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ON THE NUMBER OF HAUSDORFF GROUP TOPOLOGIES ON INFINITE ABELIAN AND FREE GROUPS*

Dedicated to the memory of Professor I. Prodanov

DIETER REMUS

In 1945 the following question was posed by A. Markov [12] dealing with free topological groups: Does every infinite group admit a nondiscrete Hausdorff group topology? He was in [13] able to give a necessary and sufficient condition for the existence of such a group topology on a countable group, but nevertheless the problem remained open. For abelian groups A. Kertész, T. Szele [10] gave a positive answer to Markov's question in 1953. In 1976 S. Shelah constructed a nonabelian group of cardinality $\aleph_1$ admitting only the trivial group topologies [24]. In the proof he had to assume CH. Considering Hausdorff group topologies, CH can be omitted, as G. Hesse has shown in [9]. At the workshop on Burneside groups held at Bielefeld, FRG, in 1977, S. Adian mentioned the existence of a countable group which is only discretely topologizable to become a Hausdorff topological group. Relating to this, Adian reports that A. Ol'shanskii has pointed out the application of his results on classifications of periodic words to Markov's question (cf. [1], [15]).

The investigation on which groups there exist exactly $2^{2^{|G|}}$ Hausdorff group topologies, the maximum number possible, was started by J. Kiltinen [11] in 1974. Following him, those groups are called highly topologizable. Kiltinen shows that every infinite abelian group is highly topologizable, using results from the theory of topological fields. K. Podewski [16], in his turn, confirmed the quoted result by altogether different methods. (Only for the real numbers that result is proven in [3] by the aid of topological vector spaces.) At the same time, J. Heine asked in [7] for the number of Hausdorff linear group topologies on infinite abelian groups. It was well-known that an abelian group can be furnished with a nondiscrete Hausdorff linear group topology if and only if it does not satisfy the minimum condition for subgroups ([5], p. 34; [10]). Now Heine proved that every abelian group of infinite rank is highly topologizable by linear group topologies. For groups of finite rank he could reduce his problem to torsion-free groups. In this case, he gave partial answers.

In the theory of minimal topological groups precompact group topologies play an important role (cf. [17]). In particular, I. Prodanov, L. Stoyanov [18] proved the deep result that every abelian minimal topological group is precompact. The lattice of (not necessarily Hausdorff) precompact group topologies has been studied in [19]. It is natural to ask for classes of groups which are highly topologizable by precompact group topologies. Before Shelah's result it was well-known that there are nonabelian groups possessing only the indiscrete topology as precompact group topology [14].

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Without making use of Kiltinen's result, S. Berhanu, W. Comfort, J. Reid [2], and the author [19, 21] have independently shown the following, of which one can find generalizations in [2], too.

Theorem 1. *Let G be an infinite abelian group. Then* (a) *G is highly topologizable by precompact group topologies.* (b) *For* $|G|>2^{\aleph_0}$ *there is no metrizable precompact group topology on G.*

If $\aleph_0 \leq |G| \leq 2^{\aleph_0}$, *then there exist exactly* $2^{|G|}$ *metrizable precompact group topologies on G.*

S. Berhanu, W. Comfort, J. Reid have additionally shown in [2]

Theorem 2. *Every infinite abelian group G admits precisely* $2^{|G|}$ *metrizable group topologies.*

The question that suggests itself is whether there exist classes of nonabelian groups for which correspondent statements are valid. Pertaining to this, the author has proven in [19] that every free group is highly topologizable. In the proof the countable case follows from the following result of R. Zobel [25], which was independently proven by K. Podewski [16] in a more general form: Let G be a countable group admitting a nondiscrete Hausdorff group topology. Then G possesses exactly $2^{2^{\aleph_0}}$ Hausdorff group topologies.

In [8, 9] G. Hesse has studied classes of countable nonabelian groups which are highly topologizable. In particular, every countable solvable, FC-nilpotent or locally nilpotent group is highly topologizable. For further results concerning the topologizability of nonabelian groups see [4, 22, 23].

Using only in the abelian case one of the results mentioned till now, it was shown in [20]

Theorem 3. *Every free group is highly topologizable by precompact group topologies.*

In [6] M. Hall proved in 1950 that the finite-index topology on a free group is Hausdorff. Now one can intensify Theorem 3, making use of the methods applied in its proof.

Theorem 4. ([20]). *Every free group F admits* $2^{2^{|F|}}$ *Hausdorff precompact group topologies finer than the finite-index topology on F.*

Finally, the following problems remain open:

(1) Does every free group F admits $2^{|F|}$ metrizable group topologies?

(2) How many metrizable precompact group topologies can a free group possess?

(3) Are there further classes of nonabelian groups which are highly topologizable (by precompact group topologies)?

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