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EXISTENCE AND COMPLETENESS OF THE WAVE OPERATORS FOR DISSIPATIVE SYSTEMS

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1. Introduction. Scattering theory of operators of Schrödinger type has been extensively studied recently. A new elegant method, treating the Schrödinger equation, has been developed by V. Enss in [1]. The investigations of Enss has been extended by B. Simon in [5] on a large number of examples. One of them is the Schrödinger equation with absorption, where the perturbed operator H acting in the Hilbert space $L^2(R^m)$ is of the form $H=-\Delta+V$ with a potential V which is no longer self-adjoint but $\operatorname{Im} V\leqq 0$. Then, if V is $-\Delta-$ bounded with relative bound smaller than 1 (see [3]), e^{-itH} will be contraction semi-group for $t\geqq 0$. In this paper we extend the results of B. Simon for elliptic dissipative systems of differential operators.

Consider the matrix-valued differential operator $H_0=\Sigma_{|\alpha|=l} A_\alpha D^\alpha$ in the Hilbert space $\mathscr{H}=L^2(R^m; C^r)$, where $D^\alpha=D_1^{\alpha_1}\ldots D_m^{\alpha_m}$, $D_j=-i\partial/\partial x_j$, $l\geqq 1$, and A_α are constant $(r\times r)$ matrices. We impose the following conditions on the symbol $A(\xi)=\Sigma_{|\alpha|=l} A_\alpha\xi^\alpha$ of the unperturbed operator H_0:

A1) $A(\xi)=\Sigma_{|\alpha|=l} A_\alpha \xi^\alpha$, $l\geqq 1$ and A_α are constant symmetric matrices;

A2) $\det A(\xi)\neq 0$ for $\xi\neq 0$;

A3) The matrix $A(\xi)$ has s disjoint eigenvalues $a_1(\xi),\ldots, a_s(\xi)$ with constant multiplicities d_j, $j=1,\ldots,s$ for $\xi\in R^m\setminus 0$.

The self-adjoint operator H_0 generates a unitary group e^{-itH_0} on the Hilbert space $\mathscr{H}$.

The perturbed operator H is defined by the equality $H=H_0+V$ where V is an operator in $\mathscr{H}$ such that $D(V)\supset D(H_0)$, $D(V^*)\supset D(H_0)$ and:

B1) $\|V\varphi\|\leqq a\|H_0\varphi\|+b\|\varphi\|$ with some $a<1$;

B2) $\operatorname{Im}(V\varphi, \varphi)\leqq 0$ for any $\varphi\in D(H_0)$;

B3) $h(R)=\|V(H_0-i)^{-1}\chi(|x|>R)\|\in L^1(R_0, \infty)$;

B4) $h^*(R)=\|V^*(H_0-i)^{-1}\chi(|x|>R)\|\in L^1(R_0, \infty)$.

Here $\chi(|x|\geqq C)$ stands for the operator of multiplication by the characteristic function of the set $\{x/|x|\geqq C\}$ and R_0 is a positive constant.

As a consequence of B1) and B2), H is a closed operator on $D(H_0)$. Moreover, iH generates a contraction semi-group $B_t=e^{-itH}$ in $\mathscr{H}$ (see [3], Th. X. 50). Finally, B3) and B4) are the usual Enss' conditions considered in [1, 2, 5].

Let $\mathscr{H}_b$ be the subspace of $\mathscr{H}$ generated by the eigen-vectors of the operator H, corresponding to real eigenvalues. Our main result is the following

Theorem 1.1. *Suppose the assumptions* (A1)—(A3), (B1)—(B4) *fulfilled. Then*

a) *The wave operator* $\Omega^+=s-\lim_{t\to-\infty} B_{-t}e^{-itH_0}$ *exists*;

b) *The limit* $W\varphi=\lim_{t\to\infty} e^{itH_0}B_t\varphi$ *exists for any* $\varphi\in\mathscr{H}_b^\perp$;

c) *The only possible finite limit point of real eigenvalues of* H *is zero and any non-zero real eigenvalue has finite multiplicity.*

2. Preliminaries. First we give some information about the operators H and H_*

Lemma 2.1. *Let iH be the generator of a contraction semi-group $B_t = e^{-itH}$ on a Hilbert space $\mathscr{H}$. Then $-iH$ is also the generator of a semi-group $C_t = e^{itH^*}$ and $C_t = B_t^*$.*

