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THE SELBERG TRACE FORMULA AND A POISSON FORMULA FOR THE p-SPECTRUM OF COMPACT HYPERBOLIC SPACE FORMS

REINHARD SCHUSTER

In this paper we derive the Selberg trace formula for the p-spectrum of the Laplace operator of compact hyperbolic space forms and a related Poisson formula. These formulas state a connection between the eigenvalue spectrum of p-forms and the geometric spectrum given by invariants of the closed geodesic lines and the parallel translation of the tangent space along them. Studying the eigenvalues of mean value operators based on point pair invariant double differential forms we use the theory of Euler-Poisson-Darboux equations. We start deriving the trace formula by the Fourier analysis of the kernels of the mean value operators and the calculation of the trace of the kernel double differential forms.

1. Introduction. A variety of results on the spectrum of the Laplace operator on a compact hyperbolic space form and its geometry as well as the relations between them follow from the Selberg trace formula. For example D. Hejhal [10] extracted successively more information in the case of two dimensions by adjusting the functions g and h (cf. Theorem B) of the trace formula using a certain "balancing process" (cf. [10]). There are a lot of papers treating the two-dimensional case of the trace formula, some results for the n-dimensional case one can find in [1, 16]. By our knowledge the trace formula for the p-spectrum has not yet treated explicitely. A. Selberg has shown in his celebrated paper [15] how to proceed in principle. One usually starts with a point pair invariant function. Generalizing this approach to the differential forms, we will use the point pair invariant geodesic double differential forms introduced by P. Günther [6]. Further on we will see that the mean value operators can be used to advantage. The theory of Euler-Poisson-Darboux (abbreviated EPD) equations turns out to be quite useful for us. The Poisson formula will be obtained by starting from the Fourier analysis of the kernel of the mean value operator used and then calculating its trace. This Poisson formula, which generalizes a result of P. Günther [9], also describes a relation between the spectrum of the Laplace operator and the geometric invariants. Related results were also given by L. Berand-Bergery [1] and H. Riggenbach [14].

Let V be a compact hyperbolic space form of dimension $n \geq 2$, that means a compact Riemannian manifold of constant curvature -1. By the Killing-Hopf theorem there exists a properly discontinous group G of isometries of the n-dimensional real hyperbolic space H_n, the elements $b \in G$ have no fixed point with the exception of the identity map id. V is isometric to H_n/G. Let Ω be the set of nontrivial free homotopy classes of V. In every $\theta \in \Omega$ there lies exactly one closed geodesic line. We denote by $l(\theta)$ and $\nu(\theta)$ its length and multiplicity, respectively. The parallel displacement along the closed geodesic line induces an isometry of the tangent space in every point of the geodesic line with the eigenvalues $\beta_1, \ldots, \beta_{n-1}$, 1 with $|\beta_l| = 1$ $(l = 1, \ldots, n-1)$. Putting $\gamma_l = \operatorname{Re} \beta_l$ we define $\sigma(\theta) = (\nu(\theta))^{-1} \Pi_{l=1}^{n-1} (\cosh l(\theta) - \gamma_l)^{-1/2}$ following (9). Moreover, let $\rho_p(\theta)$ be the p-th elementary symmetric function of the β_l $(l = 1, \ldots, n-1)$ for $p = 1, \ldots, n = 1$ and $\rho_0(\theta) = 1$, which are real numbers.

Following G. de Rham [13] we can write the Laplace operator in the form $\Delta=d\delta+\delta d$ using the differential operator d and the codifferential operator $\delta=(-1)^{pn+n+1} *d*$ for a p-form and the Hodge dualisation $*$. In the space of quadratic integrable p-forms over V (for the definition of the norm and scalar product of p-forms see 2.) there exists a complete orthonormal system of eigenforms $\{\omega_i^p\}_{i\in N}$ of Δ, we have $\Delta\omega_i^p=\mu_i^p\omega_i^p$. We can suppose the eigenforms ω_i^p to be closed ($d\omega_i^p=0$) or coclosed ($\delta\omega_i^p=0$), cf. [2]. Then we can state the Poisson formula:

Theorem A: *As an equation in $D'(R)$ we have for $p\geqq 1$*

$$\Sigma' \cos[\mu_i^p-(p-(n+1)/2)^2]^{1/2}t=\mathfrak{v}_p \cosh|p-(n+1)/2|t+1/2\,\mathrm{Vol}\,VS_{n,p}$$
$$+2^{-(n+1)/2}\sum_{\theta\in\Omega} l(\theta)\sigma(\theta)\rho_{p-1}(\theta)\{\delta_{l(\theta)}+\delta_{-l(\theta)}\}$$

with

$$(1)\qquad \mathfrak{v}_p=\begin{cases}\sum_{l=0}^{p}(-1)^{p-l}B_l & \text{for } n \text{ odd and for } (p\leqq n/2,\ n \text{ even})\\ \sum_{l=0}^{p}(-1)^{p-l}B_l+(-1)^{p+1+(n/2)}\pi^{(1-n)/2}\Gamma(\frac{n+1}{2})\,\mathrm{Vol}\,V & \text{for } p\geqq(n+2)/2,\ n \text{ even.}\end{cases}$$

B_l denotes the l-th Betti number of V, $S_{n,p}$ is given by

$$(2)\qquad S_{n,p}=\begin{cases}\binom{n-1}{p-1}(-4\pi)^{(1-n)/2}\frac{\pi^{1/2}}{\Gamma(n/2)}\prod_{\substack{u=0\\u\neq k}}^{(n-1)/2}(\frac{d^2}{dt^2}-u^2)T_1 & \text{for odd } n\\ \binom{n-1}{p-1}(-4\pi)^{(2-n)/2}\frac{1}{\Gamma(n/2)}\prod_{\substack{u=1/2\\u\neq k}}^{(n-1)/2}(\frac{d^2}{dt^2}-u^2)T_2 & \text{for even } n\end{cases}$$

with the distributions T_1, $T_2\in D'(R)$, $\langle T_1,\varphi\rangle=2\varphi(0)$,

$$\langle T_2,\varphi\rangle=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\cos\mathrm{h}\,\frac{t}{2}\,\frac{1}{\sin\mathrm{h}t}\,\frac{d}{dt}\{\varphi(t)+\varphi(-t)\}dt,\ k=|p-(n+1)/2|.$$

