\widetilde{\nabla}_x J)\xi=-A_{J\xi}x+JA_\xi x+D_x J\xi-JD_x\xi\,. \tag{3.4}$$

Taking into account (1.3), (2.1), (3.3) and (3.4), we obtain $g\,(\sigma(x,x),\xi)-2g(J\sigma(x,Jx),\xi)+g(\sigma(Jx,Jx),\xi)=0$ and therefore

$$2J\sigma(x, Jx)=\sigma(Jx, Jx)-\sigma(x, x)\,. \tag{3.5}$$

Setting $x\rightarrow x+y$ in (3.5), we find (3.1). From (1.3) and (3.1) we obtain $g(JA_{J\xi}x,y)+g(A_{J\xi}Jx,y)=g(JA_\xi Jx,y)-g(A_\xi x,y)$, which implies (3.2). Thus the theorem is proved.

Let $M\in\mathcal{GB}$, $p\in M$. If $\{x, y\}$ is a basis of a nondegenerate section $\alpha$ in $T_pM$, then

$$K(x,y)=\frac{R(x,y,x,y)}{g(x,x)\,g(y,y)-g(x,y)^2} \tag{3.6}$$

is the sectional curvature of $\alpha$. If $\alpha=(x, Jx)$ is a nondegenerate holomorphic section in $T_pM$, then the holomorphic sectional curvature $h$ of $\alpha$ is

$$(3.7)\qquad h(x)=-\frac{R(x, Jx, x, Jx)}{g(x, x)^2+g(x, Jx)^2}.$$

The holomorphic bisectional curvature $H$ of the nondegenerate holomorphic sections $(x, Jx)$ and $(y, Jy)$ is as follows [4]:

$$(3.8)\qquad H(x, y)=-\frac{R(x, Jx, y, Jy)}{\sqrt{g(x, x)^2+g(x, Jx)^2}\,\sqrt{g(y, y)^2+g(y, Jy)^2}}.$$

Theorem 3. *Let $M$ be a holomorphic submanifold of $\tilde{M}\in\mathcal{AB}$. If $p\in M$ and $\{x, Jx\}$ is an orthonormal $J$-basis of a nondegenerate holomorphic section $\alpha$ in $T_pM$, then*

$$(3.9)\qquad \tilde{h}(x)=h(x)-\frac{1}{4}\|\sigma(Jx, Jx)+\sigma(x, x)\|^2,$$

*where $\tilde{h}$ and $h$ are the holomorphic sectional curvatures of $\alpha$ with respect to $\tilde{R}$ and $R$ correspondingly.*

Proof. Let $\tilde{M}\in\mathcal{AB}$. Then the following identity [5] holds:

$$(3.10)\qquad \tilde{R}(x, Jx, x, Jx)=-g((\tilde{\nabla}_xJ)x, (\tilde{\nabla}_xJ)x),\quad x\in\mathfrak{X}M.$$

If $M$ is a holomorphic submanifold of $\tilde{M}$, then, according to lemma 1, $M\in\mathcal{AB}$ and consequently $R(x, Jx, x, Jx)=-g((\nabla_xJ)x, (\nabla_xJ)x)$. By virtue of the above identity, (2.1) and (3.5), if we put $x\in\mathfrak{X}M$ in (3.10), we obtain

$$(3.11)\qquad \tilde{R}(x, Jx, x, Jx)=R(x, Jx, x, Jx)+\frac{1}{4}(\sigma(x, x)+\sigma(Jx, Jx), \sigma(x, x)+\sigma(Jx, Jx))\cdot$$

In the case when $\{x, Jx\}$ is an orthonormal $J$-basis of a nondegenerate holomorphic section, (3.7) and (3.11) imply (3.9). Thus the theorem is proved.

Theorem 4. *Let $M$ be a holomorphic submanifold of $\tilde{M}\in\mathcal{AB}$. If $p\in M$, $x, y\in T_pM$, $\|x\|=\varepsilon_1$, $\|y\|=\varepsilon_2$, $\varepsilon_1=\pm1$, $\varepsilon_2\pm1$, $g(x, y)=g(x, Jy)=g(x, Jx)=g(y, Jy)=0$ then*

$$\begin{aligned}&-6\tilde{H}(x, y)-\varepsilon_1\varepsilon_2[\tilde{K}(x, Jy)+\tilde{K}(Jx, y)-\tilde{K}(x, y)-\tilde{K}(Jx, Jy)]\\ (3.12)\qquad &=-6H(x, y)-\varepsilon_1\varepsilon_2[K(x, Jy)+K(Jx, y)-K(x, y)-K(Jx, Jy)]\\ &+\|\sigma(Jx, y)-\sigma(x, Jy)\|^2+\|\sigma(Jx, Jy)+\sigma(x, y)\|^2+g(\sigma(Jx, Jx)+\sigma(x, x), \sigma(Jy, Jy)+\sigma(y, y)),\end{aligned}$$

*where $\tilde{H}$, $\tilde{K}$ and $H$, $K$ are the holomorphic bisectional and holomorphic sectional, curvatures with $\tilde{R}$ and $R$, respectively.*

