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AN AXIOMATIZATION OF COUNTERS

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The Translation Independence Theorem of [3] establishes that the operation translation in iterative spaces can not be eliminated by making use of the operations multiplication, pairing, iteration and the constants L, R. This is shown in the present work to be no longer true if constants W, W_1, W_2 which express axiomatically the availability of an extra counter are allowed. (Some applications of the latter result to the mathematical theory of programs are dealt with in the forthcoming paper [4]). A general method for imbedding given spaces into wider ones which do have the constants W, W_1, W_2 is proposed, too.

1. Preliminaries. We adduce some definitions and auxiliary statements which, together with some of the lemmas 1.1—1.24 of [1] (called there propositions), will be needed for the present considerations.

The 5-tuple $\mathcal{S}=(\mathcal{F}, I, \Pi, L, R)$ is an iterative operative space (operatory space, in the referred earlier papers) if $(\mathcal{F}, \circ, \leq)$ is a partially ordered semigroup with a unit I, the operation $\Pi: \mathcal{F}^2 \to \mathcal{F}$, pairing, is monotonous and $L \neq R \in \mathcal{F}$ s. t., writing (φ, ψ) for $\Pi(\varphi, \psi)$, it is the case that $(\varphi, \psi)\chi=(\varphi\chi, \psi\chi)$, $L(\varphi, \psi)=\varphi$, $R(\varphi, \psi)=\psi$ and there are operations $\langle\ \rangle\ [\ ]: \mathcal{F} \to \mathcal{F}$, translation and iteration, s. t.

$$(£) \qquad (\varphi L, \langle\varphi\rangle R) \leq \langle\varphi\rangle,$$

$$R\psi \leq \psi\psi_1 \ \&\ (\varphi L\psi, \tau\psi_1) \leq \tau \Rightarrow \langle\varphi\rangle\psi \leq \tau;$$

$$(££) \qquad (I, \varphi[\varphi]) \leq [\varphi],$$

$$(\psi, \varphi\tau) \leq \tau \Rightarrow [\varphi]\psi \leq \tau.$$

The operation $\Delta(\varphi, \psi)=\langle\varphi\rangle[\langle\psi\rangle]$ plays an important role, too.

An element φ is recursive in $\mathcal{B} \subseteq \mathcal{F}$ if it can be obtained from L, R and members of $\mathcal{B}$ by means of the initial operations $\circ$, Π, $\langle\ \rangle$, $[\ ]$, while φ is a prime recursive in $\mathcal{B}$, if it can be obtained in the same way, bar $\langle\ \rangle$.

Write A, B, C respectively for (R, RL), (LR, L) and $\Delta(B^2, A^2)$. The following two statements improve lemmas 1.21, 1.22 [1].

First Recursion Lemma. Let $\sigma=C([\langle\psi\rangle]L, \langle\varphi\rangle R^2)$ and $[\varphi, \psi]=LR[\sigma]$. Then $(I, \varphi[\varphi, \psi]\psi)=[\varphi, \psi]$ and whenever $\psi\chi \leq \chi\chi_1$ and $(\chi, \varphi\ \tau\chi_1) \leq \tau$, then $[\varphi, \psi]\chi \leq \tau$.

Proof. The equality $(I, \varphi[\varphi, \psi]\psi)=[\varphi, \psi]$ is established in lemma 1.21 [1].

Suppose that $\psi\chi \leq \chi\chi_1$, $(\chi, \varphi\tau\chi_1) \leq \tau$. Making use of lemmas 1.4, 1.13 [1] one gets

$$(\chi, \langle\psi\rangle\Delta(\chi, \chi_1))=(\chi, \langle\psi\chi\rangle[\langle\chi_1\rangle]) \leq (\chi, \langle\chi\chi_1\rangle[\langle\chi_1\rangle])=(\chi L, \langle\chi\rangle R)[\langle\chi_1\rangle]=\Delta(\chi, \chi_1)$$

which implies $[\langle\psi\rangle]\chi \leq \Delta(\chi, \chi_1)$ by (££). Therefore,

$$(\chi, \sigma(\chi, \Delta(\tau, \chi_1)))=(\chi, C([\langle\psi\rangle]\chi, \langle\varphi\rangle R\Delta(\tau, \chi_1))) \leq (\chi, C(\langle\chi\rangle[\langle\chi_1\rangle], \langle\varphi\rangle\langle\tau\rangle\langle\chi_1\rangle[\langle\chi_1\rangle]))$$

$$=(\chi, C(\langle\chi\rangle, \langle\varphi\tau\chi_1\rangle)[\langle\chi_1\rangle])=(\chi, \langle(\chi, \varphi\tau\chi_1)\rangle[\langle\chi_1\rangle]) \leq (\chi, \langle\tau\rangle[\langle\chi_1\rangle])=(\chi, \Delta(\tau, \chi_1))$$

by lemmas 1.4, 1.13, 1.19 [1], hence $[\sigma]\chi \leq (\chi, \Delta(\tau, \chi_1))$ by (££). Therefore, $[\varphi, \psi]\chi \leq LR(\chi, \Delta(\tau, \chi_1))=\tau$. The proof is completed.