Σ' means that the sum has been taken over those eigenvalues which result from closed eigenforms. The second product in $S_{n,p}$ runs over half numbers (i. e. 1/2, 3/2, 5/2, ...). The related trace formula is described by

Theorem B: *If $h(r)$ is an analytic function on $|\mathrm{Im}\,r|\leqq((n-1)/2)+\delta$ such that $h(r)=h(-r)$ and $|h(r)|\leqq A(1+|r|)^{-n-\delta}$ (A, $\delta>0$) we have with* $g(u)=(2\pi)^{-1}\int_{-\infty}^{+\infty} h(r)e^{-iru}\,dr$

$$\Sigma' h(\sqrt{\mu_i^p-(p-\tfrac{n+1}{2})^2})=\mathfrak{v}_p h(i|p-\tfrac{n+1}{2}|)+\mathrm{Vol}\,V\frac{1}{2}\langle S_{n,p},g\rangle$$
$$+2^{(1-n)/2}\sum_{\theta\in\Omega} l(\theta)\sigma(\theta)\,\rho_{p-1}(\theta)g(l(\theta)).$$

Thereby $\langle S_{n,p},g\rangle$ is given by

$$(3)\qquad \langle S_{n,p},g\rangle=\binom{n-1}{p-1}(4\pi)^{(2-n)/2}\frac{1}{\Gamma(n/2)\pi}\cdot\begin{cases}\int_0^\infty[\prod_{\substack{u=0\\u\neq k}}^{(n-1)/2}(r^2+u^2)]h(r)dr & \text{for odd } n\\ \int_0^\infty[\prod_{\substack{u=1/2\\u\neq k}}^{(n-1)/2}(r^2+u^2)]rh(r)\tanh\pi r dr & \text{for even } n.\end{cases}$$

2. Mean value operators for diferential forms. Let $r=r(\zeta, \eta)$ be the geodesic distance between the points $\zeta, \eta \in \mathrm{H}_n$. P. Günther [6] has introduced the following double differential forms: $\sigma_0=1$, $\tau_0=0$,

$$\sigma_1(\zeta, \eta)=\sin hr(\zeta, \eta)\hat{d}dr(\zeta, \eta), \; \tau_1(\zeta, \eta)=\hat{d}r(\zeta, \eta)\,.\,dr(\zeta, \eta), \tag{4}$$

$$\sigma_p=\frac{1}{p}\sigma_{p-1}\hat{\wedge}\wedge\sigma_1, \; \tau_p=\tau_1\hat{\wedge}\wedge\sigma_{p-1}.$$

$\hat{d}, \hat{\wedge}, \ldots$ show that $d, \wedge, \ldots$ refer to the first variable ζ. An isometry b of H_n induces a mapping b^* for differential forms, see [12, p. 8]. The following property of the point pair invariance is the main reason to use the geodesic double differential forms:

$$\hat{b}^*b^*(\sigma_p(b\zeta, b\eta))=\sigma_p(\zeta, \eta), \tag{5}$$

$$\hat{b}^*b^*(\tau_p(b\zeta, b\eta))=\tau_p(\zeta, \eta).$$

Let $K(\zeta, t)$ be the ball, $S(\zeta, t)$ the sphere around $\zeta\in\mathrm{H}_n$ with a radius t. dv, do shall denote the volume and the survey element, respectively. Let $(x^1, \ldots, x^n)$ be a global coordinate system of H_n, later on we will use the Poincaré model. The scalar product of the differential forms $\varphi=\varphi_{i_1\ldots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ and $\varphi'=\varphi'_{i_1\ldots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ is defined by

$$\varphi\,.\,\varphi'=p!\;\varphi_{i_1\ldots i_p}\varphi'^{i_1\ldots i_p}. \tag{6}$$

We adopt the convention of summing over repeated indices (here from 1 to n). We are lowering and raising indices by the covariant and contravariant metric tensors g_{ij} and g^{ij}, respectively. The norm then is defined by $\|\varphi\|=(\varphi\,.\,\varphi)^{1/2}$. P. Günther initiated the treatment of the spherical mean value operators

$$M^{\sigma}_{n+1}[\varphi](t, \zeta)=(-1)^p\,.\,c_0\,.\,\sin h^{1-n}t\int_{S(\zeta,t)}\sigma_p(\zeta, \eta)\,.\,\varphi(\eta)do_\eta,$$

$$M^{\tau}_{n+1}[\varphi](t, \zeta)=(-1)^p\,.\,c_0\,.\,\sin h^{1-n}t\int_{S(\zeta,t)}\tau_p(\zeta, \eta)\,.\,\varphi(\eta)do_\eta$$

with $c_0=\Gamma(n/2)2^{-1}\pi^{-n/2}$.

In this paper we will use the double differential form

$$\alpha_p(\zeta, \eta)=\sigma_p(\zeta, \eta)+\cos hr(\zeta, \eta)\tau_p(\zeta, \eta)=\frac{1}{p!}(\hat{d}d\cos hr(\zeta, \eta))^p \text{ (exterior power)} \tag{7}$$

and the related sperical mean value operator

$$M_{n+1}[\varphi](t, \zeta)=(-1)^p\,.\,c_0\,.\,\sin h^{1-n}t\int_{S(\zeta,t)}\alpha_p(\zeta, \eta)\,.\,\varphi(\eta)do_\eta. \tag{8}$$

