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LOCALLY DECOMPOSABLE RIEMANNIAN SPACES WITH A CONSTANT HOLOMORPHIC 4-CURVATURE

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The present paper deals with the locally decomposable Riemannian spaces with constant 4-curvatures of the 4-dimensional linear holomorphic subspaces of the tangent spaces. Some curvature properties are obtained.

1. Introduction. Let M be a Riemannian manifold with a metric g, having an affinor tensor f such that:

$$f^2=I,\quad g(fX,\ Y)=g(x,\ fY),\quad \nabla f=0, \tag{1}$$

where $X, Y \in \mathfrak{X}(M)$, and ∇ is the Levi-Civita connection. In this case M is called a locally decomposable Riemannian space ($M=M_1\times M_2$ of the Riemannian spaces at least locally) [5].

It is known that with respect to the separating coordinate system f has special constant components. Let

$$f=\left(\begin{array}{cccc|cccc} 1 &&&&&&& \\ & 1 &&&&&& \\ && \ddots &&&&& \\ &&& 1 &&&& \\ \hline &&&& -1 &&& \\ &&&&& -1 && \\ &&&&&& \ddots & \\ &&&&&&& -1 \end{array}\right)\begin{array}{l}\left.\vphantom{\begin{array}{c}1\\1\\ \ddots\\1\end{array}}\right\} p\\ \left.\vphantom{\begin{array}{c}1\\1\\ \ddots\\1\end{array}}\right\} q\end{array}. \tag{2}$$

Similarly to the case of the complex structure we can define the notion of a linear holomorphic subspace of the tangent space at a point of the manifold M. We shall call the linear subspace E^m in T_sM, $s\in M$ a holomorphic one if $fE^m=E^m$. Thus, if M is a manifold with a structure of the algebra of the double numbers ($p=q$), then such a linear holomorphic subspace will envelop a holomorphic surface which is a realization of some surface in a manifold over the algebra of the double numbers [2, 3].

The necessary and sufficient condition for a B-manifold to be a manifold with constant curvature of the 4-demensional linear holomorphic subspace, see [4], for the case when tr $f=0$ (i. e. $p=q$) is obtained [1].

In this paper we consider the more general case, when p and q are arbitrary natural numbers. But this time the above definition of a linear holomorphic subspace is applied without the requirement for f to be generated by the algebra of the double numbers.

Let M be a locally decomposable Riemannian space, $s\in M$ and $\{e_1, e_2, \ldots, e_p, e_1', e_2', \ldots, e_q'\}$ be a separating coordinate system in T_sM, where $fe_i=e_i$, $i=\overline{1,\ p}$ and $fe_1=-e_\alpha'$, $\alpha=\overline{1,\ q}$. Let us denote by U^+ (resp. U^-) the subspace of T_sM with a base

$\{e_1, e_2, \ldots, e_p\}$ (resp. $\{e'_1, e'_2, \ldots, e'_q\}$). Evidently $U^+ \perp U^-$ and if $x \in U^+$ and $y \in U^-$, then $\{x, y\}$ is a holomorphic section. We consider the 4-dimensional holomorphic linear subspaces $E^4 = \{x, y, z, u\}$ of T_sM of the following types:

I. $x, y \in U^+, \quad z, u \in U^-$;

II. $x \in U^+, \quad y, z, u \in U^-$;

III. $x, y, z \in U^+, \quad u \in U^-$;

IV. $x, y, z, u \in U^+$;

V. $x, y, z, u \in U^-$.

If the curvatures of all E^4 of the type I are the same, then M will be the class of the manifolds with a constant holomorphic 4-curvature of 4-dimensional linear holomorphic subspaces [1]. We denote this class by CH_{I}, for brevity. Similarly we introduce the classes

$$CH_{\mathrm{II}},\ CH_{\mathrm{III}},\ CH_{\mathrm{IV}},\ CH_{\mathrm{V}} \text{ and } CH = CH_{\mathrm{I}} \cap CH_{\mathrm{II}} \cap CH_{\mathrm{III}} \cap CH_{\mathrm{IV}} \cap CH_{\mathrm{V}}.$$

