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A DIFFERENCE METHOD FOR WEAK SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER WITH A RETARDED ARGUMENT

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1. Introduction. We consider here the initial problem for the first order partial differential equation with a retarded argument.

(1) $$z_x(x, y)=f(x, y, z(x, y), z(\alpha(x, y), \beta(x, y)), z_y(x, y)), \ (x, y)\in E$$

(2) $$z(x, y)=\gamma(x, y) \quad \text{for} \quad (x, y)\in E_0,$$

where $y=(y_1, \dots, y_n)$, $z_y(x, y)=(z_{y_1}(x, y), \dots, z_{y_n}(x, y))$
and $z(\alpha(x, y), \beta(x, y))=(z(\alpha_1(x, y), \beta_1(x, y)), \dots, z(\alpha_p(x, y), \beta_p(x, y)))$

$$\beta_i(x, y)=(\beta_i^{(1)}(x, y), \dots, \beta_i^{(n)}(x, y)), \quad i=1, \dots, p.$$

The sets E and E_0 are defined by

$$E=\{(x, y):\ 0\leqq x\leqq a;\ |y_i|\leqq b_i-M_i x, \quad i=1, 2, \dots, n\},$$

where a, $b_i>0$, $aM_i<b_i$, $i=1, 2, \dots, n$ and

$$E_0=\{(x, y):\ -\tau\leqq x\leqq 0,\ |y_i|\leqq b_i\}, \quad \tau\geqq 0.$$

We assume that

(3) $$(\alpha_i(x, y), \beta_i(x, y))\in E_0\cup E, \quad i=1, \dots, p, \quad \text{if} \quad (x, y)\in E$$

and $\alpha_i(x, y)\leqq x$ for $(x, y)\in E$, $i=1, \dots, p$.

Definition. *A continuous function u, defined on the set $E_0\cup E$ is called a weak solution of the initial problem* (1), (2) *if*:
(i) *there exists a function $\lambda\in C(\langle 0, a\rangle, R_+)$, $R_+=\langle 0, +\infty)$, such that the inequality*

$$\frac{1}{\|l\|^2}[u(x, y+l)-2u(x, y)+u(x, y-l)]\leqq\lambda(x)$$

holds for each triple (x, y), $(x, y+l)$, $(x, y-l)$ belonging to E, $x\neq 0$, $l\neq 0$;

(ii) *there exists a constant $K>0$ such that $\|u_y(x, y)\|\leqq K$ almost ewerywhere on E (we denote by $\|\cdot\|$ the norm in R^n);*
(iii) *u satisfies on E the Lipschitz condition with respect to (x, y);*
(iv) *u satisfies almost ewerywhere on E the differential equation with a retarded argument* (1) *and initial condition* (2) *for $(x, y)\in E_0$.*

Suppose that there exists a weak solution u of (1). In this paper we give sufficient conditions for the convergence of the sequence $(v^{(m)})$, where $v^{(m)}$ are solution of a difference equation corresponding to (1), to the weak solution u of (1).

A difference scheme for the classicall solutions of first order partial differential equations and partial differential-functional equations was considered in [1] and [2]. An existence and uniqueness theory for weak solutions of (1), (2) is given in [3], [4].

2. Notations and assumptions. For k, $h_i>0$, $i=1,\dots,n$ we define $x^{(j)}=jk$, $j=-n^*, -n^*+1,\dots,-1,0,1,\dots,n_0$, where $n^*k=\tau$, $n_0k\leqq a$, $(n_0+1)k>a$

$$y_i^{(j)}=jh_i,\ j=-n_i,\quad -n_i+1,\dots,-1,0,1,\dots,n_i,\quad i=1,\dots,n,$$

where $n_ih_i=b_i$. We assume that $h_i\leqq kM_i$, $i=1,\dots,n$. If $m_0, m_1,\dots,m_n$ are integers, then we write

$$m=(m_0, m_1,\dots,m_n),\ m'=(m_1,\dots,m_n)$$

and $(x^{(m_0)}, y^{(m')})=(x^{(m_0)}, y_1^{(m_1)},\dots, y_n^{(m_n)})$.

