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GENERATING FUNCTIONS FOR THE JACOBI POLYNOMIALS

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The paper deals with a theorem on triple series and certain general expansions of a different character which gives as special cases new results involving the functions of Kampé de Fériet and Appell and Jacobi polynomials. In fact, our theorem extends a result of M. Cohen [3] on double series and generalize a number of generating functions for Jacobi polynomials. Also of interest are erroneous results (2.10) and (2.13) of M. Cohen [3] which are corrected here.

Introduction: In a paper which appeared in [3], M. Cohen presented two general theorems for double series using a generalization of operators given in [4]. His approach differs from usual procedures adopted by previous workers in that he does not apply the Lagrange Theorem [7; p. 115]. Of our concern here is one of his theorem [3; p. 272 (2.1)], which we recall as Theorem 1. For r, s, α and β any arbitrary complex numbers,

$$(1.1)\qquad \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-z)^k(-y)^p(\alpha+1)_{k+rp}(\beta+1)_{p+sk}}{k!\,p!\,(\alpha+1)_{rp}(\beta+1)_{sk}(1-z)^{k+rp}(1-y)^{p+sk}}=(1-z)^{\alpha+1}(1-y)^{\beta+1}/(1-rsyz),$$

where $(a)_n=\sqrt{(a+n)}\,/\,\sqrt{(a)}$
$=(a)\,(a+1)\dots(a+n-1)$ for n a positive integer
$=1$ for $n=0$,

and $|y|<1$, $|z|<1$ and $|rsyz|<1$.

Using the above theorem for $r=-1$ in conjugation with Gauss transformation for the hypergeometric function [10; p. 33 (2.1)] and variable changes $y'=-y\,(1-z)/(1-y)$, he obtained a result for Gauss hypergeometric function ${}_2F_1$ [10; p. 29 (4)]

$$(1.2)\qquad \sum_{k=0}^{\infty}\frac{(\alpha+1)_k}{k!}\Big[\frac{-z\,(1-z-y')^s}{[(1-z)\,(1-y')]^{s+1}}\Big]^k\,{}_2F_1\left[\begin{matrix}-k,\ -\alpha-\beta-(s+1)\,k-1;\\ \alpha-k\end{matrix};y'\right]$$
$$=(1-z)^{\alpha+\beta+2}\,(1-y')^{\beta+1}\,/\,(1-z-y')^{\beta}\,(1-z-y'-sy'z),$$

which is equivalent to H. Srivastava's equation 8 [9].

Further variable changes in (1.2) yields the following known generating functions of J. Brown [2] and E. Feldheim [6] (see also [9]).

$$(1.3)\qquad \sum_{n=0}^{\infty}P_n^{(\alpha-n,\ \beta-(b+1)\,n)}(x)\,t^n=(1+w)^{-\alpha-\beta}\,(1-bw)^{-1}\,[1+\frac{2w}{(1-x)}]^{\alpha},$$

where $w=\frac{t}{2}\,(1-x)\,(1+w)^{b+1}$, and

$$(1.4)\qquad \sum_{n=0}^{\infty}P_n^{(\alpha+bn,\ \beta-(b+1)\,n)}(x)\,t^n=(1+v)^{\alpha+1}\,(1-bv)^{-1}\,[1-\frac{1}{2}\,(x-1)\,v]^{-\alpha-\beta-1},$$

where $v=t\,(1+v)^{b+1}$ and the classical Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ [10; p. 35 (34)] defined by

$$(1.5)\qquad P_n^{(\alpha,\beta)}(x)=\binom{\alpha+n}{n}\,{}_2F_1\left[\begin{matrix}-n,\ \alpha+\beta+n+1\ ; & \frac{1-x}{2}\\ \alpha+1\ ; & \end{matrix}\right],\ \forall\ n\in\{0,1,2,\dots\}.$$

Note that equation (1.3) is given erroneously in the paper of M. Cohen [3; p. 272 (1.5)].

The main aim of the present paper is to give an interesting extension of the above theorem 1 of Cohen involving triple series with essentially arbitrary complex numbers. Some special cases of our Theorem are discussed. As a result, we are led to a correct form of the corollary 2 of M. Cohen [3; p. 274 (2.10)]. Unfortunately the same technique is not helpful in proving an extension of Theorem 2 of M. Cohen [3]. However, we noticed that equation (2.13) of [3] used in the proof of theorem 2 is also not correct. Thus, a secondary aim of this paper is to correct these erroneous results of M. Cohen [3].