One observes that closed and coclosed eigenforms of the Laplace operator are at the same time eigenforms of the mean value operator M_{n+1}. To make this more precise, we will introduce a function $z(t, \lambda, \mu)$ for $t\geq 0$ as the unique determined solution of the following initial value problem of the EPD-equation

$$\frac{d^2}{dt^2}z(t, \lambda, \mu)+\lambda\cot ht\frac{d}{dt}z(t, \lambda, \mu)+\left\{\mu+\frac{\lambda^2-(n-1)^2}{4}\right\}z(t, \lambda, \mu)=0 \tag{9}$$

$$z(0, \lambda, \mu)=1, \quad \frac{d}{dt}z(t, \lambda, \mu)|_{t=0}=0.$$

It should be noted that

$$z(t, \lambda, \mu)=F\left(\frac{\lambda+\sqrt{(n-1)^2-4\mu}}{2}, \frac{\lambda-\sqrt{(n-1)^2-4\mu}}{2}, \frac{\lambda+1}{2}, \frac{1-\cos ht}{2}\right)$$

using the Gauss hypergeometric function F. Further on we define

(10) $x(t, \lambda, \mu, p)=z(t, \lambda, \mu+(p+1)(n-p)-n)$, $y(t, \lambda, \mu, p)=z(t, \lambda, \mu+p(n+1-p)-n)$,

$$q(\lambda)=p+\frac{\lambda-n-1}{2}, \quad w(t, \lambda-2, \mu, p)=-\frac{\lambda-1-q(\lambda)}{\lambda-1}\sin h^2 t y(t, \lambda, \mu, p)+\cos ht_y(t, \lambda-2, \mu, p).$$

If there is no danger of confusion, we omit the last argument p in x, y and w. By referring to [7, Satz 2] it is quite easy to establish the following result:

Proposition 1: (i) *For* $\Delta\omega=\mu\omega$, $d\omega=0$ *yields* $M_{n+1}[\omega](t, \zeta)=w(t, n-1, \mu)\omega(\zeta)$.
(ii) *For* $\Delta\omega=\mu\omega$, $\delta\omega=0$ *yields* $M_{n+1}[\omega](t, \zeta)=x(t, n-1, \mu)\omega(\zeta)$.

As a consequence of the correspondence principle of EPD theory we have the recursion formula

$$(11) \qquad z(t, \lambda, \mu)=\left(\frac{1}{\lambda+1}\sin ht\frac{d}{dt}+\cos ht\right)z(t, \lambda+2, \mu).$$

One can easily check that the same equation holds for x, y and w instead of z. The following integral equation will turn out to be quite essential.

Proposition 2: *For* $\lambda_2\geqq\lambda_1+2\geqq 2$ *we have*

$$z(t, \lambda_2, \mu)=\frac{2\sin h^{1-\lambda_2}t}{B\left(\frac{\lambda_1+1}{2}, \frac{\lambda_2-\lambda_1}{2}\right)}\int_0^t\{2(\cos ht-\cos h\rho)\}^{(\lambda_2-\lambda_1-2)/2}\sin h^{\lambda_1}\rho \,.\, z(\rho, \lambda_1, \mu)d\rho.$$

For sufficiently large λ_2 this is a consequence of the uniqueness of the solution of the initial value problem (9). For the remaining values of λ_2 we use (11) and a partial integration. After a short computation, we see that Proposition 2 also holds for x, y and w instead of z. Motivated by Propositions 1 and 2 we take for $\lambda\geqq n+3$

$$(12) \qquad M_\lambda[\varphi](t, \zeta)=\frac{2\sin h^{3-\lambda}t}{B\left(\frac{n}{2}, \frac{\lambda-n-1}{2}\right)}\int_0^t\{2(\cos ht-\cos h\rho)\}^{(\lambda-n-3)/2}\sin h^{n-1}\rho M_{n+1}[\varphi]/(\rho,\zeta)d\rho.$$

Combining (12) with (8), we can write M_λ as an integral over the ball $K(\zeta, t)$:

$$M_\lambda[\varphi](t, \zeta)=\frac{\Gamma\left(\frac{\lambda-1}{2}\right)\sin h^{3-\lambda}t}{\Gamma\left(\frac{\lambda-n-1}{2}\right)\pi^{n/2}}\int_{K(\zeta,t)}\{2(\cos ht-\cos hr(\zeta, \eta))\}^{(\lambda-n-3)/2}\alpha_p(\zeta, \eta)\cdot\varphi(\eta)dv_\eta.$$

According to the above propositions, we get

Proposition 3: (i) *For* $\Delta\omega=\mu\omega$, $d\omega=0$ *yields* $M_\lambda[\omega](t, \zeta)=w(t, \lambda-2, \mu)\omega(\zeta)$.
(ii) *For* $\Delta\omega=\mu\omega$, $\delta\omega=0$ *yields* $M_\lambda[\omega](t, \zeta)=x(t, \lambda-2, \mu)\omega(\zeta)$.

Let G be a properly discontinuous group of isometries of H_n. This shall mean that for every $\zeta\in H_n$ the set of $b\zeta$ (for all $b\in G$) has no accumulation point. Let F be a fundamental domain, that means first that the sets bF, $b\in G$, cover the space H_n and secondly that $bF\cap cF$ with $b, c\in G$, $b\neq c$, has Lebesgue measure 0. We suppose F to be compact. Without loss of generality we can suppose F to be the closure of an open, connected domain. We call a differential form φ on H_n G-automorphic, if $b^*\varphi=\varphi$ is valid for all $b\in G$. For G-automorphic differential forms φ we can rewrite the integration as an integration over a fundamental domain F:

(13) $$M_\lambda[\varphi](t,\zeta)=\sinh^{3-\lambda}t\int_F \mathfrak{M}_\lambda(t,\zeta,\eta)\,.\,\varphi(\eta)dv_\eta \quad\text{with}$$

$$\mathfrak{M}_\lambda(t,\zeta,\eta)=\frac{\Gamma(\frac{\lambda-1}{2})}{\Gamma(\frac{\lambda-n-1}{2})\pi^{n/2}}\sum_{\substack{b\in G\\ r(\zeta,b\eta)<t}}\{2(\cosh t-\cosh r(\xi,b\eta))\}^{(\lambda-n-3)/2}b^*\alpha_p(\zeta,b\eta).$$

The induced mapping b^* is to be taken with respect to the second variable of the double differential forms. Since G was supposed to be properly discontinuous only a finite number of terms in the above integral kernel would not vanish.

