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ON CURVES WHICH BOUND SPECIAL CONVEX SETS

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A class of plane closed convex curves is considered. We construct curves which bound special convex sets, especially curves which bound both a triangle set and a set of constant width. Moreover, a counterpart of antipodal pairs is introduced.

Introduction. A family of C^1, plane closed convex simple curves is considered. Each convex polygon with n sides and equal interior angles is called equiangular n-polygon. In this paper we give a construction of a curve C that for a given finite or infinite sequence of natural numbers $n_1, n_2, \ldots$ all equiangular n_i-polygons described on C have the same perimeter (the circle is the unique curve which posses the following property: all equiangular n-polygons have the same perimeter). In particular, we determine all such ovals (C^2, plane closed simple curve with positive curvature is called on oval, [4]). The construction is based on the characterisation of an oval for which all equiangular n-polygons described on it have the same perimeter [2]. If $s_1, \ldots, s_n$ are parameters of tangent points of an equiangular n-polygon described on an oval with the curvature k, then all equiangular n-polygons have the same perimeter if and only if

$$\frac{1}{k(s_1)} + \cdots + \frac{1}{k(s_n)} = \text{const.}$$

Curves which bound triangle set (see [7, 5]) and curves of constant width (see [7, 6, 1]) are contained in the class under consideration. Ovals are included to the class of closed curves of the following form

$$x(\theta) = \int_0^\theta r(u)\cos u\,du, \quad y(\theta) = \int_0^\theta r(u)\sin u\,du \quad \text{for } 0 \leqq \theta \leqq 2\pi,$$

where r is a continuous, 2π-periodic and positive function (if r is a differentiable function, then $1/r$ is the curvature), (see [4, 6]). Moreover, a counterpart of antipodal pairs is introduced.

1. The class M. Let us fix a natural number $n \geqq 2$ and a positive number h. We denote by $M(n, h)$ a class containing all functions $r\colon R \to R$ such that for all $u \in R$ the following conditions are satisfied

(1) $$r(u) > 0,$$

(2) $$r(u) = r(u + 2\pi),$$

(3) $$\sum_{i=1}^{n} r\left(u + \frac{2\pi}{n}(i-1)\right) = h,$$

(4) $$r \text{ is continuous},$$

(5) $$\int_0^{2\pi} r(u)\cos u\,du = \int_0^{2\pi} r(u)\sin u\,du = 0,$$

(6) the Fourier series expansion of $r\ \frac{1}{2}a_0+\Sigma_{j=1}^{\infty}(a_j\cos ju+b_j\sin ju)$ is uniformly convergent.

Remark 1. The condition (5) means that the Fourier coefficients a_1, b_1 of r are equal to zero.

Remark 2. The class $M(2, h)$ was considered in [6].

Let

(7) $$M=\bigcup_{n=2}^{\infty}\bigcup_{h>0}M(n, h).$$

The coefficients of $r\in M$ are characterized by the following condition:

Lemma 1. *If $r\in M(n, h)$, then*

(8) $$\begin{cases} a_0=\dfrac{2h}{n}, \\ a_j=b_j=0 \quad \textit{if } n\mid j \quad (j \textit{ is divided by } n). \end{cases}$$

Proof. Making use of (3) we get

$$\pi a_0=\int_0^{2\pi}r(u)du=\sum_{i=1}^{n}\int_{2\pi(i-1)/n}^{2\pi i/n}r(u)du=\sum_{i=1}^{n}\int_0^{2\pi/n}r(v+\frac{2\pi}{n}(i-1))\,dv$$

$$=\int_0^{2\pi/n}[\sum_{i=1}^{n}r(v+\frac{2\pi}{n}(i-1))]\,dv=\frac{2\pi}{n}h.$$

If $n\ j$, then we have

$$\int_0^{2\pi}r(u)\cos ju\,du=\sum_{i=1}^{n}\int_{2\pi(i-1)/n}^{2\pi i/n}r(u)\cos ju\,du$$

$$=\sum_{i=1}^{n}\int_0^{2\pi/n}r(v+\frac{2\pi}{n}(i-1))\cos(jv+2\pi(i-1)\ \frac{j}{n})\ dv$$

$$=\int_0^{2\pi/n}[\sum_{i=1}^{n}r(v+\frac{2\pi}{\pi}(i-1))]\cos jv\,dv$$

$$=h\int_0^{2\pi/n}\cos jv\,dv=0$$

and $a_j=0$. The equality $b_j=0$, where $n\mid j$ may be derived similarly.