2. Theorem 1. *For r, s, u, α, β and ω any arbitrary complex numbers*

$$(2.1)\qquad \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-z)^k\,(-y)^l\,(-t)^q\,(\alpha+1)_{k+rl}\,(\beta+1)_{l+sq}(\omega+1)_{q+uk}}{k!\,l!\,q!\,(\alpha+1)_{rl}\,(\beta+1)_{sq}(\omega+1)_{uk}(1-z)^{k+rl}\,(1-y)^{l+sq}\,(1-t)^{q+uk}}$$

$$=(1-z)^{\alpha+1}\,(1-y)^{\beta+1}\,(1-t)^{\omega+1}\,/\,(1+rsuyzt),$$

where $(a)_n=\sqrt{(a+n)}\,/\,\sqrt{(a)}$

$=(a)\,(a+1)\dots(a+n-1)$ *for n a positive integer*

$=1$ *for* $n=0$,

and $|y|<1$, $|z|<1$, $|t|<1$ *and* $|rsuyzt|<1$.

Proof. Consider the expression

$$(2.2)\qquad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{z^n\,y^m\,t^p}{n!\,m!\,p!}D^n\{x^{n+\alpha}(1-x^r)^m\}\,D^m\{x^{m+\beta}\,(1-x^s)^p\}\,D^p\{x^{p+\omega}\,(1-x^u)^n\}.$$

Putting the operators in polynomial form, (2.2) reduces to

$$(2.3)\qquad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{z^n\,y^m\,t^p}{n!\,m!\,p!}\sum_{k=0}^{n}\sum_{l=0}^{m}\sum_{q=0}^{p}\frac{(-n)_k\,(-m)_l\,(-p)_q}{k!\,l!\,q!}\,\frac{\sqrt{(n+\alpha+rl+1)}}{\sqrt{(\alpha+rl+1)}}$$

$$\times\frac{\sqrt{(m+\beta+sq+1)}}{\sqrt{(\beta+sq+1)}}\,\frac{\sqrt{(p+\omega+uk+1)}}{\sqrt{(\omega+uk+1)}}\,x^{\alpha+\beta+\omega+rl+uk+sq}.$$

Take (2.2) and (2.3) at $x=1$. In (2.2) only $n=m=p$ contributes and we get

$$(2.4)\qquad \sum_{n=0}^{\infty}(-rsuyzt)^n=(1+rsuyzt)^{-1}.$$

Applying the series transformation

$$(2.5)\qquad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{m}\sum_{q=0}^{p}f(n,m,p,k,l,q)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}f(n+k,\,m+l,\,p+q,\ k,\,l,\,q),$$

to (2.3), take at $x=1$ and summing the series by using a result

$$(2.6)\qquad (1-z)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n,$$

we get

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-n-k)_k\,(-m-l)_l\,(-p-q)_q\,(\alpha+1)_{k+rl}\,(\beta+1)_{l+sq}\,(\omega+1)_{q+uk}}{k!\,l!\,q!\,(\alpha+1)_{rl}\,(\beta+1)_{sq}\,(\omega+1)_{uk}\,(n+k)!\,(m+1)!\,(p+q)!}$$

$$\frac{z^k\,y^l\,t^q}{(1-z)^{k+rl}\,(1-y)^{l+sq}\,(1-t)^{q+uk}}=(1-z)^{\alpha+1}\,(1-y)^{\beta+1}\,(1-t)^{\omega+1}\,/\,(1+rsuyzt).$$

Now using a result $(n)_{-k}=(-1)^k\,/\,(1-n)_k$ we obtain the required result (2.1).