Second Recursion Lemma. Let $\varphi_1=\varphi(\psi L, \chi R)$ and $[\varphi, \psi, \chi, \rho]=\varphi_1[\varphi_1, \rho]$. Then

$$(\S) \qquad \varphi(\psi, \chi[\varphi, \psi, \chi, \rho]\rho) \leq [\varphi, \psi, \chi, \rho],$$

$$\rho\sigma \leq \sigma\sigma_1 \ \&\ \varphi(\psi\sigma, \chi\tau\sigma_1) \leq \tau \Rightarrow [\varphi, \psi, \chi, \rho]\sigma \leq \tau.$$

Proof. The inequality of (§) (in fact, equality) is established in lemma 1.22 [1]. As for the implication, the premises imply $[\varphi_1, \rho]\sigma \leqq (\sigma, \tau\sigma_1)$ by the First Recursion Lemma, hence

$$[\varphi, \psi, \chi, \rho]\sigma \leqq \varphi_1(\sigma, \tau\sigma_1) = \varphi(\psi\sigma, \chi\tau\sigma_1) \leqq \tau,$$

which completes the proof.

Notice that $\langle\varphi\rangle = [I, \varphi L, I, R]$, $[\varphi] = [I, I, \varphi, I]$, $\Delta(\varphi, \psi) = [I, \varphi, I, \psi]$ and (£), (££) are particular instances of (§). Conversely, the two Recursion Lemmas ensure that the more sophisticated iteration is expressible by means of $\langle\ \rangle$ $[\ ]$, while the more general induction principle (§) follows by (£), (££).

Lemma 1. $\langle L\rangle C = \langle I\rangle L$, $\langle R\rangle C = \langle I\rangle R$.

Proof. The equalities $L^2C = L^2$, $RC = CA^2$ and $RL = LA^2$ imply $\langle L\rangle C = \langle I\rangle L$ by (£) Similarly, $\langle R\rangle C = \langle I\rangle R$.

Lemma 2. $[C, \langle\varphi L\rangle, I, \langle R\rangle] = \langle\langle\varphi\rangle\rangle$.

Proof. Writing σ for $[C, \langle\varphi L\rangle, I, \langle R\rangle]$, it follows that

$$C(\langle\varphi L\rangle, \langle\langle\varphi\rangle\rangle\langle R\rangle) = C(\langle\varphi L\rangle, \langle\langle\varphi\rangle R\rangle) = \langle(\varphi L, \langle\varphi\rangle R)\rangle = \langle\langle\varphi\rangle\rangle,$$

hence $\sigma \leqq \langle\langle\varphi\rangle\rangle$.

On the other hand, $RL = L\langle R\rangle$ and $(\varphi L^2, L\sigma\langle R\rangle) = LC(\langle\varphi L\rangle, \sigma\langle R\rangle) = L\sigma$ imply $\langle\varphi\rangle L \leqq L\sigma$ by (£), while $\langle R\rangle R = R\langle R\rangle$ and

$$C(\langle\varphi L\rangle R, R\sigma\langle R\rangle) = RC(\langle\varphi L\rangle, \sigma\langle R\rangle) = R\sigma$$

imply $\sigma R \leqq R\sigma$ by (§), hence

$$(\langle\varphi\rangle L, \sigma R) \leqq (L\sigma, R\sigma) = \sigma,$$

which gives $\langle\langle\varphi\rangle\rangle \leqq \sigma$ by (£). Thereby the proof is completed.

Let $\rho_1 = \Delta(L, R^2)$, $\rho_2 = \rho_1 R$ and $\rho = [(L^3, RL^2, R), I, I, R]$ (right grouping of brackets).

Lemma 3. $\rho_1\rho = \langle L\rangle$, $\rho_2\rho = \langle R\rangle$.

Proof. It follows that $R^2\rho = \rho R$ and

$$(L\rho, \langle L\rangle R) = (L^2, \langle L\rangle R) = \langle L\rangle,$$

hence $\rho_1\rho \leqq \langle L\rangle$ by (§). Conversely,

$$R\rho_1\rho = \rho_1 R^2\rho = \rho_1\rho R,$$

$$(L^2, \rho_1\rho R) = (L\rho, R\rho_1\rho) = (L\rho_1\rho, R\rho_1\rho) = \rho_1\rho$$

give $\langle L\rangle \leqq \rho_1\rho$ by (£). Therefore, $\rho_1\rho = \langle L\rangle$ and similarly $\rho_2\rho = \langle R\rangle$.

Lemma 4. $\langle\varphi\rangle\rho_1 = \rho_1\langle\varphi\rangle$, $\langle\varphi\rangle\rho_2 = \rho_2\langle\varphi\rangle$.

Follows by the proof of lemma 1.24 [1].

Lemma 5. $\langle\varphi\rangle = [(\varphi L^2, \varphi LRL, R), I, I, R^2]$.