Lemma 2. *If a function $r\colon R\to R$ satisfies the conditions* (1), (2), (4), (5), (6) *and* (8), *then the equality* (3) *holds.*

Proof. If j is not divided by n, then we have the known relations

(9) $$\sum_{i=1}^{n}\cos j(u+\frac{2\pi}{n}(i-1))=\sum_{i=1}^{n}\sin j(u+\frac{2\pi}{n}(i-1))=0.$$

Using (6), (9) and (8) we get

$$\sum_{i=1}^{n}r(u+\frac{2\pi}{n}(i-1))=\sum_{i=1}^{n}(\frac{h}{n}+\sum_{j=2}^{\infty}[a_j\cos j(u+\frac{2\pi}{n}(i-1))+b_j\sin j(u+\frac{2\pi}{n}(i-1))])$$

$$=h+\sum_{j=2}^{\infty}[a_j(\sum_{i=1}^{n}\cos j(u+\frac{2\pi}{n}(i-1)))+b_j(\sum_{i=1}^{n}\sin j(u+\frac{2\pi}{n}(i-1)))]$$

$$= h + \sum_{\substack{j=2 \\ n|j}}^{\infty} n(a_j \cos ju + b_j \sin ju) = h.$$

In view of Lemmas 1 and 2 we can replace the condition (3) by the relation (8) given in terms of the Fourier coefficients of the function $r \in M$.

We introduce the following compositions in the class M:

$$(10) \qquad \begin{aligned} (r_1 + r_2)(u) &= r_1(u) + r_2(u), \\ (r_1 * r_2)(u) &= \frac{1}{\pi} \int_0^{2\pi} r_1(t) r_2(t-u)\, dt. \end{aligned}$$

Theorem 3. *If $r_j \in M(n_j, h_j)$ for $j = 1, 2$, then*

$$(11) \qquad \begin{aligned} & r_1 + r_2 \in M\Big(\text{l. c. m. } (n_1, n_2), \ \frac{n_2 h_1 + n_1 h_2}{\text{g. c. d. } (n_1, n_2)}\Big), \\ & r_1 * r_2 \in M\Big(n_1, \ \frac{2h_1 h_2}{n_2}\Big), \\ & r_1 * r_2 \in M\Big(n_2, \ \frac{2h_1 h_2}{n_1}\Big), \\ & r_1 * r_2 \in M(n_1 n_2, \ 2h_1 h_2), \end{aligned}$$

where g. c. d. = greatest common divisor,
l. c. m. = least common multiple.

Proof. Let us note that if $r_1, r_2 \in M$ and

$$r_1(u) = \frac{1}{2} a_0 + \sum_{j=2}^{\infty} (a_j \cos ju + b_j \sin ju),$$

$$r_2(u) = \frac{1}{2} c_0 + \sum_{j=2}^{\infty} (c_j \cos ju + d_j \sin ju),$$

then

$$(12) \qquad (r_1 * r_2)(u) = \frac{1}{2} a_0 c_0 + \sum_{j=2}^{\infty} [(a_j c_j + b_j d_j) \cos ju + (a_j d_j - b_j c_j) \sin ju].$$

Using (8) and (12) the relation (11) can be easily obtained.

2. Curves associated with the class M. We associate with each $r \in M$ a plane curve C_r, $\theta \to x(\theta) = (x^1(\theta), x^2(\theta))$ defined as follows

$$(13) \qquad \begin{cases} x^1(\theta) = \int_0^{\theta} r(u) \cos u\, du, \\ x^2(\theta) = \int_0^{\theta} r(u) \sin u\, du \qquad \text{for} \quad 0 \leq \theta \leq 2\pi. \end{cases}$$

We introduce the following class of curves

$$(14) \qquad C(M) = \{C_r : r \in M\}.$$

Let us note that $C_r \in C(M)$ is a closed simple convex curve. Indeed, using the relation $a_1 = b_1 = 0$ and (6), we get

$$x^1(\theta + 2\pi) = \int_0^{\theta} r(u) \cos u\, du + \int_{\theta}^{\theta+2\pi} r(u) \cos u\, du$$

$$= x^1(\theta) + \int_{\theta}^{\theta+2\pi} \Big[\frac{1}{2} a_0 + \sum_{j=2}^{\infty} (a_j \cos ju + b_j \sin ju)\Big] \cos u\, du$$

$$=x^1(\theta)+\sum_{j=2}^{\infty}(a_j\int_{\theta}^{\theta+2\pi}\cos ju\cos u\,du+b_j\int_{\theta}^{\theta+2\pi}\sin ju\cos u\,du=x^1(\theta).$$

Similarly $x^2(\theta+2\pi)=x^2(\theta)$. Thus C_r is a closed curve. The convexity of C_r is proved in [6]. It is easy to see that C_r is a simple curve.