Lemma 2.2. *Let iH be the generator of a contraction semi-group on a Hilbert space $\mathscr{H}$. Suppose $H\varphi = E\varphi$ with E real. Then $H^*\varphi = E\varphi$. In particular:*

a) *If $H\varphi = E\varphi$, $H\psi = \lambda\psi$ with $E \neq \lambda$ and at least one real, then $(\varphi, \psi) = 0$;*

b) *If $\mathscr{H}_b$ is the span of the eigenvectors of H with real eigenvalues then $\mathscr{H}_b$ and $\mathscr{H}_b^\perp$ are invariant spaces for H.*

The proofs of the lemmas above can be found in [5]. A basic point in our approach is the following form of RAGE theorem, obtained by B. Simon in [5].

Theorem 2.1. *Let $\mathscr{H}$ and $\mathscr{H}_b$ be as in Lemma 2.2 Suppose that L is a bounded operator and $L(H-i)^{-1}$ is a compact one. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \| LB_t f \|^2 dt = 0 \quad for \quad f \in \mathscr{H}_b^\perp.$$

As usual from this result and the compactness of the operator $\chi(|x|<n)(H_0-i)^{-1}$ it follows that we can find a sequence $t_n \to \infty$ with the property

$$\lim_{n\to\infty} \| \chi(|x|<n) \varphi_n \| = 0, \text{ where } \varphi_n = B_{t_n} \varphi. \tag{2.1}$$

At the end of this section we shall note that the eigenvalues $a_j(\xi) = |\xi|^l a_j(\xi/|\xi|)$, $j = 1, \dots, s$ of the matrix $A(\xi)$ and the projectors $\Pi_j(\xi) = \Pi_j(\xi/|\xi|)$ on the corresponding eigenspaces are smooth for $\xi \neq 0$, according to the assumption A3). It is easy to prove that the operator H_0 with the symbol $A(\xi)$ has an absolutely continuous spectrum.

3. The Enss' decomposition. The aim of this and the next two sections is to prove the following

Theorem 3.1. *Let H obeys the hypotheses of Theorem 1.1. Let $\{\varphi_n\}$, $\{\eta_n\}$ $n = 1, 2, 3, \dots$ be two sequences in $\mathscr{H}$ satisfying the conditions*

a) $\eta_n = (H-i)^{-2} H\varphi_n, \quad \varphi_n \in D(H)$;

b) $\| \varphi_n \| \leq 1$;

c) $\| \chi(|x|<n) \varphi_n \| \to 0, \quad as \ n \to \infty.$

Then there exists a function $\varepsilon(M) \to 0$ as $M \to \infty$, such that for each $M > 2$ one can decompose

$$\eta_n = \eta^{(1)}_{n,M,w} + \eta^{(2)}_{n,M,w} + \eta_{n,M,\text{out}} + \eta_{n,M,\text{in}} \tag{3.1}$$

so that

(A) $\| \eta^{(1)}_{n,M,w} \| \to 0 \quad as \quad n \to \infty$, M-fixed,

(B) $\| \eta^{(2)}_{n,M,w} \| \leq \varepsilon(M)$ *for each n,*

(C) $\int_0^\infty \| Ve^{-itH_0} \eta_{n,M,\text{out}} \| dt \to 0 \quad as \quad n \to \infty, \quad M\text{-fixed},$

(D′) $\int_0^\infty \| (H^*+i)^{-1} V^* e^{itH_0} \eta_{n,M,\text{in}} \| dt \to 0$ *as* $n \to \infty$, *M-fixed.*

Furthermore, if $\varphi_n = B_{t_n}\varphi$ (and $\eta_n = B_{t_n}\eta$) for some sequence $t_n \to \infty$, then $\eta_{n,M,\text{in}}$ obeys

(D) $\| \eta_{n,M,\text{in}} \| \to 0 \quad as \quad n \to \infty, \quad M\text{-fixed}.$

In this section we will describe the decomposition. Fix a constant $M>2$. Choose two functions F and ψ_M such that

(3.2) $$F(z)=z(z-i)^{-2}=(z-i)^{-1}+i(z-i)^{-2};$$

(3.3) $$\psi_M=\begin{cases} 1, & 2/M<|s|<M/2 \\ 0, & |s|<1/M \text{ or } |s|>M \end{cases}$$

and $\psi_M \in C^\infty(R^1)$.

