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ERLICH'S METHODS WITH A RAISED SPEED OF CONVERGENCE

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Let the equation

$$(1) \qquad f(x) \equiv x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

have different roots $x_1, x_2, \ldots, x_n$. In 1967 L. Erlich [1] suggested the following method for simultaneous determination of the roots of the equation (1):

$$(2) \qquad x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{f'(x_i^k) - f(x_i^k) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{x_i^k - x_j^k}}, \qquad \begin{array}{l} i = 1, 2, \ldots, n, \\ k = 1, 2, \ldots \end{array}$$

He showed that the method (2) has a rate of convergence equal to 3. Let us note that in 1964 K. Dochev and P. Barnev proposed in [4] the following expression for the equation (1):

$$x_i^{k+1} = x_i^k - f(x_i^k)\, 2 \Big(\prod_{\substack{j=1 \\ j \neq i}}^{n} (x_i^k - x_j^k) - f'(x_i^k) + f(x_i^k) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{x_i^k - x_j^k}\Big) \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} (x_i^k - x_j^k)^2$$

$i = 1, \ldots, n$; $k = 1, 2, \ldots$. In 1979 Ch. Semerdzjev proved the cubic rate of convergence of the above method [5]. The cubic rate of convergence under weaker initial conditions was obtained also by N. Kjurkchiev [6] in 1982.

In 1977 A. Nourein [2] modified the method (2) in the following way:

$$(3) \qquad x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{f'(x_i^k) - f(x_i^k) \sum_{\substack{j=1 \\ j \neq i}}^{n} (x_i^k - x_i^k + f(x_j^k)/f'(x_j^k))^{-1}}, \qquad \begin{array}{l} i = 1, 2, \ldots, n, \\ k = 1, 2 \ldots, \end{array}$$

and he proved that the rate of convergence is equal to 4.

In 1974 Alefeld and Herzberger [3] suggested the modification of Gauss-Seidel type for the method (2)

$$x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{f'(x_i^k) - f(x_i^k)\Big(\sum_{j=1}^{i-1} \frac{1}{x_i^k - x_j^{k+1}} + \sum_{j=i+1}^{n} \frac{1}{x_i^k - x_j^k}\Big)}, \qquad \begin{array}{l} i = 1, 2, \ldots, n, \\ k = 1, 2, \ldots, \end{array}$$

and they obtained that the rate of convergence is in the open interval (3, 4).

In 1981 N. Kjurkchiev proposed a method which also makes use of $\{f'(x_i^k)\}_{i=1}^n$ on the k-th step [7]

$$x_i^{k+1}=x_i^k-f(x_i^k)/(f'(x_i^k)-f(x_i^k)\sum_{\substack{j=1\\j\neq i}}^n\frac{1}{x_i^k-x_j^k}-f(x_i^k)\sum_{\substack{j=1\\j\neq i}}^n\frac{f(x_j^k)}{(x_i^k-x_j^k)^3}\prod_{\substack{s=1\\s\neq i\\s\neq j}}^n\frac{1}{x_j^k-x_s^k}),$$

$i=1, 2, \ldots, n$; $k=1, 2, \ldots$ with a rate convergence equal to 4.

The methods of type (2) and (3) (when we use the derivative) have perfect computing properties and it turned out in practice that they are inpretentious to the choice of the initial approximation.

In this paper we present the following modification of (2):

$$x_i^{k+1}=x_i^k-f(x_i^k)/(f'(x_i^k)-f(x_i^k)\sum_{\substack{j=1\\j\neq i}}^n\frac{1}{x_i^k-x_j^k-\Delta_j^{R,k}}),\quad i=1, 2, \ldots, n,\ k=0, 1, 2, \ldots, \tag{4}$$

where

$$\Delta_s^{R,k}=-f(x_s^k)/(f'(x_s^k)-f(x_s^k)\sum_{\substack{l=1\\l\neq s}}^n\frac{1}{x_s^k-x_l^k-\Delta_l^{R-1,k}}),\quad s=1, \ldots, n,$$

$$\Delta_s^{0,k}=0,\quad s=1, \ldots, n,\quad k=0, 1, \ldots$$

It is clear that when $R=0$ the method (4) coincides with (2). The rate of convergence of (4) is equal to $2R+3$.