2. Curvature tensor field of a locally decomposable Riemannian manifolds in CH_{I}**.** Let $M \in CH_{\mathrm{I}}$, $E^4 = \{e_1, e_2, e'_1, e'_2\}$ and let $K(E^4)$ be E^4_s 4-curvature. Since the sectional curvatures of the holomorphic sections vanish, then

$$K(E^4) = K(e_1, e_2) + K(e'_1, e'_2).$$

We suppose that $K(E^4) = C^1$, i. e.

$$K(e_1, e_2) + K(e'_1, e'_2) = C^1. \tag{3}$$

Let us fix e_1, e_2 and let e'_1, e'_2 run over U^-. Then

$$K(e_1, e_2) + K(e'_i, e'_j) = C^1 \quad (i, j = 1, \ldots, q). \tag{4}$$

Subtracting (4) from (3) we have

$$K(e'_1, e'_2) = K(e'_i, e'_j).$$

Consequently, the curvatures of all sections of U^- are the same $-C''_1$. Similarly we obtain that the curvatures of all sections of U^+ are the same $-C'_1$. So, we get, that M_1 (resp. M_2) is the space with a constant curvature C'_1 (resp. C''_1).

We denote by: x^+, x^- — the projections of $x \in T_sM$ respectively on U^+, U^-; R_1, R_2 — the projections of a curvature tensor R of T_sM respectively on U^+, U^-; g_1, g_2 — the projections of g on U^+, U^-. The following relation is valid:

$$R(x, y)z = R_1(x^+, y^+)z^+ \oplus R_2(x^-, y^-)z^-. \tag{5}$$

Now, making use of the algebraic structure of the curvature tensor of a Riemannian space with constant curvature, the above notations and (5) we obtain:

$$R(x, y, z, u) = \frac{(\tau R_1)}{p(p-1)}[g_1(y^+, z^+)g_1(x^+, u^+) \tag{6}$$

$$-g_1(x^+, z^+)g_1(y^+, u^+)] + \frac{\tau(R_2)}{q(q-1)}[g_2(y^-, z^-)g_2(x^-, u^-) - g_2(x^-, z^-)g_2(y^-, u^-)],$$

where $\tau(R_1)$, $\tau(R_2)$ are the scalar curvatures of

$$M_1,\ M_2, \text{ respectively } \left(C'_1 = \frac{\tau(R_1)}{p(p-1)},\quad C''_1 = \frac{\tau(R_2)}{q(q-1)}\right).$$

Using (1) and (2) we obtain the equations

$$g_1(x^+, y^+)=\frac{1}{2}\,[g(x, y)+\tilde{g}(x, y)] \tag{7}$$

$$g_2(x^-, y^-)=\frac{1}{2}\,[g(x, y)-\tilde{g}(x, y)],\quad \tilde{g}(x, y):\,=g(fx, y).$$

Usually for the pure, symmetric tensor fields A, B, the tensor field $L_{A,B}$ is defined by the condition

$$\begin{aligned} L_{A,B}(x, y, z, u)&=B(z, y)A(x, u)-A(x, z)B(y, u)+B(x, u)A(z, y)-A(y, u)B(x, z)\\ &+B(z, \tilde{y})\,A(\tilde{x}, u)-A(z, \tilde{x})B(\tilde{y}, u)+B(\tilde{x}, u)A(z, \tilde{y})-A(\tilde{y}, u)B(z, \tilde{x}),\quad \tilde{x}:\,=fx. \end{aligned} \tag{8}$$

Then from (7) and (8), the equality (6) can be rewritten in the following form

$$\begin{aligned} R(x, y, z, u)&=\frac{\tau(R_1)}{8p(p-1)}[L_{g,g}(x, y, z, u)+L_{\tilde{g},g}(x, y, z, u)]\\ &\frac{\tau(R_2)}{8q(q-1)}\,[L_{g,g}(x, y, z, u)-L_{\tilde{g},g}(x, y, z, u)]. \end{aligned} \tag{9}$$