P r o p o s i t i o n 4: *The kernel double differential form* $\mathfrak{M}_\lambda(t,\zeta,\eta)$ *is symmetric with respect to* ζ *and* η: $\mathfrak{M}_\lambda(t,\zeta,\eta)=\mathfrak{M}_\lambda(t,\eta,\zeta)$.

P r o o f: Setting $F(r(\zeta,b\eta))=\{2(\cosh t-\cosh r(\zeta,b\eta))\}^{(\lambda-n-3)/2}$ we conclude from (5)

$$\sum_{\substack{b\in G\\ r(\zeta,b\eta)<t}}F(r(\zeta,b\eta))b^*\alpha_p(\zeta,b\eta)=\sum_{\substack{b\in G\\ r(\zeta,b\eta)<t}}F(r(b^{-1}\zeta,\eta))\widehat{b^{-1*}}\alpha_p(b^{-1}\zeta,\eta).$$

If b runs through G the same will be true for b^{-1} and we could continue this equation by

$$=\sum_{\substack{b\in G\\ r(\eta,b\zeta)<t}}F(r(b\zeta,\eta))\widehat{b^*}\alpha_p(b\zeta,\eta)=\sum_{\substack{b\in G\\ r(\eta,b\zeta)<t}}F(r(\eta,b\zeta))b^*\alpha_p(\eta,b\zeta).$$

Thus the proof is completed. ■

3. Fourier analysis of the kernels of the mean value operators. There is an one-to-one correspondence between the G-automorphic forms on H_n and the differential forms on V. Using this statement we will do our calculations in H_n. The G-automorphic eigenforms of the Laplace operator on H_n are related to the eigenforms of the Laplace operator on V. We will also denote the corresponding G-automorphic eigenforms on H_n to the eigenforms ω_i^p on V by ω_i^p. It is not difficult to check that the kernel double differential form $\mathfrak{M}_\lambda(t,\zeta,\eta)$ is G-automorphic with respect to both variables ζ and η. In view of the mean value formula and the symmetry of $\mathfrak{M}_\lambda(t,\zeta,\eta)$ it is possible to expand the kernel double form with respect to the complete eigenform system $\{\omega_i^p\}_{i\in\mathsf{N}}$:

(14) $$\mathfrak{M}_\lambda(t,\zeta,\eta)=\sum_{\mu_i^p>0}{}'\Big[-\frac{\lambda-1-q(\lambda)}{\lambda-1}\sinh^2 t\,y(t,\lambda,\mu_i^p)+\cosh t\,y(t,\lambda,\mu_i^p)\Big]\,.\,\sinh^{\lambda-3}t\,\omega_i(\zeta)\omega_i(\eta)$$
$$+\sum_{\mu_i^p\geq 0}{}''\,x(t,\lambda-2,\mu_i^p)\sinh^{\lambda-3}t\;\omega_i(\zeta)\omega_i(\eta).$$

The sum Σ' is taken over eigenvalues of closed eigenforms of Δ (Σ'' for coclosed eigenforms, respectively). First one has to understand the equation (14) in L^2-sense over F with respect to η. But for $\lambda>2n+2$ one gets that (14) is pointwise relevant with respect to ζ and η by standard continuity arguments if one uses the well-known asymptotic behaviour of the eigenforms (see [5]):

$$\sum_{0\leq\mu_i^p\leq\xi}\|\omega_i^p(\zeta)\|^2=O(\xi^{n/2}).$$

This implies by partial summation

(15) $$\sum_{\xi<\mu_i^p}\|\omega_i^p(\zeta)\|^2/(\mu_i^p)^\rho=O(\xi^{(n/2)-\rho})\quad\text{for }\rho>n/2.$$

(13) $$M_\lambda[\varphi](t,\zeta)=\sinh^{3-\lambda} t \int_F \mathfrak{M}_\lambda(t,\zeta,\eta)\,.\,\varphi(\eta)dv_\eta \text{ with}$$

$$\mathfrak{M}_\lambda(t,\zeta,\eta)=\frac{\Gamma(\frac{\lambda-1}{2})}{\Gamma(\frac{\lambda-n-1}{2})\pi^{n/2}}\sum_{\substack{b\in G\\ r(\zeta,b\eta)<t}}\{2(\cos \mathrm{h}t-\cos \mathrm{h}r(\xi,b\eta))\}^{(\lambda-n-3)/2}b^*\alpha_p(\zeta,b\eta).$$

The induced mapping b^* is to be taken with respect to the second variable of the double differential forms. Since G was supposed to be properly discontinuous only a finite number of terms in the above integral kernel would not vanish.

Proposition 4: *The kernel double differential form $\mathfrak{M}_\lambda(t,\zeta,\eta)$ is symmetric with respect to ζ and η: $\mathfrak{M}_\lambda(t,\zeta,\eta)=\mathfrak{M}_\lambda(t,\eta,\zeta)$.*

Proof: Setting $F(r(\zeta,b\eta))=\{2(\cos \mathrm{h}t-\cos \mathrm{h}r(\zeta,b\eta))\}^{(\lambda-n-3)/2}$ we conclude from (5)

$$\sum_{\substack{b\in G\\ r(\zeta,b\eta)<t}} F(r(\zeta,b\eta))b^*\alpha_p(\zeta,b\eta)=\sum_{\substack{b\in G\\ r(\zeta,b\eta)<t}} F(r(b^{-1}\zeta,\eta))\widehat{b^{-1*}}\alpha_p(b^{-1}\zeta,\eta).$$

If b runs through G the same will be true for b^{-1} and we could continue this equation by

$$=\sum_{\substack{b\in G\\ r(\eta,b\zeta)<t}} F(r(b\zeta,\eta))\widehat{b^*}\alpha_p(b\zeta,\eta)=\sum_{\substack{b\in G\\ r(\eta,b\zeta)<t}} F(r(\eta,b\zeta))b^*\alpha_p(\eta,b\zeta).$$