Corollary 1. *Taking $r=u=1$ in Theorem 1, we get*

$$(2.7)\qquad \sum_{k=0}^{\infty}\frac{(\alpha+1)_k}{k!}\Big[\frac{-z}{(1-z)(1-t)}\Big]^k\sum_{q=0}^{\infty}\frac{(\omega+k+1)_q}{q!}\Big[\frac{-t}{(1-y)^s(1-t)}\Big]^q$$

$$\times{}_2F_1\left[\begin{matrix}\alpha+k+1,\ \ \beta+sq+1\,; \\ \alpha+1\end{matrix}\ ;\ \frac{-y}{(1-y)\,(1-z)}\right]=(1-z)^{\alpha+1}\,(1-y)^{\beta+1}\,(1-t)^{\omega+1}/(1+syzt),$$

which on using Gauss transformation for the hypergeometric function [10; p. 33 (21)]

$$(2.8)\qquad {}_2F_1\left[\begin{matrix}a,\ b\,; \\ c\end{matrix}\ ;\ x\right]=(1-x)^{c-a-b}\ {}_2F_1\left[\begin{matrix}c-a,\ c-b\,; \\ c\end{matrix}\ ;\ x\right],$$

letting $\frac{-y}{1-y}=\frac{(1-z)\,(1-x)}{2}$, $\alpha=a$, $\beta=-b-1$, $\omega=c$ *and using the Jacobi representation* (1.5) *gives*

$$(2.9)\qquad \sum_{q=0}^{\infty}\frac{(c+1)_q}{q!}\Big[\frac{-t\,[2-(1-z)\,(1-x)]^s}{(1-t)\,(1+x)^s}\Big]^q\sum_{k=0}^{\infty}\frac{(c+q+1)_k}{(c+1)_k}\,P_k^{(a,\,b-sq-k)}(x)\,T^k$$

$$=\frac{(1-z)^{a+1}\,[2-(1-z)(1-x)]^{b+1}\,(1-t)^{c+1}}{[2-(1-z)\,(1-x)\quad(1+szt)]\,(1+x)^b},$$

where $T=-2z/(1-z)\,(1-t)\,(1+x)$.

For $t\to 0$, (2.9) reduces to the following special case of a known result of M. Cohen [3; p. 271 (1.3)]

$$\sum_{k=0}^{\infty}\Big[\frac{-2z}{(1-z)(1+x)}\Big]^k P_k^{(a,\,b-k)}(x)=\frac{(1-z)^{a+1}\,[2-(1-z)(1-x)]^b}{(1+x)^b}.$$

Using a result [10; p. 165 (10)] in (2.7), we obtain

$$(2.10)\qquad \sum_{q=0}^{\infty}\frac{(\omega+1)_q}{q!}\Big[\frac{-t\,(1-z)^s}{(1-z+yz)^s\,(1-t)}\Big]^q$$

$$F^{2:0;1}_{1:0;1}\left[\begin{matrix}\alpha+1,\ \omega+q+1:\ldots;\ \beta+sq+1\,; \\ \omega+1\qquad\qquad :\ldots,;\ \alpha+1\end{matrix}\ ;\ \frac{-z}{(1-z)\,(1-t)},\ \frac{yz}{(1-z)(1-t)\,(1-z+yz)}\right]$$

$$=(1-z)^{\alpha-\beta}\,(1-t)^{\omega+1}\,(1-z+yz)^{\beta+1}\,/\,(1+syzt),$$

where $F^{p:q;r}_{m:n;p}$ is Kampé de Fériet's function [10; p. 63 (16)].

Setting $s=1$ and $\omega=\alpha$, it gives a reduction formula

$$(2.11)\quad F_E\Big(\alpha,\alpha,\alpha,\delta,\beta,\beta;\delta,\beta,\alpha;\frac{-z}{(1-z)\,(1-t)},\ \frac{yz}{(1-z)\,(1-t)(1-z+yz)},\ \frac{-t\,(1-z)}{(1-z)(1-z+y)}\Big)$$

$$=(1-z)^{\alpha-\beta}\,(1-t)^{\alpha}\,(1-z+yz)^{\beta}\,/\,(1+yzt),$$

where F_E is Saran's function [5; p. 66].

Setting $\alpha=\beta$ and $y=(z-1)/z$ in the above equation and using [5; p. 116 (4.1.16)] we get a known result [10; p. 34 (27)].

Result (2.7) for $s=1$ is equivalent to a reduction formula

$$(2.12)\qquad H_B(\alpha,\omega,\beta;\ \omega,\beta,\alpha;\frac{-z}{(1-z)(1-t)},\frac{-t}{(1-t)(1-y)},\frac{-y}{(1-y)(1-z)})$$
$$=(1-z)^{\alpha}(1-y)^{\beta}(1-t)^{\omega}/(1+yzt),$$

where H_B is Srivastava's function [10; p. 68 (37)].