Proof. Write σ for $[(\varphi L^2, \varphi LRL, R), I, I, R^2]$. It follows that

$$\langle\varphi\rangle = (\varphi L, \langle\varphi\rangle R) = (\varphi L, \varphi LR, \langle\varphi\rangle R^2),$$

hence $\sigma \leqq \langle\varphi\rangle$. Conversely,

$$(\varphi LR, \varphi LR^2, R\sigma R^2) = (\varphi LR, (\varphi L, R\sigma)R^2) = (\varphi LR, \sigma R^2) = R\sigma$$

implies $\sigma R \leqq R\sigma$ by (§), hence

$$(\varphi L, \sigma R) \leqq (\varphi L, R\sigma) = \sigma,$$

which finally gives $\langle\varphi\rangle \leqq \sigma$. The proof is completed.

Lemma 6. $\langle\varphi\rangle\rho = \rho\langle(\varphi L, \varphi R)\rangle$.

Proof. It follows that $R^2\rho = \rho R$ and

$$(\varphi L\rho, \varphi LR\rho, \rho\langle(\varphi L, \varphi R)\rangle R) = (\varphi L^2, \varphi RL, \rho\langle(\varphi L, \varphi R)\rangle R)$$
$$= (L^3, RL, \rho R)\langle(\varphi L, \varphi R)\rangle = \rho\langle(\varphi L, \varphi R)\rangle,$$

hence $\langle\varphi\rangle\rho\leq\rho\langle(\varphi L,\ \varphi R)\rangle$ by lemma 5 and (§). Conversely, $R\langle(\varphi L,\ \varphi R)\rangle=\langle(\varphi L,\ \varphi R)\rangle R$ and

$$(L^2\langle(\varphi L,\ \varphi R)\rangle,\ RL\langle(\varphi L,\ \varphi R)\rangle,\ \langle\varphi\rangle\rho R)=(\varphi L^2,\ \varphi RL,\ \langle\varphi\rangle\rho R)$$
$$=(\varphi L,\ \varphi LR,\ \langle\varphi\rangle R^2)(L^2,\ RL,\ \rho R)=\langle\varphi\rangle\rho$$

imply $\rho\langle(\varphi L,\ \varphi R)\rangle\leq\langle\varphi\rangle\rho$ by (§). Thereby the proof is completed.

Pull Back Lemma. Whenever φ is recursive in $\mathscr{B}$, then it is prime recursive in $\{\langle B\rangle,\ \langle\langle L\rangle\rangle,\ \langle\langle A\rangle\rangle\}\cup\langle\mathscr{B}\rangle$, where $\langle\mathscr{B}\rangle=\{\langle\psi\rangle/\psi\in\mathscr{B}\}$.

P r o o f. An easy induction on the construction of φ gives by lemmas 1.13, 1.19, 1.24, 1.20 [1] that $\langle\varphi\rangle$ is prime recursive in $\{C,\ P,\ Q,\ \langle L\rangle,\ \langle R\rangle\}\cup\langle\mathscr{B}\rangle$, hence so is φ since $\varphi=L\langle\varphi\rangle[L]$. While C is by definition $\langle B\rangle^2[\langle A\rangle^2]$, a more careful analysis of the definitions of P, Q in lemma 1.24 and the proofs of lemmas 1.22. 1.21 [1] shows that P, Q are prime recursive in $\langle\langle I\rangle\rangle C,\ \langle\langle L\rangle\rangle\langle\langle R\rangle\rangle,\ \langle\langle A\rangle\rangle$. (Take $\sigma=\Delta(L^2,\ A)$, $\sigma_1=\langle I\rangle\sigma\langle I\rangle$, and show that $\sigma_1=\sigma$ by (§) and the equality $\Delta(\varphi,\ \psi)=[(\varphi L,\ \varphi\psi L,\ R),\ I,\ I,\ \psi^2]$ which is analogous to that of lemma 5. Then $Q=[\sigma_1(R,\ L),\ L,\ I,\ R]$. However, it can be proved by making use of (§) that $\langle\langle I\rangle\rangle C=G(L,\ GR)$ with $G=\langle\langle L\rangle\rangle[\langle\langle R\rangle\rangle]$, while $\langle\langle R\rangle\rangle=\langle\langle L\rangle\rangle\langle\langle A\rangle\rangle$ follows by lemma 1.13 [1]. The proof is completed.