Let $f_M \in \mathrm{J}(R^m)$ be a function with the properties $0\leq f_M(x)$, $\int f_M(x)\,dx=1$ $\operatorname{supp}\hat{f}_M\subset B_{\varepsilon_M}=\{\xi/|\xi|\leq\varepsilon_M\}$ for a suitable $\varepsilon_M>0$, specified below.

Denote by K the compact subset of $R^m\setminus 0$: $K=\{\xi/\ 1/Mb_0\leq|\xi|^l\leq M/a_0\}$.

Since

$$0<a_0=\min_{\substack{\omega\in S^{m-1}\\ 1\leq k\leq s}}|a_k(\omega)|\leq|a_j(\omega)/\leq\max_{\substack{\omega\in S^{m-1}\\ 1\leq k\leq s}}|a_k(\omega)|=b_0<\infty$$

for any $j=1,\ldots s$, we obtain that $\operatorname{supp}\psi_M(a_j(\omega))\subset K$.

Choose $\varepsilon_M>0$ such that $K+B_{\varepsilon_M}\subset O\subset\bar{O}\subset R^m\setminus 0$, where O is a bounded open set in R^m. Denote by U_α the unit cube, centred at $\alpha\in Z^m$ and let χ_α be the characteristic function of U_α. Consider the function $f_\alpha=f_M*\chi_\alpha$.

Thus, we obtain the partition of the unity in the "x"-space $1=\Sigma_{\alpha\in Z^m}f_\alpha(x)$, such that for any $\alpha\in Z^m$ the function $f_\alpha(x)$ is localized (not strictly) about α. In order to construct a suitable decomposition in "ξ"-space, choose two functions G_{out} and G_{in} in $C_0^\infty(R^m)$, such that the following two properties hold: $C_{\text{in}}(v)+G_{\text{out}}(v)=1$ if $0<A<|v|<B$, where A and B are positive constants, such that

$$\{v_j(\xi)=\operatorname{grad}a_j(\xi);\ \xi\in O\}\subset\{v;\ A<|v|<B\}.$$

This is possible, because Euler's equality $la_j(\xi)=\langle\xi,\operatorname{grad}a_j(\xi)\rangle$ guarantees that $v_j(\xi)\neq 0$ for $\xi\neq 0$.

(3.4) $$G_{\text{in}}(v)=0\quad(G_{\text{out}}(v)=0)\ \text{ if }\ 0<A<|v|<B$$

and the angle between v $(-v)$ and $(1,0,\ldots,0)$ is smaller than 45^0. For $\alpha\in Z^m$, let R_α be a rotation, taking α to $(|\alpha|,0,\ldots,0)$ and finally, let $g^{\text{out}}_{\alpha,j}=G_{\text{out}}(R_\alpha(v_j(\xi)))$ and $g^{\text{in}}_{\alpha,j}=G_{\text{in}}(R_\alpha(v_j(\xi)))$.

Consider the pseudodifferential operators:

(3.5) $$P^{\text{ex}}_n=\sum_{j=1}^{s}\sum_{|\alpha|>n/2}g^{\text{ex}}_{\alpha,j}(D)\,\psi_M(a_j(D))\,F(a_j(D))\,\Pi_j(D)\,f_\alpha(x)\,\Pi_j(D)$$

$$\text{ex}=\begin{cases}\text{in}\\ \text{out}\end{cases}$$

Lemma 3.1. *P^{out}_n and P^{in}_n are well-defined bounded operators, satisfying the following inequalities:*

$$\|P^{\text{ex}}_n\|\leq C,\ \|(H_0-i)P^{\text{ex}}_n\|\leq C,\ \|(H_0-i)P^{\text{ex}*}_n\|\leq C$$

for $\text{ex}=\begin{cases}\text{in}\\ \text{out}\end{cases}$, *with constant C, independent of n.*

The proof of this lemma is similar to the proof of the corresponding result in [5]. For the proof of the last inequality it is important to observe, that the last operator

$\Pi_j(D)$ in (3.5) can be replaced by $\Pi_j(D)\gamma(D)$, where γ is a suitable function in $C_0^\infty(R^m\setminus 0)$, such that $\gamma(\xi)=1$ for $\xi\in\bar{O}$.