Let $\Gamma=\{m=(m_0, m'): (x^{(m_0)}, y^{(m')})\in E\}$.

$$\Gamma_0=\{m=(m_0, m'):(x^{(m_0)}, y^{(m')})\in E_0\},\quad \Gamma^{(i)}=\{m\in\Gamma: m_0=i\}.$$

$$E^*=\{(x^{(m_0)}, y^{(m')}): m=(m_0, m')\in\Gamma\}.$$

$$E_0^*=\{(x^{(m_0)}, y^{(m')}): m=(m_0, m')\in\Gamma_0\}$$

We define functions $\tilde{\varphi}_{i,k}$, $\tilde{\psi}_{ih}=(\tilde{\psi}_{i,h_1}^{(1)},\dots,\tilde{\psi}_{i,h_n}^{(n)})$, $i=1,\dots,p$, in the following way: for $(x, y)\in E$

$$\tilde{\varphi}_{i,k}(x, y)=[\tfrac{1}{k}\alpha_i(x, y)];\quad \tilde{\psi}_{i,h_s}^{(s)}(x, y)=\begin{cases}[\frac{1}{h_s}\beta_i^{(s)}(x, y)] & \text{for}\quad \beta_i^{(s)}(x, y)\geqq 0\\ 1+[\frac{1}{h_s}\beta_i^{(s)}(x, y)] & \text{for}\quad \beta_i^{(s)}(x, y)<0\\ & s=1,\dots,n\end{cases}$$

Denote, for $m\in\Gamma$,

$$\tilde{\varphi}_{i,k}(x^{(m_0)}, y^{(m')})=\varphi_i(m);\quad \tilde{\psi}_{i,h_s}^{(s)}(x^{(m_0)}, y^{(m')})=\psi_i^{(s)}(m),$$

$$\psi_i(m)=(\psi_i^{(1)}(m),\dots,\psi_i^{(n)}(m));\quad i=1,\dots,p.$$

We have $(\varphi_i(m), \psi_i(m))\in\Gamma_0\cup\Gamma_1$ if $m\in\Gamma$ and $\varphi_i(m_0, m')\leqq m_0$.

We define the difference operators $\Delta_0, \Delta_1,\dots,\Delta_n$: $\Delta_0v(x, y)=\frac{1}{k}[v(x+k, y)-v(x,y)]$, $\Delta_iv(x,y)=\frac{1}{h_i}[v(x, y)-v(x, y_1,\dots,y_{i-1}, y_i-h_i, y_{i+1},\dots,y_n)]$, $i=1,\dots,n$ and $\Delta v(x, y)=(\Delta_1v(x, y),\dots,\Delta_nv(x, y))$.

Let v be a function defined on $E\cup E_0$.

We write $v^{(m)}=v(x^{(m_0)}, y^{(m')})$.

We consider here the following difference method for the Cauchy problem (1), (2)

$$\Delta_0v^{(m)}=f(x^{(m_0)}, y^{(m')}, v^{(\varphi(m), \psi(m))}, \Delta v^{(m)})\quad\text{for}\quad m\in\Gamma\setminus\Gamma^{(n_0)} \tag{4}$$

$$v^{(m)}=\gamma(x^{(m_0)}, y^{(m')})\ \text{for}\ m\in\Gamma_0,$$

where $v^{(\varphi(m), \psi(m))}=(v^{(\varphi_1(m), \psi_1(m))},\dots,v^{(\varphi_p(m), \psi_p(m))})$.

We introduce

Assumption H. Suppose that:

(i) the function f of the variables (x, y, u, q), $u=(u_0, u_1,\dots,u_p)$, $q=(q_1,\dots,q_n)$ is continuous for $(x, y)\in E$, $u\in R^{p+1}$, $q\in R^n$;

(ii) the derivatives $f_{u_i}(i=0, 1,\dots,p)$, $f_{q_i}(i=1,\dots,n)$ exist and they are continuous on E^{p+1+n};

(iii) for $(x, y, u, q)\in E\times R^{p+1+n}$ we have

$$0 \leqq f_{u_i}(x, y, u, q) \leqq L_i, \ i=0, 1, \dots, p; \ f_{q_i}(x, y, u, q) \leqq 0, \ i=1, \dots, n;$$

(iv) for each $(x, y, u, q) \in E \times R^{p+1+n}$

$$\sum_{j=1}^{n} \frac{1}{h_j} f_{q_j}(x, y, u, q) + \frac{1}{k} \geqq 0;$$

(v) $\alpha=(\alpha_1, \dots, \alpha_p)$, $\beta=(\beta_1, \dots, \beta_p)$ are continuous on E and satisfy (3).