M o d i f i c a t i o n L e m m a. *Let $X,\ L_1,\ R_1\in\mathscr{F}$ and $L_1X=L$, $R_1X=R$. Take $(\varphi,\ \psi)_1=X(\varphi,\ \psi)$ by definition. Then $\mathscr{S}_1=(\mathscr{F},\ I,\ \Pi_1,\ L_1,\ R_1)$ is an iterative operative space and φ is (prime) recursive in $\{L_1,\ R_1,\ (L,\ R)_1\}\cup\mathscr{B}$ iff φ is (prime) recursive$_1$ in $\{L,\ R,\ (L_1,\ R_1)\}\cup\mathscr{B}$.*

P r o o f. It follows that

$$(\varphi,\ \psi)_1\chi=X(\varphi,\ \psi)\chi=X(\varphi\chi,\ \psi\chi)=(\varphi\chi,\ \psi\chi)_1,$$
$$L_1(\varphi,\ \psi)_1=L_1X(\varphi,\ \psi)=L(\varphi,\ \psi)=\varphi$$

and similarly $R_1(\varphi,\ \psi)_1=\psi$, hence $\mathscr{S}_1$ is an operative space. For all φ the elements $\langle\varphi\rangle_1=[(L,\ R)_1,\ \varphi L_1,\ I,\ R_1]$, $[\varphi]_1=(L,\ R)_1[\varphi(L,\ R)_1]$ exist and meet respectively (£) (by the Second Recursion Lemma) and (££). The operations $\langle\ \rangle$, $[\ ]$ are in turn expressible by means of $\langle\ \rangle_1$, $[\ ]_1$ since $(\varphi,\ \psi)=X_1(\varphi,\ \psi)_1$, $LX_1=L_1$ and $RX_1=R_1$, where $X_1=(L_1,\ R_1)$. Thereby the proof is completed.

2. Translation Elimination. In this section we establish a general translation elimination theorem which will make it possible to implement translation by an extra counter. The availability of additional counting facilities is expressed by the constants W, W_1, W_2 and their axioms given below; semantics will be discussed in the next section.

Translation Elimination Theorem. Let $W,\ W_1,\ W_2\in\mathscr{F}$ s. t. $W_1W=L$, $W_2W=R$ $W(L,\ R)=W$, $LW=W(L^2,\ LR)$ and $RW=W(RL,\ R^2)$. Let

$$\mathscr{C}=\{\sigma\in\mathscr{F}/W_i\sigma=\sigma W_i,\quad i=1,\ 2\},$$

assumming that $L,\ R,\ (L,\ R),\ \langle I\rangle\in\mathscr{C}$. Then whenever $\mathscr{B}\subseteq\mathscr{C}$ and φ is recursive in $\{W,\ W_1,\ W_2\}\cup\mathscr{B}$, then φ is prime recursive in $\{W,\ W_1,\ W_2\}\cup\mathscr{B}$. (The reverse implication is immediate.)

P r o o f. It follows that $W_i(L,\ R)=(W_iL,\ W_iR)$ and the more general equalities

$$W_i(\varphi,\ \psi)=(W_i\varphi,\ W_i\psi),$$
$$LW(\varphi,\ \psi)=W(L\varphi,\ L\psi),\ RW(\varphi,\ \psi)=W(R\varphi,\ R\psi)$$

hold for $i=1,\ 2$ and all $\varphi,\ \psi$.

The set $\mathscr{C}$ is obviously closed under the operations multiplication and pairing. In order to show that it is also closed under translation, suppose that $\sigma\in\mathscr{C}$. Then the equalities

$$\sigma LW_i=W_i\sigma L=LW_i\langle\sigma\rangle,$$
$$RW_i=W_iR,\ W_i\langle\sigma\rangle R=RW_i\langle\sigma\rangle,\quad i=1,\ 2$$

imply $\langle\sigma\rangle W_i = \langle I\rangle W_i\langle\sigma\rangle$ by (£) (or lemma 1.12 [1]). Therefore,

$$W_i\langle\sigma\rangle = W_i\langle I\rangle\langle\sigma\rangle = \langle I\rangle W_1\langle\sigma\rangle = \langle\sigma\rangle W_i$$

for $i=1, 2$ by lemma 1.13 [1], hence $\langle\sigma\rangle \in \mathscr{C}$.

Our next aim is to prove that $\langle\sigma\rangle$ is prime recursive in W, W_1, W_2, σ for all $\sigma\in\mathscr{C}$. Making use of lemma 1.7 [1], one gets

$$[W_2]W_2 = R[(W_2L,\ W_2R)] = R[W_2(L,\ R)] = R[W_2].$$

Writing ρ for $W(L,\ R^2)$, it follows that

$$R^2[\rho] = R\rho[\rho] = \rho(RL,\ R^2)[\rho] = \rho(R,\ R^2[\rho]),$$

hence $R[\rho]R \leqq R^2[\rho]$ by lemma 1.6 [1]. Conversely,

$$\rho(I,\ W(I,\ R[\rho]R)) = W(I,\ RW(I,\ R[\rho]R)) = W(I,\ W(R,\ R^2[\rho]R))$$
$$= W(I,\ \rho[\rho]R) = W(I,\ R[\rho]R)$$

gives $R[\rho] \leqq W(I,\ R[\rho]R)$ by lemma 1.6 [1], hence

$$R^2[\rho] \leqq RW(I,\ R[\rho]R) = W(R,\ R^2[\rho]R) = \rho[\rho]R = R[\rho]R.$$

Therefore, $R^2[\rho] = R[\rho]R$.