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(14) $$\mathfrak{M}_\lambda(t,\zeta,\eta)=\sum_{\mu_i^p>0}{}' \left[-\frac{\lambda-1-q(\lambda)}{\lambda-1}\sin \mathrm{h}^2 t y(t,\lambda,\mu_i^p)+\cos \mathrm{h}t y(t,\lambda,\mu_i^p)\right].\sin \mathrm{h}^{\lambda-3}t\omega_i(\zeta)\omega_i(\eta)$$
$$+\sum_{\mu_i^p\geq 0}{}'' x(t,\lambda-2,\mu_i^p)\sin \mathrm{h}^{\lambda-3}t\ \omega_i(\zeta)\omega_i(\eta).$$

The sum Σ' is taken over eigenvalues of closed eigenforms of Δ (Σ'' for coclosed eigenforms, respectively). First one has to understand the equation (14) in L^2-sense over F with respect to η. But for $\lambda>2n+2$ one gets that (14) is pointwise relevant with respect to ζ and η by standard continuity arguments if one uses the well-known asymptotic behaviour of the eigenforms (see [5]):

$$\sum_{0\leq\mu_i^p\leq\xi}\|\omega_i^p(\zeta)\|^2=O(\xi^{n/2}).$$

This implies by partial summation

(15) $$\sum_{\xi<\mu_i^p}\|\omega_i^p(\zeta)\|^2/(\mu_i^p)^\rho=O(\xi^{(n/2)-\rho}) \text{ for } \rho>n/2.$$

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)=\frac{\delta_{ij}}{(x^n)^2}, \quad i, j=1, \ldots, n, \tag{19}$$

δ_{ij} being the Kronecker symbol. To go further, we recall that

$$\cos hr(\zeta, \eta)=1+((x^1-y^1)^2+\cdots+(x^{n-1}-y^{n-1})^2+(x^n-y^n)^2)/2x^ny^n \tag{20}$$

holds for the geodesic distance $r(\zeta, \eta)$ of the points $\zeta=(x^1, \ldots, x^n)$, $\eta=(y^1, \ldots, y^n) \in \mathrm{H}_n$. It is known (see [14]) that there exists a special Poincaré coordinate system $S(b)$ for every isometry b of H_n in which

$$y^i=e^{l(b)}\sum_{k=1}^{n-1} \alpha^i_k x^k, \quad i=1, \ldots, n-1, \; y^n=e^{l/b}x^n \tag{21}$$

is valid for $\eta=(y^1, \ldots, y^n)=b\zeta$, $\zeta=(x^1, \ldots, x^n)$ with an orthogonal $(n-1)$-matrix (α^i_k) From this we deduce

$$b^*(dy^i)=e^{l/b}\sum_{k=1}^{n-1} \alpha^i_k (dx^k), \quad i=1, \ldots, n-1, \; b^*(dy^n)=e^{l(b)}dx^n \tag{22}$$

for the corresponding differentials. According to H. Riggenbach [14], the eigenvalues $\beta_1, \ldots, \beta_{n-1}$ of the free homotopy class θ corresponding to the conjugacy class determined by $b \in G$ coincide with those of the matrix (α^i_k). For this reason we can express the weights ρ_p and σ in terms of α^i_k. We will always use those special Poincaré coordinate systems $S(b^m_l)$ with respect to b^m_l. Then it is not difficult to check (see [14]) that

$$F_l=\{\zeta=(x^1, \ldots, x^n), \; 1 \leq x^n \leq e^{l(b_l)}\} \tag{23}$$

s a fundamental domain of G_l. Using (23) and the volume element $dv_\zeta=(x^n)^{-n}dx^1 \ldots dx^n$, we get

$$\int_{F_l} E dv_\zeta = \int_{x^n=1}^{\exp l(b_l)} \int_{(x^1, \ldots, x^{n-1}) \in \mathrm{R}^{n-1}} E \, (x^n)^{-n}dx^1 \ldots dx^{n-1}dx^n \tag{24}$$

with

$$E=\{2(\cos ht-\cos hr(\zeta, b^m_l\zeta))\}_+^{(\lambda-n-3)/2} \, tr\, b^{m*}_l \alpha_p(\zeta, b^m_l\zeta).$$

Now we shall turn to the calculation of $tr\, b^{m*}_l \alpha_p(\zeta, b^m_l\zeta)$. From (7), (20) we get

$$\alpha_p(\zeta, \eta)=\psi^{(1)}(\zeta, \eta)+(x^i-y^i)(x^j-y^j)\psi^{(2)}_{ij}(\zeta, \eta)+\psi^{(3)}(\zeta, \eta) \tag{25}$$

with

$$\psi^{(1)}(\zeta, \eta)=\frac{1}{p!}\{-\delta_{ij}\frac{1}{x^n y^n}dx^i dy^j\}^p \tag{26}$$

$$+\frac{1}{(p-1)!}\{-\delta_{ij}\frac{1}{x^n y^n}dx^i dy^j\}^{p-1}\{\frac{\cos hr(\zeta, \eta)}{x^n y^n}-\frac{1}{(x^n)^2}-\frac{1}{(y^n)^2}\} \,\widehat{\wedge}\, \wedge dx^n dy^n,$$

$$\psi^{(2)}_{ij}(\zeta, \eta)=-\frac{1}{(p-2)!}\{-\delta_{kl}\frac{1}{x^n y^n}dx^k dy^l\}^{p-2}\frac{1}{(x^n y^n)^3} \,\widehat{\wedge}\, \wedge dx^i dy^n \,\widehat{\wedge}\, dx^n dy^j. \tag{27}$$