We define $L = \sum_{i=0}^{p} L_i$, $h=(h_1, \dots, h_n)$.

If Assumption H is satisfied, then for the weak solution u of (1), (2) there exists a function η of the variables (x, y, k, h), such that

(5) $\Delta_0 u(x, y) = f(x, y, u(x, y), u(\alpha(x, y), \beta(x, y)), \Delta u(x, y)) + \eta(x, y, k, h)$ for $(x, y) \in E$

$$u(x, y) = \gamma(x, y) \quad \text{for} \quad (x, y) \in E_0.$$

We define $S_x = \{y \in R^n: (x, y) \in E\}$ and $\omega(x, k, h) = \sup_{y \in S_x} |\eta(x, y, k, h)|$, $x \in \langle 0, a \rangle$

3. The convergence of the difference method. Now we prove

Theorem. *If*:

(i) *Assumption H is satisfied;*

(ii) $h_i \leqq k M_i$, $i=1, \dots, n$;

(iii) u *is a weak solution of* (1), (2);

(iv) $\lim_{k,h \to 0} \int_0^a \omega(x, k, h) dx = 0$,

then the difference method (4) *is convergent, i. e.*

$$\lim_{k,h \to 0} (u^{(m)} - v^{(m)}) = 0.$$

Proof. Let $r^{(m)} = u^{(m)} - v^{(m)}$ for $m \in \Gamma$ and

$$t^{(i)} = \min_{m \in \Gamma^{(i)}} r^{(m)}, \quad s^{(i)} = \max_{m \in \Gamma^{(i)}} r^{(m)}, \quad i=0, 1, \dots, n_0. \tag{6}$$

At first we prove that there exist points

$(\overline{u}^{(i)}, \overline{q}^{(i)})$, $(\widetilde{u}^{(i)}, \widetilde{q}^{(i)}) \in R^{p+1+n}$, $i=0, 1, \dots, p$, and $c_i = (c_{i1}, \dots, c_{in})$; $d_i = (d_{i1}, \dots, d_{in})$ such that $(i, c_i) \in \Gamma^{(i)}$, $(i, d_i) \in \Gamma^{(i)}$ and

$$\frac{1}{k}[t^{(i+1)} - t^{(i)}] \geqq -|\eta(x^{(i)}, y^{(c_i)}, k, h)| + f_{u_0}(x^{(i)}, y^{(c_i)}, \overline{u}^{(0)}, \overline{q}^{(0)}) \, t^{(i)} \tag{7}$$

$$+ \sum_{j=1}^{p} f_{u_j}(x^{(i)}, y^{(c_i)}, \overline{u}^{(j)}, \overline{q}^{(j)}) \, t^{(\varphi_j(i, c_i))}, \quad i=0, 1, \dots, n_0 - 1$$

and

$$\frac{1}{k}[s^{(i+1)} - s^{(i)}] \leqq |\eta(x^{(i)}, y^{(d_i)}, k, h)| + f_{u_0}(x^{(i)}, y^{(d_i)}, \widetilde{u}^{(0)}, \widetilde{q}^{(0)}) s^{(i)} \tag{8}$$

$$+ \sum_{j=1}^{p} f_{u_j}(x^{(i)}, y^{(d_i)}, \widetilde{u}^{(j)}, \widetilde{q}^{(j)}) \, s^{(\varphi_j(i, d_i))}.$$