Definition 1. *Each convex polygon with n sides and equal interior angles will be called an equiangular n-polygon.*

Theorem 4. *If $r\in M(n, h)$, then all equiangular n-polygons described on C_r will have the same perimeter.*

Proof. The perimeter L of a polygon with interior angles $\pi-\alpha_1$, $\pi-\alpha_2, \ldots$, $\pi-\alpha_n$ described on C_r is expressed as follows (see [2], (10)):

$$L(\theta)=\sum_{i=1}^{n}[d_i(\theta)\operatorname{tg}\frac{\alpha_i}{2}-D_i(\theta)], \tag{15}$$

where

$$\begin{cases} d_i(\theta)=\det\begin{bmatrix} x^1(\theta+\alpha_1+\cdots+\alpha_{i-1})-x^1(\theta+\alpha_1+\cdots+\alpha_i) & \cos\theta \\ x^2(\theta+\alpha_1+\cdots+\alpha_{i-1})-x^2(\theta+\alpha_1+\cdots+\alpha_i) & \sin\theta \end{bmatrix} \\ D_i(\theta)=\det\begin{bmatrix} x^1(\theta+\alpha_1+\cdots+\alpha_{i-1})-x^1(\theta+\alpha_1+\cdots+\alpha_i) & -\sin\theta \\ x^2(\theta+\alpha_1+\cdots+\alpha_{i-1})-x^2(\theta+\alpha_1+\cdots+\alpha_i) & \cos\theta \end{bmatrix} \end{cases} \tag{16}$$

for $i=1, 2, \ldots, n$.

Let $\alpha_i=\pi-2\pi/n$ for $i=1, 2, \ldots, n$. Then the formulae (16) can be rewritten in the following form

$$\begin{cases} d_i(\theta)=\int_0^{\frac{2\pi}{n}} r(v+\theta+\frac{2\pi}{n}(i-1))\sin v\,dv, \\ D_i(\theta)=\int_0^{\frac{2\pi}{n}} r(v+\theta+\frac{2\pi}{n}(i-1))\cos v\,dv, \end{cases} \tag{17}$$

for $i=1, 2, \ldots, n$.

Hence we get

$$L(\theta)=\sum_{i=1}^{n}[d_i(\theta)\operatorname{tg}\frac{\pi}{n}-D_i(\theta)]=\sum_{i=1}^{n}[\operatorname{tg}\frac{\pi}{n}\int_0^{2\pi/n} r(v+\theta+\frac{2\pi}{n}(i-1))\sin v\,dv \tag{18}$$

$$+\int_0^{2\pi/n} r(v+\theta+\frac{2\pi}{n}(i-1))\cos v\,dv]=\frac{1}{\cos\frac{\pi}{n}}\sum_{i=1}^{n}\int_0^{2\pi/n} r(v+\theta+\frac{2\pi}{n}(i-1))\cos(v-\frac{\pi}{n})\,dv$$

$$=\frac{n}{\cos\frac{\pi}{n}}\int_0^{2\pi/n}\cos(v-\frac{\pi}{n})\,dv=2h\operatorname{tg}\frac{\pi}{n}.$$

Theorem 5. *Let a function $r: R\to R$ satisfy the conditions* (1), (2), (4), (5), (6). *We assume that a curve C_r given by* (13) *has the following property: all equiangular n-polygons described on C_r have the same perimeter. Then $r\in M$.*

Proof. Making use of (18) and (6) we obtain

$$L(\theta)=\frac{1}{\cos\frac{\pi}{n}}\sum_{i=1}^{n}\int_0^{2\pi/n} r(v+\theta+\frac{2\pi}{n}(i-1))\cos(v-\frac{2\pi}{n})\,dv \tag{19}$$

$$= 2n \operatorname{tg} \frac{\pi}{n} \left[\frac{1}{2} a_0 - \sum_{\substack{j=2 \\ n|j}}^{\infty} \frac{1}{j^2-1} (a_j \cos j\theta + b_j \sin j\theta)\right].$$

The Fourier series expansion of L implies the following equivalence

$$\mathsf{L}(\theta) = \text{const} \Leftrightarrow a_j = b_j = 0 \quad \text{for } n \mid j. \tag{20}$$

The relation (20) and Lemma 2 imply that $r \in M$.

The following theorem is a simple consequence of Theorem 5.

Theorem 6. *If $r \in M(3, h)$, then a curve C_r bounds a triangle set if and only if the Fourier coefficients of r satisfy the condition $a_{3m} = b_{3m} = 0$ for $m = 1, 2, \ldots$.*

Let a finite or infinite sequence $3 \leq n_1 < n_2 < \cdots$ of natural numbers be given. There exists a function $r \in M$ such that

$$r(u) = \frac{a_0}{2} + \sum_{j=2}^{\infty} (a_j \cos ju + b_j \sin ju),$$

where $a_j = 0$ if $n_1 \mid j$, $n_2 \mid j, \ldots$.