Putting $\alpha=\beta$ in the above equation and using a result of H. Exton [5; p. 116 (4.1.15)], we obtain

$$(2.13)\qquad F_4(\beta,\omega;\omega,\beta;\frac{-z(1-y)^2}{(1-z+yz)(1-y+yt)},\frac{yt}{(1-z+yz)(1-y+yt)})$$
$$=\frac{(1-z+yz)^{\beta}}{(1+yzt)}(1+\frac{yt}{1-y})^{\omega},$$

where F_4 is Appell's function [5; p. 24 (4.1.4)].

In (2.13), substituting $\theta=1-z+yz$ and $\Phi=1+\frac{yt}{1-y}$, we get a result of W. Bailey [1; p. 102 Problem 20 (iii)].

In (2.9), using a result of H. Srivastava, H. Manocha [10; p. 108 (15)], we get

$$(2.14)\qquad \sum_{q=0}^{\infty}\frac{(c+1)_q}{q!}\left[\frac{-t(1-z)\,[2-(1-z)\,(1-x)]^s}{(1+x)^{s-1}A}\right]^q$$
$$F^{1:\,1;\,2}_{1:\,0;\,1}\left[\begin{matrix}a+1:-q;c+q+1,\ a+b-sq+1; & \frac{2z}{A},\frac{(1-x)z}{A}\\ c+1:\ldots; \qquad\qquad a+1 & ;\end{matrix}\right]$$
$$=(1-z)^{a-c}[2-(1-z)(1-x)]^{b+1}A^{c+1}/[2-(1-z)(1-x)(1+szt)](1+x)^{b+c+1},$$

where $A=2-(1-z)\,[2-(1+x)\,(1-t)]$.

For $a=c$, it gives

$$(2.15)\qquad \sum_{q=0}^{\infty}\left[\frac{-t(1-z)\,(1+x)\,[2-(1-z)\,(1-x)]^s(1-t+zt)^s}{[2-(1-z)\,\{2-(1+x)\,(1-t)\}]^{s+1}}\right]^q$$
$$P_q^{(c,\,b-(s+1)q)}(1+\frac{2(1-x)z}{(1+x)(1-t+zt)})$$
$$=\frac{[2-(1-z)(1-x)]^{b+1}\quad(1-t+zt)^{b+c+1}}{[2-(1-z)\,(1-x)\,(1+szt)]\,[2-(1-z)\,\{2-(1+x)(1-t)\}]^{b}}.$$

Corollary 2. *On taking $r=u=-1$ in Theorem 1 and then using transformation* (2.8) *and variable changes $y'=-y(1-z)/(1-y)$, we obtain*

$$(2.16)\qquad \sum_{k=0}^{\infty}\frac{(\alpha+1)_k}{k!}\left[\frac{-z(1-t)}{(1-z)(1-y')}\right]^k\sum_{q=0}^{\infty}\frac{(\omega-k+1)_q}{q!}\left[\frac{-t\,(1-z-y')^s}{(1-t)(1-z)^s(1-y')^s}\right]^q$$
$$\times{}_2F_1\left[\begin{matrix}-k,\ -\alpha-\beta-sq-k-1;\\ -\alpha-k\end{matrix};\ y'\right]=(1-z)^{\alpha+\beta+2}(1-t)^{\omega+1}(1-y')^{\beta+1}/(1-z-y')^{\beta}$$
$$(1-z-y'+sy'zt).$$

Going through the proof of Corollary 2, we see that the correct form of the result (2.10) of M. Cohen [3] should be

$$(2.17)\qquad \sum_{k=0}^{\infty} \frac{(\alpha+1)_k}{k!} \left[\frac{-z\,(1-z-y')^s}{[(1-z)\,(1-y')]^{s+1}}\right]^k {}_2F_1 \left[\begin{matrix} -k,\ -\alpha-\beta-(s+1)\,k-1\,; \\ -\alpha-k \quad ; \end{matrix}\ y'\right]$$

$$=(1-z)^{\alpha+\beta+2}(1-y')^{\beta+1}/(1-z-y')^{\beta}(1-z-y'-sy'\,z).$$

The correctness of (2.17) can easily be checked from our result (2.16) which for $t=s=0$ reduces to a special case of (2.17) and we obtain