We prove (7). Let index i, $0 \leqq i \leqq n_0 - 1$, be fixed. It follows from (6) that there exist $b_i = (b_{i1}, \dots, b_{in})$ and $c_i = (c_{i1}, \dots, c_{in})$ such that $(i, b_i) \in \Gamma^{(i)}$, $(i+1, c_i) \in \Gamma^{(i+1)}$ and

(9) $$t^{(i)}=r^{(i,b_i)}=u^{(i,b_i)}-v^{(i,b_i)};\ t^{(i+1)}=r^{(i+1,c_i)}=u^{(i+1,c_i)}-v^{(i+1,c_i)}.$$

Thus, we have

(10) $$\frac{1}{k}[t^{(i+1)}-t^{(i)}]=\frac{1}{k}[r^{(i+1,c_i)}-r^{(i,c_i)}]+\frac{1}{k}[r^{(i,c_i)}-r^{(i,b_i)}].$$

In view of Assumption H and (4), (5) we obtain

$$\frac{1}{k}[r^{(i+1,c_i)}-r^{(i,c_i)}]=\Delta_0 u^{(i,c_i)}-\Delta_0 v^{(i,c_i)}$$

$$=f(x^{(i)}, y^{(c_i)}, u^{(i,c_i)}, u^{(\varphi(i,c_i),\psi(i,c_i))}, \Delta u^{(i,c_i)})+\eta(x^{(i)}, y^{(c_i)}, k, h)-f(x^{(i)}, y^{(c_i)}, v^{(i,c_i)}, v^{(\varphi(i,c_i),\psi(i,c_i))}, \Delta v^{(i,c_i)})$$

$$\geqq -|\eta(x^{(i)}, y^{(c_i)}, k, h)|+f_{u_0}(x^{(i)}, y^{(c_i)}, \bar{u}^{(0)}, \bar{q}^{(0)})(u^{(i,c_i)}-v^{(i,c_i)})$$

$$+\sum_{j=1}^{p} f_{u_j}(x^{(i)}, y^{(c_i)}, \bar{u}^{(j)}, \bar{q}^{(j)})(u^{(\varphi_j(i,c_i),\psi_j(i,c_i))}-v^{(\varphi_j(i,c_i),\psi_j(i,c_i))})$$

$$+\sum_{j=1}^{n} f_{q_j}(x^{(i)}, y^{(c_i)}, \bar{\bar{u}}^{(j)}, \bar{\bar{q}}^{(j)})(\Delta_j u^{(i,c_i)}-\Delta_j v^{(i,c_i)})$$

for some $(\bar{u}^{(j)}, \bar{q}^{(j)}), (\bar{\bar{u}}^{(j)}, \bar{\bar{q}}^{(j)})\in R^{p+1+n}$.

Since for $j=1,\ldots,n$

$$\Delta_j u^{(i,c_i)}-\Delta_j v^{(i,c_i)}=\frac{1}{h_j}[r^{(i,c_i)}-r^{(i,c_{i1},\ldots,c_{ij-1},c_{ij}-1,c_{ij+1},\ldots,c_{in})}]$$

$$r^{(i,c_{i1},\ldots,c_{ij-1},c_{ij}-1,c_{ij+1},\ldots,c_{in})}\geqq r^{(i,b_i)}$$

$$u^{(i,c_i)}-v^{(i,c_i)}=r^{(i,c_i)} \quad\text{and}\quad r^{(i,c_i)}-r^{(i,b_i)}\geqq 0,$$

we obtain

(11) $$\frac{1}{k}[r^{(i+1,c_i)}-r^{(i,c_i)}]\geqq -|\eta(x^{(i)}, y^{(c_i)}, k, h)|+f_{u_0}(x^{(i)}, y^{(c_i)}, \bar{u}^{(0)}, \bar{q}^{(0)})r^{(i,c_i)}$$

$$+\sum_{j=1}^{p} f_{u_j}(x^{(i)}, y^{(c_i)}, \bar{u}^{(j)}, \bar{q}^{(j)})\, r^{(\varphi_j(i,c_i),\psi_j(i,c_i))}$$

