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FUNCTIONAL RELATIONS BY MEANS OF RIEMANN-LIOUVILLE OPERATOR

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The aim of this paper is to show how some interesting results involving series summation and the digamma function are established by means of Riemann-Liouville operator of fractional calculus. We derive the relation

$$\frac{\Gamma(\lambda)}{\Gamma(\upsilon)}\sum_{n=1}^{\infty}\frac{\Gamma(\upsilon+n)}{n\Gamma(\lambda+n)}\,{}_3F_2\begin{pmatrix}\alpha, & \beta, & \lambda & ; & \dfrac{x}{a}\\ \gamma, & \lambda+n & & & \end{pmatrix}$$

$$=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k\,k!}\left(\frac{x}{a}\right)^k[\psi(\lambda+k)-\psi(\lambda-\nu+k)],\ \mathrm{Re}(\lambda),\ \mathrm{Re}(\nu)\geq 0$$

and mention some particular cases.

1. Introduction. The so-called "fractional calculus" [1, 8, 10] really deals with the differentiation and integration of arbitrary orders. Two fundamental operators of the fractional calculus are: The Riemann-Liouville operator of order ν,

$$R_\nu f(x)={}_0D_x^{-\nu}f(x)=\frac{1}{\Gamma(\nu)}\int_0^x(x-t)^{\nu-1}f(t)\,dt,\quad \mathrm{Re}(\nu)\geq 0 \tag{1}$$

and, the Weyl operator,

$$W_\nu f(x)={}_xM_\infty^{-\nu}f(x)=\frac{1}{\Gamma(\nu)}\int_x^\infty(t-x)^{\nu-1}f(t)\,dt,\quad \mathrm{Re}(\nu)>0. \tag{2}$$

Thus a fractional integral is a direct generalization of the elementary concept of a repeated integral. Various extensions and generalizations of these operators have been given by many authors, such as I. Sneddon [12], S. Kalla and R. Saxena [4] and S. Kalla [3]. Recently, there has been a great interest in the study of these operators and their possible applications.

B. Ross [11] has proved the following result

$$\ln 4=\sum_{n=1}^{\infty}\frac{1.3.5\ldots(2n-1)}{n\,2^n\,n!} \tag{3}$$

by means of fractional calculus. In a following work Kalla and Ross [5] have given the functional relation

$$\psi(\lambda)-\psi(\lambda-\nu)=\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}, \tag{4}$$

$$\mathrm{Re}(\lambda)>\mathrm{Re}(\nu)\geq 0,$$

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where $\psi(\lambda)$ is the digamma function (2, 7). Following the same line, Kalla & Al-Saqabi [6] developed a functional relation involving the digamma and Gauss' hypergeometric function,

$$(5)\quad \frac{\Gamma(\lambda)}{\Gamma(\lambda)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}\,{}_2F_1(-\mu,\ \lambda\,;n+\lambda\,;\ -\frac{x}{a})=\sum_{k=0}^{\infty}\frac{(-\mu)_k}{k!}\,(-\frac{x}{a})^k[\psi(\lambda+k)-\psi(\lambda-\nu+k)]$$

$$\operatorname{Re}(\lambda)>\operatorname{Re}(\nu)\geqq 0.$$

In this paper, we establish a more general functional relation, which includes as particular cases many results, including those cited above. The symmetry of these results suggests that, the technique can be easily extended to establish similar results for the generalized hypergeometric function ${}_pF_q$ and other functions. We mention some other special cases of our main result.

2. The functional relation. We start with the result [9]

$$(6)\quad \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\,t^{\lambda-1}\,{}_2F_1\quad(\alpha,\ \beta\,;\ \gamma\,;\ \frac{t}{a})\,dt$$

$$=\frac{x^{\lambda+\nu-1}\,\Gamma(\lambda)}{\Gamma(\lambda+\nu)}\,{}_3F_2\begin{pmatrix}\alpha,\ \beta,\ \lambda & ; & \frac{x}{a}\\ \gamma,\ \lambda+\upsilon & & \end{pmatrix},\quad \operatorname{Re}(\lambda)>\operatorname{Re}(\nu)\geqq 0,$$

where ${}_3F_2$ is the generalized hypergeometric function. Differentiating both sides of (6) with respect to λ according to Leibnitz's rule, we get

$$(7)\quad \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\ln t\,t^{\lambda-1}\,{}_2F_1\ \ (\alpha,\ \beta\,;\ \gamma\,;\frac{t}{a})\,dt$$

$$=\frac{x^{\lambda+\nu-1}\,\Gamma(\lambda)}{\Gamma(\upsilon+\nu)}\,[\ln x\,{}_3F_2\begin{pmatrix}\alpha,\ \beta,\ \lambda & ; & \frac{x}{a}\\ \gamma,\ \lambda+\nu & & \end{pmatrix}$$

$$\times\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k(\lambda)_k}{(\gamma)_k(\lambda+\nu)_k\,k!}\,(\frac{x}{a})^k\,\{\psi(\lambda+k)-\psi(\lambda+\nu+k)\}].$$