Thus, if $M\in CH_{\rm I}$, then (9) is valid. Conversely, is (9) is valid, we can obtain directly

$$K(E^4)=\frac{\tau(R_1)}{p(p-1)}+\frac{\tau(R_2)}{q(q-1)}=C_1'+C_1'', \tag{10}$$

which implies $M\in CH^{\rm I}$. Hence we proved the following

Theorem 1. *Let M be a locally decomposable Riemannian space. Then $M\in CH_1$ if and only if, its curvature tensor R satisfies (9).*

This result is more general than the result from 10.1 in [1].

3. Locally decomposable Riemannian space in CH.

1. Let $M\in CH_{\rm I}$ and $K(E^4)=C^{\rm I}$, Then M_1, M_2 are manifolds with constant curvatures C_1', C_1'' respectively and from (10) it follows

$$C_1'+C_1''=C^{\rm I}.$$

2. Theorem 2. *Let $M\in CH_{\rm II}$, $p\geqq 5$ (resp. $M\in CH_{\rm III}$, $q\geqq 5$) and $K(E^4)=C^{\rm II}$ (resp $C^{\rm III}$). Then M_2 (resp. M_1) is a manifold with a constant sectional curvature C_2 (resp. C_3) and $C_2=1/3C^{\rm II}$, $C_3=1/3C^{\rm III}$.*

Proof. Let $M\in CH_{\rm II}$. Then

$$K(E^4)=K(e,e_i')+K(e,\ e_j')+K(e,\ e_k')+K(e_i',\ e_j')+K(e_i',\ e_k')+K(e_j',\ e_k').$$

Consequently $K(e_i', e_j')+K(e_i', e_k')+K(e_j', e_k')=C^{\rm II}$, i. e. M_2 is a space with a constant 3-curvature.

The following theorem has been proved in [4].

Theorem. *Let M be an n-dimensional Riemannian space and $3\leqq m\leqq n-2$. Then the sectional curvatures (2-curvatures) are expressed by means of m-curvatures as follows:*

$$\begin{aligned} \binom{m}{2}K_{12}&=\sum_{3\leqq i<j\leqq m+2}K_{12\ldots\hat{i}\ldots\hat{j}\ldots m+2}+\binom{m-1}{2}K_{34\ldots m+2}\\ &-\binom{m-2}{2}\Big(\sum_{3\leqq i\leqq m+2}K_{1\hat{2}\ldots\hat{i}\ldots m+2}+\sum_{3\leqq i\leqq m+2}K_{\hat{1}2\ldots\hat{i}\ldots m+2}\Big), \end{aligned} \tag{11}$$

where K_{12} *is the sectional curvature of the section* $\{u_1, u_2\}$ *and* $K_{1\ldots\hat{i}\ldots\hat{j}\ldots m+2}$ *is an m-curvature of the linear subspace spanned by* $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{m+2}$, $u_i \in T_s(M)$, $i \in \{1, 2, \ldots, n\}$.

That is why M_2 is a space with a constant sectional curvature — C_2. Moreover from (11) for $m=3$ and 5 we have

$$C_2 = 1/3\, C^{II}.$$

Similarly we can prove the assertion that $M \in CH_{III}$.

In the same way we can obtain the following theorem:

Theorem 3. *Let* $M \in CH_{IV}$, $p \geqq 6$ (resp. $M \in CH_V$, $q \geqq 6$) *and* $K(E^4) = C^{IV}$ (*resp.* C^V) *Then* M_1 (*resp.* M_2) *is a manifold with a constant curvature* C_4 *(resp.* C_5*) and*

$$C_4 = 1/6\, C^{IV}, \quad C_5 = 1/6\, C^V.$$

Finally, we have

Theorem 4. *If* $M \in CH$, $p \geqq 6$, $q \geqq 6$, *then* M *is flat.*

Proof. The assertion of the theorem follows from (10), theorem 2, theorem 3.

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