$\{ \; \}^p$ thereby denotes the exterior power. If we use the convention of summing over repeated indices in connection with the Poincaré coordinate systems, we will always sum from 1 to $n-1$ ($i, j, k, l=1, \dots, n-1$). If $p-1$ or $p-2$ in $1/(p-1)!$ or $1/(p-2)!$ are negative we omit the related term. $\psi^{(3)}(\zeta, \eta)$ consists of those terms in which one and only one of the two differentials dx^n, dy^n appears. Applying (17), (19) and (22), we get $tr\, b_l^{m*}\psi^{(3)}(\zeta, b_l^m \zeta)=0$. Consequently $\psi^{(3)}(\zeta, \eta)$ is of no further interest for us. For $\zeta=(x^1, \dots, x^n)$, $\eta=b_l^m\zeta=(y^1, \dots, y^n)$ we have

$$(28)\qquad tr\, b_l^{m*}\{(x^i-y^i)(x^j-y^j)\psi_{ij}^{(2)}(\zeta, b_l^m\zeta)\}=(x^i-y^i)(x^j-y^j)A_{ij}$$

with $A_{ij}=tr\, b_l^{m*}\psi_{ij}^{(2)}(\zeta, b_l^m \zeta)$.

(22) and (27) show that A_{ij} does not depend on $x^1, \dots, x^{n-1}$. In order to calculate the right hand side of (24) the coordinate transformation

$$(29)\qquad z^i=x^i-y^i(x^1, \dots, x^{n-1}), \quad i=1, \dots, n-1,$$

will turn out to be quite useful. Indeed, the determinant of this mapping is different from 0. If that was not the case then the 0 would have been an eigenvalue of the linear map (29) and consequently $(x^1, \dots, x^{n-1})^T=e^{l(b_l^m)}(\alpha^i_k(b_l^m))(x^1, \dots, x^{n-1})^T$ would hold for a corresponding eigenvector $(x^1,\dots, x^{n-1})^T$. But this can't be the case for $b_l^m\neq 0$ because of the fact that an orthogonal matrix preserves $(x^1)^2+\dots+(x^{n-1})^2$. Let D be the absolute value of the determinant of the inverse mapping $(z^1, \dots, z^{n-1})\to(x^1, \dots, x^{n-1})$. If do denotes the survey element of the sphere $(z^1)^2+\dots+(z^{n-1})^2=\rho^2$ in $\mathbb{R}^{n-1}$ we obtain

$$\int_{(x^1, \dots, x^{n-1})\in \mathbb{R}^{n-1}} tr\,(b_l^m)^*\{(x^i-y^i)(x^j-y^j)\psi_{ij}^{(2)}(\zeta, b_l^m\zeta)\}\{2(\cos\text{h}t-\cos\text{h}r(\zeta, b_l^m\zeta))\}_+^{(\lambda-n-3)/2}$$

$$\cdot\, dx^1\dots dx^{n-1}=\frac{1}{D}\int_{(z^1, \dots, z^{n-1})\in \mathbb{R}^{n-1}} z^iz^jA_{ij}\{2(\cos\text{h}t-\cos\text{h}r(\zeta, b_l^m\zeta))\}_+^{(\lambda-n-3)/2}dz^1\dots dz^{n-1}$$

$$=\frac{1}{D}\int_{\rho=0}^{\infty}\frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n-1)/2}}\rho^{2-n}\{2(\cos\text{h}t-\cos\text{h}l(b_l^m)-\frac{\rho^2}{2x^ny^n})\}_+^{(\lambda-n-3)/2}\int_{(z^1)^2+\dots+(z^{n-1})^2=\rho^2} z^iz^jA_{ij}do.$$

Thereby we have used $\cos\text{h}r(\zeta, b_l^m\zeta)=1+((z^1)^2+\dots+(z^{n-1})^2+(x^n-y^n)^2)/2x^ny^n=(\rho^2/2x^ny^n)+\cos\text{h}l(b_l^m)$.

Using the well-known equation

$$\int_{(z^1)^2+\dots+(z^{n-1})^2=\rho^2} z^iz^jdo=\int_{(z^1)^2+\dots+(z^{n-1})^2=\rho^2}\delta^{ij}\frac{\rho^2}{n-1}do,$$

we obtain by transforming back to $(x^1, \dots, x^{n-1})$-coordinates

$$\int_{(x^1, \dots, x^{n-1})\in \mathbb{R}^{n-1}} tr\,(b_l^m)^*\{(x^i-y^i)(x^j-y^j)\psi_{ij}^{(2)}(\zeta, b_l^m\zeta)\}$$
$$\times\{2(\cos\text{h}t-\cos\text{h}r(\zeta, b_l^m\zeta))\}_+^{(\lambda-n-3)/2}dx^1\dots dx^{n-1}$$

$$=\int_{(x^1, \dots, x^{n-1})\in \mathbb{R}^{n-1}}\{2(\cos\text{h}t-\cos\text{h}r(\zeta, b_l^m\zeta))\}_+^{(\lambda-n-3)/2}\,2x^ny^n\frac{1}{n-1}$$
$$\times(\cos\text{h}r(\zeta, b_l^m\zeta)-\cos\text{h}l(b_l^m))\,tr\,(b_l^m)^*\delta^{ij}\psi_{ij}^{(2)}(\zeta, b_l^m\zeta)dx^1\dots dx^{n-1}.$$

Combining this with (25) and (27), we get

$$\int_{F_l} E dv_\zeta = \int_{F_l} \{2(\cos ht - \cos hr(\zeta, b_l^m\zeta))\}_+^{(\lambda-n-3)/2} tr(b_l^m)^* \{\psi^{(1)}(\zeta, b_l^m\zeta) + 2x^n y^n \frac{1}{n-1}(\cos hr(\zeta, b_l^m\zeta) - \cos hl(b_l^m))\delta^{ij}\psi_{ij}^{(2)}(\zeta, b_l^m\zeta)\} dv_\zeta. \tag{30}$$