For convenience, we designate the right-hand side above by $K(x,\lambda,\ \nu,\ \alpha,\ \beta,\ a)$. The above formula, by definition (1) of Riemann-Liouville operator can be written as

$$(8)\quad {}_0D_x^{-\nu}\,(x^{\lambda-1}\ln x\,{}_2F_1(\alpha,\ \beta\,;\ \lambda\,;\ \frac{x}{a}))=K(x,\ \lambda,\ \nu,\ \alpha,\ \beta,\ a).$$

Due to the property of analyticity and continuity at $\nu=0$, we can interchange the roles of $-\nu$ and ν. Hence, for differentiation of $x^{\lambda-1}\ln x\,{}_2F_1\ (\alpha,\ \beta\,;\ \gamma\,;\ x/a)$ to an arbitrary order we have

$$(9)\quad {}_0D_x^{\nu}\,(x^{\lambda-1}\ln x\,{}_2F_1(\alpha,\ \beta\,;\ \gamma\,;\ \frac{x}{a})=\frac{x^{\lambda-\nu-1}\,\Gamma(\lambda)}{\Gamma(\lambda-\nu)}\,[\ln x\,{}_3F_2\begin{pmatrix}\alpha,\ \beta,\ \lambda & ; & \frac{x}{a}\\ \gamma,\ \lambda-\nu & & \end{pmatrix}$$

$$+\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k(\lambda)_k}{(\gamma)_k(\lambda-\nu)_k\,k!}\,(\frac{x}{a})^k\,\{\psi(\lambda+k)-\psi(\lambda-\nu+k)\}]=K(x,\ \lambda,\ -\nu,\ \alpha,\ \beta,\ a).$$

We proceed to solve an integral equation of the Volterra type

$$(10)\quad \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\,dt=x^{\lambda-1}\ln x\,{}_2F_1(\alpha,\ \beta\,;\ \gamma\,;\ \frac{x}{a}),\ \operatorname{Re}(\lambda)>\operatorname{Re}(\nu)\geqq 0.$$

This is an integral equation of convolution type and we would like to solve it by the use of fractional operators, showing the power, elegance and simplicity of the method used. By definition (1), equation (10) can be written as

(11) $${}_0D_x^{-\nu} f(x) = x^{\lambda-1} \ln x \; {}_2F_1 (\alpha, \beta; \gamma; \frac{x}{a}).$$

Operating on both sides with ${}_0D_x^{\nu}$, we get

(12) $$f(x) = {}_0D_x^{\nu} (x^{\lambda-1} \ln x \; {}_2F_1 (\alpha, \beta; \gamma; \frac{x}{a})).$$

Hence, the result (9) gives us at once the solution to (10)

(13) $$f(x) = \frac{x^{\lambda-\nu-1}(\lambda)}{\Gamma(\lambda-\nu)} [\ln x \; {}_3F_2 \left(\begin{matrix} \alpha, \beta, \lambda \\ \gamma, \lambda-\nu \end{matrix} ; \frac{x}{a} \right)$$

$$+ \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k (\lambda)_k}{(\lambda)_k (\lambda-\nu)_k \, k!} \left(\frac{x}{a}\right)^k \{\psi(\lambda+k) - \psi(\lambda-\nu+k)\}].$$

We verify this result by substituting (13) into (10) in terms of the argument t. Write a series expansion for $\ln t$ as follows:

(14) $$t = x + t - x = x\left(1 + \frac{t-x}{x}\right),$$

where x and t are real and $x > 0$. Then

(15) $$\ln t = \ln x + \ln\left(1 + \frac{t-x}{x}\right).$$

When $\left|\frac{t-x}{x}\right| < 1$, we can expand $\ln\left(1+\frac{t-x}{x}\right)$ into a Taylor series expansion. Thus

(16) $$\ln t = \ln x - \sum_{n=1}^{\infty} \frac{(x-t)^n}{n x^n}$$

with the interval of convergence $0 < t \leq 2x$.

When (16) and (13) are substituted in (10), we get the following, on evaluating the corresponding β-type integrals

(17) $$\frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)} \; {}_3F_2 \left(\begin{matrix} \alpha, \beta, \lambda \\ \gamma, \lambda+n \end{matrix} ; \frac{x}{a} \right)$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k \, k!} \left(\frac{x}{a}\right)^k [\psi(\lambda+k) - \psi(\lambda-\nu+k)], \quad \mathrm{Re}(\lambda) > \mathrm{Re}(\nu) \geq 0.$$

3. Special cases:

(i) If we set $\beta = \gamma$ and replace a by $-a$ then (17) reduces to (5).

(ii) Further if $\alpha \to 0$ in (17), then it becomes the known result of Kalla and Ross (4).

(iii) For $\gamma = \lambda$, (17) becomes

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)} \; {}_2F_1 \left(\alpha, \beta; \lambda+n; \frac{x}{a}\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\lambda)_k \, k!} \left(\frac{x}{a}\right)^k [\psi(\lambda+k) - \psi(\lambda-\nu+k)],$$

and when $x \to a$

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)\Gamma(\lambda+n-\alpha-\beta)}{n\Gamma(\lambda+n-\alpha)\Gamma(\lambda+n-\beta)}=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\lambda)_k\, k!}[\psi(\lambda+k)-\psi(\lambda-\nu+k)],\ \mathrm{Re}\,(\lambda+n-\alpha-\beta)>0.$$

By specializing the parameters more special cases can be derived from our general. formula (17).

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