Proof. If $\tilde{M}\in\mathcal{AB}$ then the following identity [5] is valid $2\tilde{R}(x, Jx, y, Jy)+2\tilde{R}(x, Jy, y, Jx)+\tilde{R}(x, Jy, x, Jy)+\tilde{R}(Jx, y, Jx, y)=-2g((\tilde{\nabla}_xJ)x, (\tilde{\nabla}_yJ)y)-2g((\tilde{\nabla}_xJ)y, (\tilde{\nabla}_yJ)x)-g((\tilde{\nabla}_xJ)y, (\tilde{\nabla}_xJ)y)-g((\tilde{\nabla}_yJ)x, (\tilde{\nabla}_yJ)x)$, $x, y\in\mathfrak{X}(\tilde{M})$.

Let $M$ be a holomorphic submanifold of $\tilde{M}$. Then using the last identity in a way similar to the case in theorem 3, we get the identity

$$\begin{aligned}&2\tilde{R}(x, Jx, y, Jy)+2\tilde{R}(x, Jy, y, Jx)+R(x, Jy, x, Jy)+\tilde{R}(Jx, y, Jx, y)=2\tilde{R}(x, Jx, y, Jy)\\ (3.13)\qquad &+2R(x, Jy, y, Jx)+R(x, Jy, x, Jy)+R(Jx, y, Jx, y)-2g(\sigma(x, Jx)-J\sigma(x, x), \sigma(y, Jy)\\ &-J\sigma(y, y))-g(\sigma(x, Jy)+\sigma(Jx, y)-2J\sigma(x, y), \sigma(x, Jy)+\sigma(Jx, y)-2J\sigma(x, y)).\end{aligned}$$

Making use of (3.1) and (3.5) in (3.13), we obtain

$$2\tilde{R}(x,Jx,y,Jy)+2\tilde{R}(x,Jy,y,Jx)+\tilde{R}(x,Jy,x,Jy)+\tilde{R}(Jx,y,Jx,y)=2R(x,Jx,y,Jy)$$
$$(3.14)\qquad +2R(x,Jy,y,Jx)+R(x,Jy,x,Jy)+R(Jx,y,Jx,y)+\frac{1}{2}g(\sigma(Jx,Jx)$$
$$+\sigma(x,x),\sigma(Jy,Jy)+\sigma(y,y))+g(\sigma(Jx,Jy)+\sigma(x,y),\sigma(Jx,Jy)+\sigma(x,y)).$$

In (3.14) we make the change $y\to Jy$ and summing up with (3.14) we find

$$(3.15)\quad 6\tilde{R}(x,Jx,y,Jy)+\tilde{R}(x,Jy,x,Jy)+\tilde{R}(Jx,y,Jx,y)+\tilde{R}(x,y,x,y)+\tilde{R}(Jx,Jy,Jx,Jy)$$
$$=6R(x,Jx,y,Jy)+R(x,Jy,x,Jy)+R(Jx,y,Jx,y)+R(x,y,x,y)+R(Jx,Jy,Jx,Jy)$$
$$+g(\sigma(Jx,Jx)+\sigma(x,x),\sigma(Jy,Jy)+\sigma(y,y))+g(\sigma(Jx,Jy)+\sigma(x,y),\sigma(Jx,Jy)+\sigma(x,y))$$
$$+g(\sigma(Jx,y)-\sigma(x,Jy),\sigma(Jx,y)-\sigma(x,Jy)).$$

Taking into account (3.6), (3.8), (3.15) and the assumptions of the theorem, we obtain (3.12). Thus the theorem is proved.

## REFERENCES

. K. J a n o, M. K o n. *GR* Submanifolds of Kaehlerian and Sasakian manifolds. (*Progress in Mathematics*, **30**). Basel, 1983.
2. A. G r a y. Minimal varieties and almost Hermitian submanifolds. *Michigan Math. J.*, **12**, 1965 273—287.
3. K. I. G r i b a c h e v, D. G. M e k e r o v, G. D. D j e l e p o v. Generalized *B*-manifold. *G. R. Acad. bulg. Sci.*, **38**, 1985, 299-302.
4. K. I. G r i b a c h e v, G. D. D j e l e p o v, D. G. M e k e r o v. On some subclasses of generalized *B*-manifolds. *C. R. Acad. bulg. Sci.*, **38**, 1985, 437-441.
5. K. I. G r i b a c h e v, D. G. M e k e r o v, G. D. D j e l e p o v. On the geometry of almost *B*-manifolds. *C. R. Acad. bulg. Sci.*, **38**, 1985, 563-567.
6. A. P. Н о р д е н. Об одном классе четырехмерных *A*-пространств. *Известия ВУЗ, Матем.*, **4**, 1960, 145—157.
7. К. И. Г р и б а ч е в. Подмногообразия на *B*-многообразия. *Научни трудове татем.*, ВПИ, Пловдив, **22**, кн. 2, 1984.

*Institute for Agriculture*
*Plovdiv Bulgaria*

*Received 18. 7. 1985*