Applying (26), (27), we get by direct calculation

$$tr(b_l^m)^* \{\psi^{(1)}(\zeta, b_l^m\zeta) + 2x^n y^n \frac{1}{n-1}(\cos hr(\zeta, b_l^m\zeta) - \cos hl(b_l^m))\delta^{ij}\psi_{ij}^{(2)}(\zeta, b_l^m\zeta)\} \tag{31}$$
$$= (-1)^p \rho_p(b_l^m) + (-1)^p \frac{2(p-1)}{n-1}(\cos hr(\zeta, b_l^m\zeta) - \cos hl(b_l^m))\rho_{p-1}(b_l^m)$$
$$+ (-1)^{p-1}(\cos hr(\zeta, b_l^m\zeta) - 2\cos hl(b_l^m))\rho_{p-1}(b_l^m).$$

Thereby we have used $tr(b_l^m)^*\{\delta_{ij}dx^i dy^j\}^p = p!\, \rho_p(b_l^m) e^{pl(b_l^m)}(x^n)^{2p}$. We are now ready to start the computation of $\int_{F_l} E dv_\zeta$. This will be done by applying the following

Proposition 6: *Let $g(\tau)$ be a continuous function defined for $\tau \geq 0$. Then we have*

$$\int_{F_l} g(\cos hr(\zeta, b_l^m\zeta))\, dv_\zeta = \frac{l(b_l^m)}{\nu(b_l^m)} \prod_{s=1}^{n-1} \left(\frac{\cos hl(b_l^m)}{\cos hl(b_l^m) - \gamma_s(b_l^m)}\right)^{1/2} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^\infty g(\cos hl(b_l^m)$$
$$\times (1+\rho^2))\rho^{n-2} d\rho$$

if the integral on the right-hand side exists.

In [9, Proposition 3.1] a similar assertion is stated and the proof can be done similar to those of [14, 8.4.]. If we use the integrand on the right-hand side of (30) instead of function g of Proposition 6 in view of (31) we will obtain

$$S = \sum_{\substack{l \in L \\ m \in Z \setminus \{0\}}} l(b_l^m)\, \sigma(b_l^m) \{\{2(\cos ht - \cos hl(b_l^m))\}_+^{(\lambda-2)/2} \frac{2p-n-\lambda+1}{2(\lambda-2)} \rho_{p-1}(b_l^w)$$
$$+ \{2(\cos ht - \cos hl(b_l^m))\}_+^{(\lambda-4)/2}(\rho_{p-1}(b_l^m)\cos ht + \rho_p(^{ml}))\}.$$

If we consider the correspondence between the free homotopy classes of H_n/G and the conjugacy classes of G we can also express this last equation in terms of the free homotopy classes:

$$S = \sum_{\theta \in \Omega} l(\theta)\sigma(\theta)\{\{2(\cos ht - \cos hl(\theta))\}_+^{(\lambda-2)/2} \frac{2p-n-\lambda+1}{2(\lambda-2)} \rho_{p-1}(\theta) \tag{32}$$
$$+ \{2(\cos ht - \cos hl(\theta))\}_+^{(\lambda-4)/2}(\rho_{p-1}(\theta)\cos ht + \rho_p(\theta))\}.$$

Now we shall turn to the reverse of the coin, i. e. we will apply the Fourier analysis of $\mathfrak{M}_\lambda$ by taking the trace for $\zeta = \eta$ and integrating over F. Therefore we use $\int_F \omega_i(\zeta) . \omega_i(\zeta) dv_\zeta = 1$. If we include the summand for $b = id$ following the same considerations as those done above we will obtain finally

(33)
$$\pi^{-\frac{n}{2}} \frac{\Gamma(\frac{\lambda-1}{2})}{\Gamma(\frac{\lambda-n-1}{2})} \binom{n}{p} \operatorname{Vol} V\{2(\cos ht-1)\}^{(\lambda-n-3)/2}$$

$$+\frac{\Gamma(\frac{\lambda-1}{2})}{\Gamma(\frac{\lambda-2}{2})} 2^{(1-n)/2}\pi^{-1/2} \sum_{\theta\in\Omega} l(\theta)\sigma(\theta)\{\{2(\cos ht-\cos hl(\theta))\}_+^{(\lambda-2)/2} \frac{2p-n-\lambda+1}{2(\lambda-2)} \rho_{p-1}(\theta)$$

$$+\{2(\cos ht-\cos hl(\theta))\}_+^{(\lambda-4)/2}(\rho_{p-1}(\theta)\cos ht+\rho_p(\theta))\}$$

$$=\sum_{\mu_i^p>0}{}' \left[-\frac{\lambda-1-q(\lambda)}{\lambda-1} \sin h^2 t y(t, \lambda, \mu_i^p, p)+\cos hty(t, \lambda-2, \mu_i^p, p)\right]\sin h^{\lambda-3} t$$

$$+\sum_{\mu_i^p\geq 0}{}'' x(t, \lambda-2, \mu_i^p, p)\sin h^{\lambda-3} t.$$

This equation includes the values λ, $\lambda-2$ and the weights ρ_p, ρ_{p-1} for the degrees p and $p-1$. It would be nice if we could derive an equation which contains only λ, ρ_p and from which we could obtain the original equation. This goal is reached by

(34)
$$G(p, \lambda)=\sum_{\mu_i^p\geq 0}{}' \sin h^{\lambda-1} ty(t, \lambda, \mu_i^p, p)=\mathfrak{v}_p \sin h^{\lambda-1} ty(t, \lambda, 0, p)+\mathfrak{T}(t, \lambda, p)$$

$$+\frac{\Gamma(\frac{\lambda+1}{2})}{\Gamma(\frac{\lambda}{2})} 2^{(1-n)/2}\pi^{-1/2} \sum_{\theta\in\Omega} l(\theta)\sigma(\theta)\rho_{p-1}(\theta)\{2(\cos ht-\cos hl(\theta))\}_+^{(\lambda-2)/2}$$