Now we can write down the needed decomposition

$$\text{(3.6)}\qquad \eta^{(1)}_{n,M,w}=(F(H)-F(H_0))\varphi_n+\sum_{j=1}^{s}\sum_{|\alpha|\leq\frac{n}{2}}\psi_M(H_0)F(H_0)\Pi_j(D)f_\alpha\Pi_j(D)\varphi_n;$$

$$\eta^{(2)}_{n,M,w}=(I-\psi_M(H_0))F(H_0)\varphi_n;\quad \eta_{n,M,\mathrm{out}}=P_n^{\mathrm{out}}\varphi_n;\quad \eta_{n,M,\mathrm{in}}=P_n^{in}\varphi_n.$$

Since $\operatorname{supp}((\psi_M F)(a_j(D))\Pi_j(D)f_\alpha\Pi_j(D)\varphi_n)^\frown\subset\operatorname{supp}\psi_M(a_j(\xi))\subset O$ and $g^{\mathrm{in}}_{\alpha,j}(\xi)+g^{\mathrm{out}}_{\alpha,j}(\xi)=1$ for $\xi\in O$, the equality (3.1) holds.

4. Free time evolution of the incoming and outgoing states. The important role in the investigation of the incoming and outgoing states plays the following

Lemma 4.1. *There exists a constant $\delta>0$, depending only on the matrix $A(\xi)$, such that*

$$\text{(4.1)}\qquad \|\chi(|x|<\delta(n+t))e^{-itH_0}(H_0-i)P_n^{\mathrm{out}}\|\leq C_N(1+n+t)^{-N};$$

$$\text{(4.2)}\qquad \|\chi(|x|<\delta(n+t))e^{itH_0}(H_0-i)P_n^{\mathrm{in}}\|\leq C_N(1+n+t)^{-N};$$

$$\text{(4.3)}\qquad \|\chi(|x|<\delta(n+t))e^{itH_0}(H_0-i)P_n^{\mathrm{in}*}\|\leq C_N(1+n+t)^{-N}$$

for $t\geq 0$, $n=1,2,\ldots$ with a constant C_N, independent of t and n.

Proof: Denote $P^{\mathrm{out}}_{\alpha,j}=g^{\mathrm{out}}_{\alpha,j}(D)F(a_j(D))\psi_M(a_j(D))\Pi_j(D)f_\alpha\Pi_j(D)$. In order to verify the first estimate (4.1) it is sufficient to prove that

$$|e^{-itH_0}(H_0-i)P^{\mathrm{out}}_{\alpha,j}h|(x)\leq C_N(t+|\alpha|)^{-N}\|h\|$$

for $t\geq 0$, $\alpha\in Z^m$ and $|x|<\delta(|\alpha|+t)$ with suitable $\delta>0$.

It is easy to compute that

$$\text{(4.4)}\qquad [e^{-itH_0}(H_0-i)P^{\mathrm{out}}_{\alpha,j}h](x)=(2\pi)^{-m}\int e^{i\langle x-\alpha,\xi\rangle-ita_j(\xi)}h_{\alpha,j}(\xi)\,d\xi,$$

where $h_{\alpha,j}(\xi)=e^{i\langle\alpha,\xi\rangle}(a_j(\xi)-i)g^{\mathrm{out}}_{\alpha,j}(\xi)\psi_M F\Pi_j\widehat{f_\alpha\Pi_j h}(\xi)$.

We have $h_{\alpha,j}\in C_0^\infty(R^m\setminus 0)$ and $\operatorname{supp}h_{\alpha,j}\subset K$. In order to integrate by parts in (4.4) consider the operator

$$L_j(x,t,\alpha,\xi,D_\xi)=|x-\alpha-tv_j(\xi)|^{-2}\sum_{k=1}^{m}(x-\alpha-tv_j(\xi))_k\frac{1}{i}\frac{\partial}{\partial\xi_k}$$

where $v_j(\xi)=\operatorname{grad}a_j(\xi)$. It is evident that $L_j e^{i\langle x-\alpha,\xi\rangle-ita_j(\xi)}=e^{i\langle x-\alpha,\xi\rangle-ita_j(\xi)}$. Let us estimate the vector $x-\alpha-tv_j(\xi)$ for $\xi\in\operatorname{supp}h_{\alpha,j}$.