Making use of the above equalities, one gets for $\sigma\in\mathscr{C}$

$$LW_1[W_2]\sigma L\rho[\rho] = W_1\sigma L\rho[\rho] = \sigma LW_1\rho[\rho] = \sigma L^2[\rho] = \sigma L$$

and

$$RW_1[W_2]\sigma L\rho[\rho] = W_1R[W_2]\sigma L\rho[\rho] = W_1[W_2]W_2\sigma L\rho[\rho] = W_1[W_2]\sigma LW_2\rho[\rho]$$
$$= W_1[W_2]\sigma LR^2[\rho] = W_1[W_2]\sigma LR[\rho]R = W_1[W_2]\sigma L\rho[\rho]R,$$

which implies

$$\langle\sigma\rangle = \langle I\rangle W_1[W_2]\sigma LR[\rho]$$

by (£).

The multiplier $\langle I\rangle$ in the last equality can be skipped. Actually, $R[W_2] = [W_2]W_2$ and $(L[W_2],\ [W_2]W_2) = (I,\ R[W_2]) = [W_2]$ imply $\langle I\rangle[W_2] \leqq [W_2]$ by (£), while

$$(I,\ W_2\langle I\rangle[W_2]) = (I,\ \langle I\rangle W_2[W_2]) = (L,\ \langle I\rangle R)[W_2] = \langle I\rangle[W_2]$$

gives $[W_2] \leqq \langle I\rangle[W_2]$ by (££); therefore,

$$\langle\sigma\rangle = \langle I\rangle W_1[W_2]\sigma LR[\rho] = W_1\langle I\rangle[W_2]\sigma LR[\rho] = W_1[W_2]\sigma LR[\rho]$$

for all $\sigma\in\mathscr{C}$. In particular, $\langle I\rangle$ is prime recursive in W, W_1, W_2.

Let us prove now that $\langle W\rangle$, $\langle W_1\rangle$, $\langle W_2\rangle$ are prime recursive in W, W_1, W_2.

The equalities $W_1L = LW_1$, $W_1R = RW_1$ give $\langle W_1\rangle = \langle I\rangle W_1$ by (£), and similarly $\langle W_2\rangle = \langle I\rangle W_2$.

The equalities

$$LW(\langle L\rangle,\ \langle R\rangle) = W(L\langle L\rangle,\ L\langle R\rangle) = W(L^2,\ RL) = W(L,\ R)L = WL,$$

$$RW(\langle L\rangle,\ \langle R\rangle) = W(R\langle L\rangle,\ R\langle R\rangle) = W(\langle L\rangle R,\ \langle R\rangle R) = W(\langle L\rangle,\ \langle R\rangle)R$$

imply $\langle W\rangle = \langle I\rangle W(\langle L\rangle,\ \langle R\rangle)$ by (£).

Suppose that $\mathscr{B} \subseteq \mathscr{C}$ and φ is recursive in $\{W,\ W_1,\ W_2\} \cup \mathscr{B}$. Then φ is prime recursive in

$$\mathscr{B}_1 = \{\langle B\rangle,\ \langle\langle L\rangle\rangle,\ \langle\langle A\rangle\rangle,\ \langle W\rangle,\ \langle W_1\rangle,\ \langle W_2\rangle\} \cup \langle\mathscr{B}\rangle$$

by the Pull Back Lemma. Taking into account that B, $\langle L\rangle$, $\langle A\rangle \in \mathscr{C}$, one finally concludes that all the members of $\mathscr{B}_1$ are prime recursive in $\{W,\ W_1,\ W_2\} \cup \mathscr{B}$, hence so is φ. Thereby the proof is completed.

3. Translation Implementation: Examples. In this section we describe and illustrate the idea of translation implementation based on the Translation Elimination Theorem.

Given a space $\mathscr{S}$, we would like to construct a wider space $\mathscr{S}_1$ so that: (1) $\mathscr{S}$ is imbedded into $\mathscr{S}_1$, i. e. isomorphic with a subspace $\mathscr{S}^\sim$ of $\mathscr{S}_1$; (2) $\mathscr{S}_1$ has the constants W, W_1, W_2; (3) All $\varphi^\sim$ in $\mathscr{F}^\sim$ commutate with W_1, W_2; (4) $\mathscr{S}^\sim$ is conservative in $\mathscr{S}_1$ in the sense that whenever $\varphi^\sim$ is resursive$_1$ in $\{W, W_1, W_2\} \cup \mathscr{B}^\sim$, then φ is recurisve in $\mathscr{B}$. That would ensure by the Translation Elimination Theorem that φ is recursive in $\mathscr{B}$ iff $\varphi^\sim$ is prime recursive$_1$ in $\{W, W_1, W_2\} \cup \mathscr{B}^\sim$ for all $\varphi \in \mathscr{F}$, $\mathscr{B} \subseteq \mathscr{F}$.

Now we are giong to show how this technique works for particular operative spaces, leaving the problem of abstract translation implementation to the next section.