Theorem. *Let $0<q<1$, $d=\min_{i\neq j}|x_i-x_j|$ and $c>0$ be a number such that*

$$d>2c(1+q(2n-1)),\quad c^2n[(d-c)(d-2c-2cq)(1-cq\frac{3cq(n-1)}{(d-c)(d-2c-2cq)})]^{-1}\leq 1.$$

If the initial approximations $\{x_i^0\}_{i=1}^n$ of the roots $\{x_i\}_{i=1}^n$ of the equation (1) *satisfy the inequalities $|x_i-x_i^0|\leq cq$, $i=1, \ldots, n$ then the following estimate*

$$|x_i^k-x_i|\leq cq^{(2R+3)^k},\quad i=1, \ldots, n,\quad k=0, 1, \ldots$$

holds.

Proof. The proof goes by induction on k. For $k=0$ the theorem is evidently true. Let us assume that the inequalities

$$|x_i^k-x_i|\leq cq^{(2R+3)^k},\quad i=1, \ldots, n \tag{5}$$

hold. We shall consider the $k+1$ approximations

$$x_i^{k+1}-x_i=x_i^k-x_i-f(x_i^k)/(f'(x_i^k)-f(x_i^k)\sum_{\substack{\beta_0=1\\\beta_0\neq i}}^n\frac{1}{x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k}})^{-1}$$

$$=x_i^k-x_i-(\frac{1}{x_i^k-x_i}+\sum_{\substack{\beta_0=1\\\beta_0\neq i}}^n\frac{1}{x_i^k-x_{\beta_0}}-\sum_{\substack{\beta_0=1\\\beta_0\neq i}}^n\frac{1}{x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k}})^{-1} \tag{6}$$

$$=x_i^k-x_i-\frac{x_i^k-x_i}{1+(x_i^k-x_i)\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\left(\frac{1}{x_i^k-x_{\beta_0}}-\frac{1}{x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k}}\right)}$$

$$=(x_i^k-x_i)\Big(1-\Big(1+(x_i^k-x_i)\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\frac{x_{\beta_0}-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})}\Big)^{-1}\Big)$$

$$=\frac{(x_i^k-x_i)^2}{1+(x_i^k-x_i)\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\frac{x_{\beta_0}-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})}}\cdot\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\frac{x_{\beta_0}-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})}.$$

The last sum can be rewritten in the form

$$\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\frac{x_{\beta_0}-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})}=\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\frac{x_{\beta_0}-x_{\beta_0}^k+\Big(f'(x_{\beta_0}^k)/f(x_{\beta_0}^k)-\sum\limits_{\beta_1\neq\beta_0}^{n}\frac{1}{x_{\beta_0}^k-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k}}\Big)^{-1}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})}$$

$$=\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\Big(x_{\beta_0}-x_{\beta_0}^k+\Big(\frac{1}{(x_{\beta_0}^k-x_{\beta_0})}+\sum\limits_{\beta_1\neq\beta_0}^{n}\frac{1}{x_{\beta_0}^k-x_{\beta_1}}-\sum\limits_{\beta_1\neq\beta_0}^{n}\frac{1}{x_{\beta_0}^k-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k}}\Big)^{-1}\Big)/(x_i^k$$

$$-x_{\beta_0})(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})=\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}(x_{\beta_0}-x_{\beta_0}^k)\Big(1-\Big(1+(x_{\beta_0}^k-x_{\beta_0})\sum\limits_{\beta_1\neq\beta_0}^{n}\frac{x_{\beta_1}-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k}}{(x_{\beta_0}^k-x_{\beta_1})(x_{\beta_0}^k-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k})}\Big)^{-1}\Big)/$$

$$(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})=\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\Big((x_{\beta_0}^k-x_{\beta_0})^2\sum\limits_{\beta_1\neq\beta_0}^{n}\frac{x_{\beta_1}-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k}}{(x_{\beta_0}^k-x_{\beta_1})(x_{\beta_0}^k-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k})}\Big)/$$

$$(x_{\beta_0}-x_i^k)(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})\Big(1+(x_{\beta_0}^k-x_{\beta_0})\sum\limits_{\substack{\beta_1=1\\ \beta_1\neq\beta_0}}^{n}\frac{x_{\beta_1}-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k}}{(x_{\beta_0}^k-x_{\beta_1})(x_{\beta_1}^k-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k})}\Big).$$

It should be noted that the sum in the numerator in the last expression is similar to the initial sum with the only difference that we have $R-1$ instead of k. Using that dependence recursively wet get successively