$$(2.18)\qquad \sum_{k=0}^{\infty} \frac{(\alpha+1)_k}{k!} \left[\frac{-z}{(1-z)(1-y')}\right]^k {}_2F_1 \left[\begin{matrix} -k,\ -\alpha-\beta-k-1\,; \\ -\alpha-k \quad ; \end{matrix}\ y'\right]$$

$$= (1-z)^{\alpha+\beta+2}\,(1-y')^{\beta+1}/(1-z-y')^{\beta+1}.$$

Using Jacobi representation (1.5) in (2.16), we get

$$(2.19)\qquad \sum_{k=0}^{\infty} \left[\frac{z(1-t)}{(1-z)(1-y')}\right]^k \sum_{q=0}^{\infty} \frac{(\omega-k+1)_q}{q!} \left[\frac{-t\,(1-z-y')^s}{(1-t)\,(1-z)^s\,(1-y')^s}\right]^q$$

$$\times P_k^{(-\alpha-k-1,\ -\beta-sq-k-1)}\,(1-2y')$$

$$= (1-z)^{\alpha+\beta+2}\,(1-t)^{\omega+1}\,(1-y')^{\beta+1}/(1-z-y')^{\beta}\,(1-z-y'+sy'\,zt).$$

For $t\to 0$, it reduces to

$$(2.20)\qquad \sum_{k=0}^{\infty} P_k^{(-\alpha-k-1,\ -\beta-k-1)}(1-2y')\,\xi^k = \frac{(1-z)^{\alpha+\beta+2}\,(1-y')^{\beta+1}}{(1-z-y')^{\beta+1}},$$

where $\xi=z/(1-z)\,(1-y')$.

Corollary 3. *On taking $r=-1$, $u=1$ in Theorem 1, using Euler's transformation* [10; p. 33 (19)] *and a result* [10; p. 166, problem 11], *we get*

$$(2.21)\qquad \sum_{q=0}^{\infty} \frac{(\omega+1)_q}{q!} \left[\frac{-t}{(1-yz)^s\,(1-t)}\right]^q$$

$$F_1\left(\omega+q+1,\ \alpha+1,\ \beta+sq+1\,;\ \omega+1\,;\frac{-z}{(1-z)\,(1-t)},\ \frac{-yz}{(1-yz)\,(1-t)}\right)$$

$$= (1-z)^{\alpha+1}\,(1-t)^{\omega+1}\,(1-yz)^{\beta+1}/(1-syzt),$$

where F_1 is Appell's function [5; p. 23 (1.4.1)].

Setting $s=1$, it gives

$$(2.22)\qquad F_F\left(\alpha,\alpha,\alpha,\beta,\omega,\beta\,;\beta,\alpha,\alpha\,;\frac{-t}{(1-yz)(1-t)},\ \frac{-z}{(1-z)(1-t)},\ \frac{-yz}{(1-yz)\,(1-t)}\right)$$

$$= (1-t)^{\alpha}\,(1-yz)^{\beta}(1-z)^{\omega}/(1-yzt),$$

where F_F is Saran's function [5; p. 66].

Again, setting $\beta=\alpha$ in the above equation and using [8; p. 71 (4.2.17)], we get

$$(2.23)\ H_3\left(\alpha,\omega\,;\alpha\,;\frac{yzt}{(1+yzt)^2},\ \frac{zt}{(1+yzt)(1-t+zt)}\right)= (1+yzt)^{\alpha}(1-t+zt)^{\omega}/(1-yzt)\,(1-t)^{\omega},$$

where H_3 is Horn's function [5; p. 36].

Finally, it may be remarked that following the procedure as in Theorem 1, an extension of Theorem 2 of M. Cohen [3] is not possible. However, we noticed that a result (2.13) used in the proof of the Theorem 2 is erroneous. His result (2.13) in the corrected form is

$$(2.24)\ \sum_{k=0}^{m}\sum_{p=0}^{n} \frac{(-n)_p\,(-m)_k\,\sqrt{(n+\alpha+sk+1)}\,\sqrt{(m+\beta+p+1)}}{k!\,p!\,\sqrt{(\alpha+sk+1)}\,\sqrt{(\beta+p+1)}\,(\alpha+sk+p)} = n!\,m!\,(\beta+1)_m/(\alpha)\,(\alpha/s+1)_m.$$

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Received 26. 9. 1986