Given an object domain M, the standard functional operative space over M consists of partial single-valued functions $\varphi : M \dashrightarrow M$, while the standard relational space consists of partial multiple-valued functions φ: $M \to 2^M$ (with $\varphi(s)\uparrow$ if $\varphi(s)=\emptyset$) also regarded as relations $\varphi \subseteq M^2$. Other interesting spaces consist of fuzzy and probabilistic functions or relations over M. All these are first order spaces, as opposed to the higher order spaces s. a. $\mathscr{S}_1$ of [2] which consist of operator-like rather than function-like elements.

In order to construct an operative space one has to augment at first the domain M with a **splitting scheme**, injective functions f_1, f_2: $M \to M$ with disjoint ranges. This is only possible for infinite domains. For an arbitrary nonempty M that can be arranged by introducing a counter, i. e. taking the wider set $\omega \times M$ which does have a natural splitting scheme, e. g. $f_1 = \lambda ns.(2n, s)$, $f_2 = \lambda ns.(2n+1, s)$.

It should be mentioned also that whenever f_1, f_2 is a splitting scheme for M and N is a nonempty set, then $\lambda sx.(f_1(s), x)$, $\lambda sx.(f_2(s), x)$ is a splitting scheme for $M \times N$. We say that M and $M \times N$ have the same splitting schemes in such a case.

Example 1. (Analogue to example 2 of [5], chapter 2.) Let M be a set with a splitting scheme f_1, f_2. Take $\mathscr{F} = \{\varphi/\varphi\colon M \dashrightarrow 2^M\}$, $\varphi \leq \psi$ if $\varphi \subseteq \psi$, $\varphi\psi = \lambda s.\psi(\varphi(s))$, $(\varphi, \psi)(f_1(s)) = \varphi(s)$, $(\varphi, \psi)(f_2(s)) = \psi(s)$ and $(\varphi, \psi)(s)\uparrow$ otherwise, $I = \lambda s.s$, $L = f_1$ and $R = f_2$. Then $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ is an iterative operative space and its operations $\langle\ \rangle$, $[\ ]$ are explicitly characterized as follows:

$$\langle\varphi\rangle(f_2^n(f_1(s))) = f_2^n(f_1(\varphi(s)))$$

and $\langle\varphi\rangle(s)\uparrow$ otherwise, $t \in [\varphi](s)$ iff there are $n \in \omega$, $r_0, \ldots, r_{n-1} \in f_2(M)$ and $r_n \in f_1(M)$ s. t. $r_0 = s$, $r_{i+1} \in \varphi(f_2^{-1}(r_i))$ for all $i < n$, and $t = f_1^{-1}(r_n)$.

Example 2. (Analogue to example 1 of [5], chapter 2.) The subspace of example 1 consisting of all single-valued functions. Translation and iteration are characterized as above, with $t \in [\varphi](s)$, $r_{i+1} \in \varphi(f_2^{-1}(r_i))$ replaced by $t = [\varphi](s)$, $r_{i+1} = \varphi(f_2^{-1}(r_i))$.

Example 3. Example 1 with $M \times \omega$ substituted for M, using the same splitting scheme. In other words, an extra counter is added.

Example 4. Example 2 with $M \times \omega$ substituted for M, preserving the splitting scheme.

To implement translation in example 2 we shall use the additional counting facilities of example 4; example 1 is likewisely imbedded into example 3 and other first order spaces are treated in the same manner. So assume from now on that $\mathscr{S} = (\mathscr{F}, I, \Pi, L, R)$ and $\mathscr{S}_1 = (\mathscr{F}_1, I_1, \Pi_1, L_1, R_1)$ are respectively the spaces of examples 2, 4.

Lemma 7. *Attach to each $\varphi \in \mathscr{F}$ the element $\varphi^\sim = \lambda sn.(\varphi(s), n)$ in $\mathscr{F}_1$. Then $\mathscr{S}^\sim = (\mathscr{F}^\sim, I_1, \Pi_1 \restriction \mathscr{F}^{\sim 2}, L_1, R_1)$ is a subspace of $\mathscr{S}_1$ isomorphic with $\mathscr{S}$.*

Proof. It is immediate that $I^\sim = I_1$, $L^\sim = L_1$, $R^\sim = R_1$ and $\varphi^\sim \leq \psi^\sim$ iff $\varphi \leq \psi$. One also gets

$$\varphi^\sim\psi^\sim = \lambda sn.\psi^\sim(\varphi^\sim(s, n)) = \lambda sn.\psi^\sim(\varphi(s), n) = \lambda sn.(\psi(\varphi(s)), n) = \lambda sn.(\varphi\psi(s), n) = (\varphi\psi)^\sim$$

and similarly $(\varphi^\sim, \psi^\sim)_1 = (\varphi, \psi)^\sim$. The equalities $\langle\varphi^\sim\rangle_1 = \langle\varphi\rangle^\sim$ and $[\varphi^\sim]_1 = [\varphi]^\sim$ follow by the explicit characterizations of translation and iteration, which completes the proof.