$$+\sum_{j=1}^{n} f_{q_j}(x^{(i)}, y^{(c_i)}, \bar{\bar{u}}^{(j)}, \bar{\bar{q}}^{(j)})\frac{1}{h_j}(r^{(i,c_i)}-r^{(i,c_{i1},\ldots,c_{ij-1},c_{ij}-1,c_{ij+1},\ldots,c_{in})})$$

and

(12) $$\sum_{j=1}^{n} f_{q_j}(x^{(i)}, y^{(c_i)}, \bar{\bar{u}}^{(j)}, \bar{\bar{q}}^{(j)})\frac{1}{h_j}(r^{(i,c_i)}-r^{(i,c_{i1},\ldots,c_{ij-1},c_{ij}-1,c_{ij+1},\ldots,c_{in})})$$

$$+\frac{1}{k}[r^{(i,c_i)}-r^{(i,b_i)}]\geqq (r^{(i,c_i)}-r^{(i,b_i)})[\sum_{j=1}^{n} f_{q_j}(x^{(i)}, y^{(c_i)}, \bar{\bar{u}}^{(j)}, \bar{\bar{q}}^{(j)})\frac{1}{h_j}+\frac{1}{k}]\geqq 0.$$

The estimations (11), (12) and (10) imply (7).

In a similar way we prove (8).

In virtue of (7) we have:

(13) $$t^{(i+1)}\geqq[1+kf_{u_0}(x^{(i)}, y^{(c_i)}, \bar{u}^{(0)}, \bar{q}^{(0)})]\,t^{(i)}$$

$$+k\sum_{j=1}^{p} f_{u_j}(x^{(i)}, y^{(c_i)}, \bar{u}^{(j)}, \bar{q}^{(j)})\, t^{(\varphi_j(i,c_i))}-k|\eta(x^{(i)}, y^{(c_i)}, k, h)|;\quad i=0, 1,\ldots, n_0-1$$

and

$$t^{(i)}=0 \quad \text{for } i\leq 0.$$

Inequality (13) together with the condition $\varphi_j(i, c_i)\leq i$, $i=0, 1, \ldots, n_0$; $j=1, \ldots, p$, imply

(14) $$t^{(i)}\geq -k\,|\,\eta(x^{(0)}, y^{(c_0)}, k, h)\,(1+kL)^{i-1}-k\,|\eta(x^{(1)}, y^{(c_1)}, k, h)|\,(1+kL)^{i-2}$$
$$-\cdots-k\,|\,\eta\,(x^{(i-2)}, y^{(c_{i-2})}, k, h)\,|\,(1+kL)-k\,|\eta(x^{(i-1)}, y^{(c_{i-1})}, k, h)\,|.$$

As a result of (6) and (14) we get

$$u^{(m)}-v^{(m)}\geq -k\,|\,\eta(x^{(0)}, y^{(c_0)}, k, h)\,|\,(1+kL)^{m_0-1}$$
$$-\cdots-k\,|\,\eta(x^{(m_0-2)}, y^{(c_{m_0-2})}, k, h)\,|(1+kL)-k\,|\eta(x^{(m_0-1)}, y^{(c_{m_0-1})}, k, h)\,|$$

and

(15) $$u^{(m)}-v^{(m)}\geq -M[k|\eta(x^{(0)}, y^{(c_0)}, k, h)|+k\,|\,\eta(x^{(1)}, y^{(c_1)}, k, h)\,|$$
$$+\cdots+k\,|\eta(x^{(m_0-1)}, y^{(c_{m_0-1})}, k, h)\,|]\geq -M[k\omega(x^{(0)}, k, h)$$
$$+k\omega(x^{(1)}, k, h)+\cdots+k\omega(x^{(m_0-1)}, k, h)]=-MI(k, h),$$

where $M=\exp(aL)\geq(1+kL)^{m_0}$.
In a similar way we prove

(16) $$u^{(m)}-v^{(m)}\leq MI(k, h).$$

It follows from assumption (iv) of the theorem that

(17) $$\lim_{k,h\to 0} I(k, h)=0.$$

From (15), (16) and (17) we obtain

$$\lim_{k,h\to 0}|\,u^{(m)}-v^{(m)}\,|=0.$$

This completes the proof.

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