(35) with
$$u(t, \lambda, p)=\pi^{-n/2} \frac{1}{\Gamma(\frac{n}{2})} \binom{n-1}{p-1} \operatorname{Vol} V$$

$$\times \sum_{u=0}^{\frac{n}{2}-\frac{1}{2}-k} 2^{-2u} \frac{\Gamma(\frac{n}{2}-u)\Gamma(\frac{\lambda+1}{2})}{\Gamma(\frac{\lambda-n+1}{2}+u)} \prod_{v=0}^{u} ((\frac{n+1}{2}-v)^2-k^2)\{2(\cos ht-1)\}^{\frac{\lambda-n-1}{2}+u},$$

$$k=|p-\frac{n+1}{2}|.$$

$\mathfrak{v}_p$ is given by (1). For $p=1$ (34) is known from [9]. Let us suppose that (34) is proved for a certain value of p. We consider $G(p, \lambda-2)\cosh t-(\lambda-1-q(\lambda))/(\lambda-1)\times G(p, \lambda)$. In this way we get $G(p+1, \lambda-2)$ if we use (33) and

$$(-B_p+\mathfrak{v}_{p-1})\, x(t, \lambda-2, 0, p) \sin h^{\lambda-3} t+\pi^{-n/2}\frac{\Gamma(\frac{\lambda-1}{2})}{\Gamma(\frac{\lambda-n-1}{2})} \binom{n}{p} \operatorname{vol} V\{2(\cos ht-1)\}^{(\lambda-n-3)/2}$$

(36)
$$\mathfrak{T}-(t, \lambda-2, p)\cos ht+\frac{\lambda-q(\lambda)-1}{\lambda-1} \mathfrak{T}(t, \lambda, p)$$

$$=-\mathfrak{v}_p x(t, \lambda-2, 0, p)\sin h^{\lambda-3} t+\mathfrak{T}(t, \lambda-2, p+1).$$

Thereby we have to take use of the property of the spectrum of positive eigenvalues of Δ which is corresponding to coclosed p-forms coincides with the part of the spectrum of positive eigenforms of Δ which is corresponding to closed $(p+1)$-forms (Telescopage theorem of Mac Kean-Singer, cf. [2]), and also that $x(t, \lambda, \mu, p)=y(t, \lambda, \mu, p+1)$. Thus we obtain

$$\sum_{\mu_i^p>0}'' x(t, \lambda-2, \mu_i^p, p) \sin h^{\lambda-3}t = \sum_{\mu_i^{p+1}>0}' y(t, \lambda-2, \mu_i^{p+1}, p+1) \sinh^{\lambda-3} t.$$

(36) can be proved straight-forward, but the calculations are nevertheless rather long. Thereby it is useful to consider the cases $k>1/2$, $k=1/2$, $k=0$ for $k=|p-(n+1/2)|$ separately. In order to prove the Poisson formula, we start with the equation (34) for $\lambda=2n+2$ and apply the differential operator $\frac{d}{dt}\wedge^{(n)}$ with $\wedge=\frac{1}{\sin ht}\frac{d}{dt}$ on it. By using (11) we get

$$\frac{d}{dt}\wedge^{(n)}\{\sin h^{2n+1} t y(t, 2n+2, \mu_i^p)\} = \frac{\Gamma(2n+2)}{\Gamma(n+1)} 2^{-n} y(t, 0, \mu_i^p) \tag{37}$$

$$= \frac{\Gamma(2n+2)}{\Gamma(n+1)} \cos\sqrt{\mu_i^p-(\frac{n+1}{2}-p)^2}\, t.$$

It is not difficult to check that we have as distributions in $D'(\mathbf{R})$

$$\frac{d}{dt}\wedge^{(n)} \operatorname{sign} t\{2(\cos ht-1)\}^{\frac{n+1}{2}+u} = \pi^{\frac{n-1}{2}-u} 2^n \Gamma(\frac{n+3}{2}+u) T_{n-2u} \tag{38}$$

with

$$\langle T_m, \varphi\rangle = \begin{cases} (-2\pi)^{(1-m)/2} \wedge^{(m-1)/2}\{\varphi(t)+\varphi(-t)\}|_{t=0} & \text{for } m \text{ odd} \\ (-2\pi)^{-m/2} \int_{-\infty}^{+\infty} \cos h\frac{t}{2} \wedge^{m/2}\{\varphi(t)+\varphi(-t)\}dt & \text{for } m \text{ even.} \end{cases} \tag{39}$$

$\varphi \in D(\mathbf{R})$ is a test function. By (2) one can check that

$$S_{n,p} = \frac{1}{\Gamma(\frac{n}{2})}\binom{n-1}{p-1} \sum_{u=0}^{\frac{n-1}{2}-k} (4\pi)^{-u} \Gamma(\frac{n}{2}-u) \prod_{v=1}^{u} [(\frac{n+1}{2}-v)^2-k^2] T_{n-2u}.$$

Further on we have

$$\frac{d}{dt}\wedge^{(n)} \operatorname{sign} t\{2(\cos ht-\cos hl(\theta))\}_+^n = \Gamma(n+1)2^n\{\delta_{l(\theta)}+\delta_{-l(\theta)}\}. \tag{40}$$

Summarizing (37), (38) and (40), we deduce the Poisson formula stated in theorem A from (34) by applying $\frac{d}{dt}\wedge^{(n)}$.

In order to prove the trace formula we use an even test function $g \in D(\mathbf{R})$ and its Fourier transform $h(r)=\int_{-\infty}^{+\infty} g(u)e^{-iru}du$. Then h is an even function, too. Applying elementary integrals we can write $\langle T_{1,g}\rangle = 2\pi^{-1}\int_0^\infty h(r)dr$, $\langle T_{2,g}\rangle = \pi^{-1}\int_0^\infty \tan h\, \pi r\, h(r)dr$. In view of (2) it is quite easy to establish (3). The fact that the Fourier transform of $\cos x$ is $(2\pi)^{-1}(\delta_x+\delta_{-x})/(2\pi)^{-1}$ implies $\langle \cos x\cdot, g\rangle = \langle(\delta_x+\delta_{-x})/2, h\rangle$. If we use theorem A with respect to a test function $g \in D(\mathbf{R})$ we will get theorem B with this premise at first. By a standard approximation argument (cf. [4, 10]) we complete the proof of theorem B.

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