Lemma 8. *Whenever* $\varphi \in \mathcal{F}$, $\mathcal{B} \subseteq \mathcal{F}$ *and* φ *is recursive in* $\mathcal{B}$, *then* $\varphi^{\sim}$ *is prime recursive*$_1$ *in* $\{W, W_1, W_2\} \cup \mathcal{B}^{\sim}$, *where* $W_1 = \lambda sn.(s, 2n)$, $W_2 = \lambda sn.(s, 2n+1)$, $W(s, 2n) = (f_1(s), n)$ *and* $W(s, 2n+1) = (f_2(s), n)$.

Proof. The elements W, W_1, W_2 meet the assumptions of the Translation Elimination Theorem and $\mathcal{F}^{\sim} \subseteq \mathcal{C}$ (in fact, $\mathcal{C} = \mathcal{F}^{\sim}$). Whenever φ is recursive in $\mathcal{B}$, then $\varphi^{\sim}$ is recursive$_1$ in $\mathcal{B}^{\sim}$ by lemma 7, hence $\varphi^{\sim}$ is prime recursive$_1$ in $\{W, W_1, W_2\} \cup \mathcal{B}^{\sim}$ by the Translation Elimination Theorem. The proof is completed.

Lemma 9. *Let* $\mathcal{B}_1 \subseteq \mathcal{F}_1$, $\mathcal{B}_1^* \subseteq \mathcal{F}$ *and a* $\psi_1^* \in \mathcal{B}_1^*$ *correspond to each* $\psi_1 \in \mathcal{B}_1$ *so that* $\overline{m}\psi_1^*(s) = \overline{n}(t)$, *if* $\psi_1(s, m) = (t, n)$, *and* $\overline{m}\psi_1^*(s)\uparrow$, *if* $\psi_1(s, m)\uparrow$, *where* $\overline{m} = LR^m$. *Then for every* $\varphi_1 \in \mathcal{F}_1$ *recursive*$_1$ *in* $\{W, W_1, W_2\} \cup \mathcal{B}_1$ *there is a* $\varphi_1^* \in \mathcal{F}$ *recursive in* $\mathcal{B}_1^*$ *to correspond to* φ_1 *as above. In particular, whenever* $\varphi^{\sim}$ *is recursive*$_1$ in $\{W, W_1, W_2\} \cup \mathcal{B}^{\sim}$, *then* φ *is recursive in* $\mathcal{B}$.

Proof. Take $L_1^* = \langle L \rangle$, $R_1^* = \langle R \rangle$, $W_1^* = \Delta(L, R^2)$, $W_2^* = W_1^* R$ and $W^* = [(L^3, RL^2, R), I, I, R]$. Whenever φ_1^*, ψ_1^* correspond to φ_1, ψ_1, then take $(\varphi_1\psi_1)^* = \varphi_1^*\psi_1^*$, $(\varphi_1, \psi_1)_1^* = C(\varphi_1^*, \psi_1^*)$, $\langle\varphi_1\rangle_1^* = G\langle\varphi_1^*\rangle G$, $[\varphi_1]_1^* = C[\varphi_1^* C]$. (See section 1 for definitions of C, G.)

Suppose that $\varphi^{\sim}$ is recursive$_1$ in $\{W, W_1, W_2\} \cup \mathcal{B}^{\sim}$. Taking $\mathcal{B}^{\sim *} = \langle \mathcal{B} \rangle$, it follows that an element $\varphi^{\sim *}$ recursive in $\langle \mathcal{B} \rangle$ corresponds to $\varphi^{\sim}$, hence φ will be recursive in $\mathcal{B}$ since $\varphi = L\varphi^{\sim *}(I, I)$. Thereby the proof is completed.

Translation Implementation Theorem (for example 2). Let $\varphi \in \mathcal{F}$, $\mathcal{B} \subseteq \mathcal{F}$. Then φ is recursive in $\mathcal{B}$ iff $\varphi^{\sim}$ is prime recursive$_1$ in $\{W, W_1, W_2\} \cup \mathcal{B}^{\sim}$.

It follows by lemmas 8, 9.

4. Abstract Translation Implementation. Given an arbitrary iterative operative space $\mathcal{S} = (\mathcal{F}, I, \Pi, L, R)$, we construct in this section a corresponding space $\mathcal{S}_1$ s. t. conditions (1)–(4) of section 3 are met.

The presentation of the space of example 4 back in the space of example 2 (lemma 9) hints that it may be possible to construct $\mathcal{S}_1$ from materials provided by $\mathcal{S}$; in other words, to organize the additional counting facilities within the given space $\mathcal{S}$.

Following this idea, at first we construct a closely related space $\mathcal{S}_0$ (if $\langle I \rangle = I$ in $\mathcal{S}$, then $\mathcal{S}_0 = \mathcal{S}$), then change the pairing scheme Π_0, L_0, R_0 of $\mathcal{S}_0$ by the Modification Lemma to get $\mathcal{S}_1$. What remains to be seen is that (1)–(3) take place; (4) will follow by the very construction of $\mathcal{S}_1$.