$$\text{(4.5)}\qquad |x-\alpha-tv_j(\xi)|\geq|\alpha+tv_j(\xi)|-\delta(|\alpha|+t)=(|\alpha|^2+|tv_j|^2+2\langle\alpha,tv_j\rangle)^{1/2}-\delta(|\alpha|+t)\geq\rho(|\alpha|+t)$$

for sufficiently small $\rho>0$, according to (3.4). After integrating by parts in (4.4) we obtain

$$\text{(4.6)}\qquad |e^{-itH_0}(H_0-i)P^{\mathrm{out}}_{\alpha,j}h|(x)\leq\int_K|L^{*N}h_{\alpha,j}|(\xi)\,d\xi.$$

On the other hand, by using the inequality (4.5), one can prove that

(4.7) $$|L^{*N} h_{\alpha,j}|(\xi)\leq C_N(t+|\alpha|)^{-N} \sum_{|\beta|\leq N} |D^\beta h_{\alpha,j}|(\xi).$$

From (4.6) and (4.7), applying Schwartz inequality, we obtain

$$|\chi(|x|<\delta(|\alpha|+t))\, e^{-itH_0}(H_0-i)\, P^{\text{out}}_{\alpha,j} h|(x)$$

$$\leq C'_N(t+|\alpha|)^{-N} \Big\| \sum_{|\beta|\leq N} D^\beta_\xi e^{i\langle\alpha,\xi\rangle}(a_j(\xi)-i)\, g^{\text{out}}_{\alpha,j}(\xi)\psi_M F \Pi_j f_\alpha \widehat{\Pi_j h}(\xi)\Big\|.$$

We use Leibniz rule to differentiate the term

$$[(a_j(\xi)-i)\, g^{\text{out}}_{\alpha,j}(\xi)\,\psi_M F\Pi_j(\xi)]\,[e^{i\langle\alpha,\xi\rangle} \widehat{f_\alpha \Pi_j h}(\xi)]$$

and the fact, that $\sup_\alpha \| D^\beta_\xi g^{\text{out}}_{\alpha,j}(a_j-i)\psi_M F\Pi_j\| < \infty$.

5. Proof of Theorem 3.1. In this section we shall establish the properties (A), (B), (C), (D′) and (D) of the decomposition (3.1). We begin with the

P r o o f o f (*A*): Consider the first term $\eta^{(1)}_{n,M,w}$ in (3.6).

$$\|(F(H)-F(H_0))\,\varphi_n\| \leq \|F(H)-F(H_0)\| \;\; \|\chi(|x|<n)\,\varphi_n\|$$

$$+\|[(H-i)^{-1}-(H_0-i)^{-1}]\,\chi(|x|>n)\| \;\; \|\varphi_n\| + \|[(H-i)^{-2}-(H_0-i)^{-2}]\,\chi(|x|>n)\| \;\; \|\varphi_n\|.$$

The first two summands tend to zero, according to assumption (c) of Theorem 3.1 and condition (B3) from section 1. By using the Vitali convergence theorem one can verify the convergence to zero of the third summand.

For the second term in (3.6) we have that

$$\Big\|\sum_{j=1}^{s} \sum_{|\alpha|\leq \frac{n}{2}} \psi_M(a_j(D))F(a_j(D))\,\Pi_j(D)\, f_\alpha \Pi_j(D)\,\varphi_n\Big\|$$

$$\leq C\sum_{j=1}^{s} \Big\|[f_M * \chi(x\in \bigcup_{|\alpha|\leq \frac{n}{2}} U_\alpha)]\,\Pi_j(D)\gamma(D)\varphi_n\Big\|.$$

The condition (c) of Theorem 3.1 and the fact that the operator of convolution with the l-function $\widecheck{(\Pi_j\gamma)}(x)$ "doesn't delocalize too much" the function φ_n (for a strong proof see [4, 5]) complete the proof of the property (A).

P r o o f o f (*B*): The estimate (B) is obtained immediately: $\|\eta^{(2)}_{n,M,w}\| \leq \|(1-\psi_M)F\|_\infty \|\varphi_n\| \leq \|(1-\psi_M)F\|_\infty = \varepsilon(M)$. From (3.2) and (3.3) it follows that $\lim_{M\to\infty}\varepsilon(M)=0$.