(7) $$\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\frac{x_{\beta_0}-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k}}{(x_i^k-x_{\beta_0})(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})}$$

$$=\sum\limits_{\substack{\beta_0=1\\ \beta_0\neq i}}^{n}\frac{(x_{\beta_0}^k-x_{\beta_0})^2}{(x_{\beta_0}-x_i^k)(x_i^k-x_{\beta_0}^k-\Delta_{\beta_0}^{R,k})\Big(1+(x_{\beta_0}^k-x_{\beta_0})\sum\limits_{\beta_1\neq\beta_0}^{n}\frac{x_{\beta_1}-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k}}{(x_{\beta_0}^k-x_{\beta_1})(x_{\beta_0}^k-x_{\beta_1}^k-\Delta_{\beta_1}^{R-1,k})}\Big)}$$

$$\times \sum_{\beta_1 \neq \beta_0}^{n} \frac{(x_{\beta_1}^k - x_{\beta_1})^2}{(x_{\beta_1} - x_{\beta_0}^k)(x_{\beta_0}^k - x_{\beta_1}^k - \Delta_{\beta_1}^{R-1,k})(1 + (x_{\beta_1}^k - x_{\beta_1}) \sum_{\beta_2 \neq \beta_1}^{n} \frac{x_{\beta_2} - x_{\beta_2}^k - \Delta_{\beta_2}^{R-2,k}}{(x_{\beta_1}^k - x_{\beta_2})(x_{\beta_1}^k - x_{\beta_2}^k - \Delta_{\beta_2}^{R-2,k})})} \times \cdots$$

$$\times \sum_{\beta_{R-1} \neq \beta_{R-2}}^{n} \frac{(x_{\beta_{R-1}}^k - x_{\beta_{R-1}})^2}{(x_{\beta_{R-1}} - x_{\beta_{R-2}}^k)(x_{\beta_{R-2}}^k - x_{\beta_{R-1}}^k - \Delta_{\beta_{R-1}}^{1,k})(1 + (x_{\beta_{R-1}}^k - x_{\beta_{R-1}})}$$

$$\cdot \frac{}{\sum\limits_{\beta_R \neq \beta_{R-1}}^{n} \frac{x_{\beta_R} - x_{\beta_R}^k}{(x_{\beta_{R-1}}^k - x_{\beta_R})(x_{\beta_{R-1}}^k - x_{\beta_R}^k)})} \times \sum_{\beta_R \neq \beta_{R-1}}^{n} \frac{x_{\beta_R} - x_{\beta_R}^k}{(x_{\beta_{R-1}}^k - x_{\beta_R})(x_{\beta_{R-1}}^k - x_{\beta_R}^k)}.$$

Let us estimate from above the absolute value of the items in (7). To this end we show first that the following estimates

$$|\Delta_s^{p,k}| \leqq 2cq, \quad s = 1, 2, \ldots, n, \quad p = 0, 1, \ldots, R \tag{8}$$

hold. The proof can be done by induction on p. When $p=0$, the inequalities (8) are satisfied since $\Delta_s^{0,k}=0$ for $s=1, 2, \ldots, n$. Let us assumed that $|\Delta_s^{m-1,k}| \leqq 2cq$, $s=1, 2, \ldots, n$. Then we get for $\Delta_s^{m,k}$, $s=1, 2, \ldots, n$,

$$\Delta_s^{m,k} = -f(x_s^k)\,(f'(x_s^k) - f(x_s^k) \sum_{\substack{l=1 \\ l \neq s}}^{n} \frac{1}{x_s^k - x_l^k - \Delta_l^{m-1,k}})^{-1}$$

$$= -(x_s^k - x_s)\,(\sum_{l=1}^{n} \frac{x_s^k - x_s}{x_s^k - x_l} - (x_s^k - x_s) \cdot \sum_{\substack{l=1 \\ l \neq s}}^{n} \frac{1}{x_s^k - x_l^k - \Delta_l^{m-1,k}})^{-1}.$$

It follows from the last equality and (5)

$$|\Delta_s^{m,k}| \leqq cq(1 - cq \sum_{\substack{l=1 \\ l \neq s}}^{n} (\frac{1}{|x_s^k - x_l|} + \frac{1}{|x_s^k - x_l^k| - |\Delta_l^{m-1,k}|}))^{-1}.$$