Lemma 10. *Take* $\mathcal{F}_0 = \{\varphi \in \mathcal{F} / \langle I \rangle \varphi \langle I \rangle = \varphi\}$, $\leq$ *and* $\circ$ *as in* $\mathcal{F}$, $\Pi_0 = \Pi \restriction \mathcal{F}_0^2$, $I_0 = \langle I \rangle$, $L_0 = \langle I \rangle L$ *and* $R_0 = \langle I \rangle R$. *Then* $\mathcal{S}_0 = (\mathcal{F}_0, I_0, \Pi_0, L_0, R_0)$ *is an iterative operative space and* $\langle\varphi\rangle_0 = \langle\varphi\rangle$, $[\varphi]_0 = [\varphi]\langle I \rangle$.

Proof. Straightforward.

Lemma 11. *Let* $X = C$, $L_1 = \langle L \rangle$, $R_1 = \langle R \rangle$ *and* $\mathcal{S}_1 = (\mathcal{F}_1, I_1, \Pi_1, L_1, R_1)$ *be obtained from* $\mathcal{S}_0$ *by the Modification Lemma. Then* $\mathcal{S}$ *is isomorphic with the subspace* $\mathcal{S}^{\sim}$ *of* $\mathcal{S}_1$ *based on* $\mathcal{F}^{\sim} = \langle \mathcal{F} \rangle$, *attaching* $\varphi^{\sim} = \langle\varphi\rangle$ *to* φ.

Proof. The elements X, L_1, R_1 are in $\mathcal{F}_0$ and meet the assumptions of Modification Lemma by lemma 1, hence $\mathcal{S}_1$ can actually be constructed.

It is immediate that $I^{\sim} = I_1$, $L^{\sim} = L_1$ and $R^{\sim} = R_1$, while $\varphi^{\sim} \leq \psi^{\sim}$ iff $\varphi \leq \psi$ follows by $\varphi = L\langle\varphi\rangle(I, I)$.

Taking into account that $(\varphi, \psi)_1 = C(\varphi, \psi)$,

$$\langle\varphi\rangle_1 = [(L_0, R_0)_1, \varphi L_1, I_0, R_1]_0 = [C, \varphi\langle L \rangle, I, \langle R \rangle], \; [\varphi]_1 = (L_0, R_0)_1[\varphi(L_0, R_0)_1]_0 = C[\varphi C]$$

for all $\varphi \in \mathcal{F}_0 = \mathcal{F}_1$, one gets $(\varphi\psi)^{\sim} = \varphi^{\sim}\psi^{\sim}$, $(\varphi, \psi)^{\sim} = (\varphi^{\sim}, \psi^{\sim})_1$, $\langle\varphi\rangle^{\sim} = \langle\varphi^{\sim}\rangle_1$, $[\varphi]^{\sim} = [\varphi^{\sim}]_1$ for all $\varphi \in \mathcal{F}$ respectively by lemmas 1.13, 1.19 [1], 2 and 1.20 [1]. Thereby the proof is completed.

Remark. It can be shown that $\langle\varphi\rangle_1 = G\langle\varphi\rangle G$, too.

Lemma 12. *The space* $\mathcal{S}_1$ *and the elements* $W_1=\Delta(L, R^2)$, $W_2=W_1R$ *and* $W=[(L^3, RL^2, R), I, I, R]$ *meet the assumptions of the Translation Elimination Theorem and all the members of* $\mathcal{F}^\sim$ *commutate with* W_1, W_2.

Proof. The elements W, W_1, W_2 are in $\mathcal{F}_1$, $W_1W=L_1$ and $W_2W=R_1$ by lemma 3, $W(L_1, R_1)_1=W$ by (§), $L_1W=W(L_1^2, L_1R_1)_1$ and $R_1W=W(R_1L_1, R_1^2)_1$ by lemma 6. All $\varphi^\sim\in\mathcal{F}^\sim$ commutate with W_1, W_2 by lemma 4 and so do the elements L_1, R_1, $(L_1, R_1)_1$, $\langle I_1\rangle_1$ since they are in $\mathcal{F}^\sim$ by lemma 11. The proof is completed.

Translation Implementation Theorem. Let $\varphi\in\mathcal{F}$ and $\mathcal{B}\subseteq\mathcal{F}$. Then φ is recursive in $\mathcal{B}$ iff $\varphi^\sim$ is prime recursive$_1$ in $\{W, W_1, W_2\}\cup\mathcal{B}^\sim$.

Proof. Whenever φ is recursive in $\mathcal{B}$, then $\varphi^\sim$ is recursive$_1$ in $\mathcal{B}^\sim$ by lemma 11, hence prime recursive$_1$ in $\{W, W_1, W_2\}\cup\mathcal{B}^\sim$ by the Translation Elimination Theorem. The reverse implication follows by the construction of $\mathcal{S}_1$ and the equality $\varphi=L\varphi^\sim.(I, I)$. Thereby the proof is completed.

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