P r o o f o f (C): We start with the inequality $\|Ve^{-itH_0}\eta_{n,M,\text{out}}\| \leq \|V(H_0-i)^{-1}\chi(|x|>\delta(n+t))\| \;\; \|(H_0-i)P^{\text{out}}_n\| \;\; \|\varphi_n\| + \|V(H_0-i)^{-1}\| \;\; \|\chi(|x|<\delta(n+t))\,e^{-itH_0}(H_0-i)\,.\,P^{\text{out}}_n\| \;\; \|\varphi_n\|$. Then, the property (C) follows from the assumption (B3), Lemma 3.1 and Lemma 4.1.

P r o o f o f (D′): This proof is based on the inequality

$$\|(H^*+i)^{-1}V^* e^{itH_0}\eta_{n,M,in}\| \leq \|(H^*+i)^{-1}V^*\| \;\; \|(H_0-i)^{-1}\| \;\; \|\chi(|x|<\delta(n+t))\,e^{itH_0}(H_0-i)\,P^{\text{in}}_n\| \;\; \|\varphi_n\| + \|(H^*+i)^{-1}\| \;\; \|V^*(H_0-i)^{-1}\chi(|x|>\delta(n+t))\| \;\; \|(H_0-i)\,P^{\text{in}}_n\| \;\; \|\varphi_n\|$$

and the assumption (B4), Lemma 3.1 and Lemma 4.1.

Proof of (D): We shall follow essentialy the argument, raised by V. Enss (see [5]). The equality

$$\eta_{n,M,\text{in}}=P_n^{\text{in}}\, e^{-it_nH_0}\varphi+P_n^{\text{in}}\,(B_{t_n}-e^{-it_nH_0})(H_0-i)^{-1}(H_0-i)\varphi$$

shows that the property (D) is a consequence of two facts:

$$\lim_{n\to\infty}\|P_n^{\text{in}}(e^{-it_nH_0}-B_{t_n})(H-i)^{-1}\|=0 \tag{5.1}$$

$$s-\lim_{n\to\infty}P_n^{\text{in}}\, e^{-it_nH_0}=0. \tag{5.2}$$

Consider (5.1). We have

$$\|(H^*+i)^{-1}(B_{t_n}^*-e^{it_nH_0})P_n^{\text{in}*}\varphi\|\le\int_0^{t_n}\|(H^*+i)^{-1}B_{t_n-u}^*\,V^*e^{iuH_0}P_n^{\text{in}*}\|\,\|\varphi\|du$$

$$\le\int_0^{\infty}\|(H^*+i)^{-1}V^*e^{iuH_0}P_n^{\text{in}*}\|du\,.\,\|\varphi\|.$$

From the proof of (D') it follows that (5.1) holds. The property (5.2) follows from (5.1). Indeed, for any $R>0$ we have $P_n^{\text{in}}(H_0-i)e^{-it_nH_0}\varphi\to 0$ for every function φ with $\operatorname{supp}\varphi\subset B_R$. Hence, it is true for any φ. This completes the proof of Theorem 3.1.

6. Proof of Theorem 1.1. In this last section we shall prove Theorem 1.1, applying Enss' decomposition principle (Th. 3.1).

The existence of the operator Ω^+ can be proved easily using Cook's method (see [4]).

Let us turn to the proof of the existence of the limit $W\varphi$ for $\varphi\in\mathscr{H}_b^{\perp}$. It is sufficient to establish the existence of the limit $W\eta$ only for the functions $\eta=(H-i)^{-2}H\varphi$, where $\varphi\in D(H)\cap\mathscr{H}_b^{\perp}$ (see [5]). Fix the functions φ and η and let t_n be the sequence from (2.1), where $\varphi_n=B_{t_n}\varphi$, $\eta_n=B_{t_n}\eta$. We can assume that $\|\varphi\|=1$. Denote $\alpha_n=\sup_{t>0}\|(B_t-e^{-itH_0})\eta_n\|$. Applying Theorem 3.1, we have

$$\|(B_t-e^{-itH_0})\eta_n\|\le 2\|\eta^{(1)}_{n,M,w}\|+2\|\eta^{(2)}_{n,M,w}\|$$

$$+\Big\|\int_0^t-\frac{d}{du}(B_{t-u}e^{-iuH_0}\eta_{n,M,\text{out}})du\Big\|+2\|\eta_{n,M,\text{in}}\|$$

$$\le 2\,\varepsilon(M)+\int_0^{\infty}\|Ve^{-iuH_0}\eta_{n,M,\text{out}}\|du+o(n).$$