A curve C_r has the following property for each fixed n_i all equiangular n_i-polygons described on C_r have the some perimeter. Thus we have

Theorem 7. *For a given finite or infinite sequence $3 \leq n_1 < n_2 < \cdots$ there exists a curve C_r such that for each fixed n_i all n_i-polygons described on C_r have the same perimeter.*

Example. We give an example of a curve which bounds a triangle set and a set of constant width simultaneously [7]. Let

$$r(u) = \frac{1}{2} a_0 + a_5 \cos 5u + b_5 \sin 5u, \tag{21}$$

where

$$a_0 > 0 \text{ and } \frac{1}{4} a_0^2 > a_5^2 + b_5^2. \tag{22}$$

The conditions (22) and $r > 0$ are equivalent. Thus the curve C_r is expressed as follows

$$24x^1(\theta) = 5b_5 + 12a_0 \sin\theta - 3b_5 \cos 4\theta + 3a_5 \sin 4\theta - 2b_5 \cos 6\theta + 2a_5 \sin 6\theta$$

$$24x^2(\theta) = 12a_0 - a_5 - 12a_0 \cos\theta + 3a_5 \cos 4\theta + 3b_5 \sin 4\theta - 2a_5 \cos 6\theta - 2b_5 \sin 6\theta.$$

The formula (19) implies that all equiangular n-polygons ($n \neq 5$) described on C_r have the same perimeter $na_0 \operatorname{tg}(\pi/n)$.

Remark 3. If a Fourier series expansion of $r \in M$ is a trigonometric polynom of a certain degree $m \geq 3$, then for each fixed $n > m$ all equiangular n-polygons described on C_r will have the same perimeter.

3. On counterpart of antipodal pairs. We recall notions of an oval and antipodal pairs, [4]. C^2, plane closed convex curve with positive curvature is called an oval. Two points of an oval are called an antipodal pair if the tangent lines at these two points are parallel and the curvatures are equal. The following result is due to W. Blaschke and W. Süss [4]: On every oval there are at least three antipodal pairs.

Let C, $s \to x(s)$ for $0 \leq s \leq L$ be a positively oriented oval. By $f(s)$ we denote the length of an oriented arc contained between two points $x(s)$ and $x(s_1)$ such that the tangent lines at these points are parallel. The extrema of f could be only at points of an antipodal pair, [3]. We introduce a counterpart of antipodal pairs for closed curves of the following form

$$\theta \to x(\theta) = \left(\int_0^\theta r(u) \cos u \, du, \int_0^\theta r(u) \sin u \, du\right) \quad \text{for } 0 \leq \theta \leq 2\pi, \tag{23}$$

where r is a continuous, 2π-periodic and positive function. It is easy to see that the tangent lines at $x(\theta)$ and $x(\theta+\pi)$ are parallel. Let us denote by $f(\theta)$ the length of an oriented arc contained between $x(\theta)$ and $x(\theta+\pi)$.

Definition 2. *A pair of points* $(x(\theta), x(\theta+\pi))$ *of a closed curve C given by* (13) *such that the function f reaches its extremum at* $f(\theta)$ *will be called a* $*$*-antipodal pair.*

Theorem 8. *There exist at least three* $*$*-antipodal pairs on a closed curve C given by* (23).

Proof. If C_r is given by (13), then the function f will be expressed by the formula $f(\theta)=\int_{\theta}^{\theta+\pi} r(u)\,du$. Hence we get $f'(\theta)=r(\theta+\pi)-r(\theta)$. Now we use the idea of the proof of Blaschke-Süss theorem (see [4], p. 202). The equality $f'(\theta+\pi)=-f(\theta)$ implies that there exists θ_0 such that $f'(\theta_0)=f'(\theta_0+\pi)=0$ and f' changes its sign in a neighborhood of θ_0. We may take $\theta_0=0$. Let assume that $f'(\theta)\geqq 0$ in some right-hand neighbourhood of 0, then $f'(\theta)\leqq 0$ in a right-hand neighbourhood of π and $f'(\theta)\geqq 0$ in a left-hand neighbourhood of π.
We have

$$\int_0^{\pi} f'(u)\sin u\,du=\int_0^{\pi} r(u+\pi)\sin u\,du-\int_0^{\pi} r(u)\sin u\,du$$

$$=-\int_{\pi}^{2\pi} r(u)\sin u\,du-\int_0^{\pi} r(u)\sin u\,du=-\int_0^{2\pi} r(u)\sin u\,du=0$$

since our curve is closed. It follows that f' has at least two zeros in $(0, \pi)$. They determine the required $*$-antipodal pairs.

A consequence of the above theorem is Blaschke-Süss theorem. Indeed, each oval with the curvature k can be represented in the form (13), where $r=1/k$. Moreover, if the function f reaches its extremum at some point θ, then $k(\theta)=k(\theta+\pi)$.

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