From

$$|x_s^k - x_l^k| = |x_s^k - x_s + x_s - x_l + x_l - x_l^k| \geqq |x_s - x_l| - |x_s - x_s^k| - |x_l - x_l^k| \geqq d - 2cq, \tag{9}$$

$$|x_s^k - x_l| \geqq d - cq$$

we finally obtain

$$|\Delta_s^{m,k}| \leqq cq(1 - cq(n-1)\,(\frac{1}{d-c} + \frac{1}{d-2c-2cq}))^{-1} \leqq cq\,(1 - \frac{2cq(n-1)}{d-2c(1+q)})^{-1}$$

end for the raliclity of the inequality $|\Delta_s^{m,k}| \leqq 2cq$ the following conditions are sufficient

$$(1 - \frac{2cq(n-1)}{d-2c(1+q)})^{-1} \leqq 2 \quad \text{and} \quad 1 - \frac{2cq\,(n-1)}{d-2c(1+q)} > 0.$$

These two conditions are satisfied if $d>2c(1+q(2n-1))$. Now we are able to estimate the sums in (7). In view of (5), (8) and (9) we get for an arbitrary item in (7)

$$\frac{1}{|x_\mu - x_\nu^k||x_\nu^k - x_\mu^k - \Delta_\mu^{s,k}||1+(x_\mu^k - x_\mu)\sum\limits_{\lambda\neq\mu}^{n}\frac{x_\lambda - x_\lambda^k-\Delta_\lambda^{s-1,k}}{(x_\mu^k-x_\lambda)(x_\mu^k-x_\lambda^k-\Delta_\lambda^{s-1,k})}|}$$

$$\leq \frac{1}{|x_\mu - x_\nu^k|(|x_\nu^k - x_\mu^k| - |\Delta_\mu^{s,k}|)(1-|x_\mu^k - x_\mu|\sum\limits_{\lambda\neq\mu}^{n}\frac{|x_\lambda - x_\lambda^k| + |\Delta_\lambda^{s-1,k}|}{(d-c)(d-2c-2cq)})}$$

$$\leq ((d-c)(d-2c-2cq)(1-cq(n-1)\frac{3cq}{(d-c)(d-2c-2cq)}))^{-1} = A.$$

It hollowy from the last inequality, (5), (6) and (7) that

$$|x_i^{k+1}-x_i|\leq [cq^{(2R+3)^k}]^{2+2R+1}\cdot(nA)^{R+1}=cq^{(2R+3)^{k+1}}(c^2nA)^{R+1}\leq cq^{(2R+3)^{k+1}},$$

since $c^2nA\leq 1$, according to the assumptions. So, the theorem is proved.

Numerical results. In order to show the computation properties of the method we carried out a series of numerical experiments.

The polinomial $x^9+3x^8-3x^7-9x^6+3x^5+9x^4+99x^3+297x^2-100x-300$ has the following different roots:

$$x_1=3,\ x_2=-1,\ x_3=1,\ x_4=2i,\ x_5=-2i,\ x_6=2+i,\ x_7=2-i,\ x_8=-2+i,\ x_9=-2-i.$$

For numerical determination of these roots we will apply the method (4) using the initial approximations

$$x_1^0=-3.2+0.2i,\quad x_2^0=-1.2-0.2i,\quad x_3^0=0.1+1.7i,\quad x_4^0=-1.9+1.3i,\quad x_5^0=-1.8-0.8i,$$
$$x_6^0=2.3+1.1i,\quad x_7^0=1.9-0.7i,\quad x_8^0=1.2+0.2i,\quad x_9^0=0.2-2.2i,$$

when $R=0, 1, 3, 6$ and 9. Let us denote by $\sigma(R,k)=\sum_{i=1}^{9}||x_i^{k,R}-x_i^{k-1,R}|$ the error between two consistent approximation on the step $k-1$ and $k\ \{x_i^{k-1,R}\}_{i=1}^{9}$, $\{x_i^{k,R}\}_{i=1}^{9}$ by fixed R equal to 0, 1, 3, 6, 9. The next table shows $\sigma(R,k)$ obtained for different R and k.

k \ R	0 L. Erlich	1	3	6	9
1	0.269907844 5005.10¹	0.267935929 7802.10¹	0.267912902 4201.10¹	0.267912462 3440.10¹	0.267912462 6439.10¹
2	0.142893377 0351.10⁰	0.929246993 3326.10⁻²	0.448638099 7840.10⁻⁴	0.151098417 4056.10⁻⁷	0.474303602 7916.10⁻¹¹
3	0.314259694 4109.10⁻⁴	0.596944211 1547.10⁻¹⁴	0.174483820 2929.10⁻¹⁴	0.277983314 2895.10⁻¹⁵	0.243631196 6690.10⁻¹⁴
4	0.165991577 6040.10⁻¹⁴				

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