The properties (A), (B), (C) and (D) guarantee that $0\le\overline{\lim}\,\alpha_n\le 2\,\varepsilon(M)$, so $\lim\alpha_n=0$. This leads immediately to the existence of the limit $W\eta$. Indeed, let $t>t_n$, $s>t_n$. Then

$$\|e^{itH_0}B_t\eta-e^{isH_0}B_s\eta\|\le\|e^{itH_0}B_t\eta-e^{it_nH_0}B_{t_n}\eta\|+\|e^{isH_0}B_s\eta-e^{it_nH_0}B_{t_n}\eta\|\le 2\alpha_n.$$

Hence, the sequence $e^{itH_0}B_t\eta$ is convergent.

Finally, consider the point (c) of Theorem 1.1. Suppose the contrary. Then we can find an orthonormal sequence φ_n, such that $H\varphi_n=E_n\varphi_n$, $E_n\in R$, $E_n\to E\neq 0,\ \pm\infty$. According to Lemma 2.2, we have $H\varphi_n=E_n\varphi_n=H^*\varphi_n$. Set $\eta_n=(E_n-i)^{-2}E_n\varphi_n$. Applying Theorem 3.1 and the fact that $\|\eta_n\|\le C$ and $\lim\|\eta_n\|=|(E-i)^{-2}E|\neq 0,\ \infty$, we get $\|\eta_n\|^2=(\eta_n,\ \eta^{(1)}_{n,M,w})+(\eta_n,\ \eta^{(2)}_{n,M,w})+(\eta_n,\ \eta_{n,M,\text{out}})+(\eta_n,\ \eta_{n,M,\text{in}})$.

The first term tends to zero according to (A). The second one can be estimated as follows $|(\eta_n, \eta^{(2)}_{n,M,w})| \leq C\varepsilon(M)$.

Consider the third term. We have

$$|(\eta_n, \eta_{n,M,\text{out}})| = |(B_t^* \eta_n, \eta_{n,M,\text{out}})| = |(\eta_n, B_t \eta_{n,M,\text{out}})| = \lim_{t\to\infty} |(\eta_n, (B_t - e^{-itH_0})\eta_{n,M,\text{out}})|$$

$$\leq C \sup_{t>0} \|(B_t - e^{-itH_0})\eta_{n,M,\text{out}}\| \leq C \int_0^\infty \|Ve^{-itH_0}\eta_{n,M,\text{out}}\| \, dt \to 0.$$

Here we used the fact that $w - \lim_{t\to\infty} e^{-itH_0}\eta_n = 0$ because H_0 has an absolutely continuous spectrum. The fourth term admits a similar estimate:

$$(\eta_n, \eta_{n,M,\text{in}})| = |(B_t\eta_n, \eta_{n,M,\text{in}})| = \lim_{t\to\infty} |(\eta_n, (B_t^* - e^{itH_0})\eta_{n,M,\text{in}})|$$

$$= |E_n - i| \lim_{t\to\infty} |((H-i)^{-1}\eta_n, (B_t^* - e^{itH_0})\eta_{n,M,\text{in}})|$$

$$\leq C|E_n - i| \sup_{t>0} \|(H^* + i)^{-1}(B_t^* - e^{itH_0})\eta_{n,M,\text{in}}\| \leq C|E_n^{-1}| \sup_{t>0} |(H^* t_i)^{-1}(B_t^* - e^{itH_0})\eta_{n,M,\text{in}}|$$

$$\leq C|E_n - i| \int_0^\infty \|(H^* + i)^{-1} V^* e^{itH_0}\eta_{n,M,\text{in}}\| dt \to 0.$$

We conclude that $\lim \|\eta_n\|^2 \leq C\varepsilon(M)$ for any $M > 2$. Hence, $\lim_{n\to\infty} \|\eta_n\| = 0$, which contradicts the assumptions.

This completes the proof of Theorem 1.1.

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