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PURE C-SEMI-SYMMETRIC J-CONNECTIONS ON AN ALMOST COMPLEX MANIFOLD

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For a symmetric J-connection on a complex manifold (M, J) with a complex structure J an important group of transformations of the linear connection is the H-projective group [4]. In [1] G. Ganchev and the author have introduced the class of C-semi-symmetric W-complex connections and the HS-projective group of transformations of the linear connection generalizing the H-projective group.

In this paper we study a subgroup of the HS-projective group. We find a new characterization for a Kaehler manifold of constant holomorphic curvature and for a Kaehler manifold with B-metric of constant totally real curvatures.

1. Let (M, J) be a 2n-dimensional almost complex manifold and $\mathfrak{X}M$ denote the algebra of smooth vector fields on M. Throughout the paper X, Y, Z, V denote smooth vector fields on M. A linear connection ∇ on (M, J) is said to be C-semi-symmetric W-complex [1] if the torsion tensor T and the covariant derivative H of J are given by:

$$T(X,Y)=\nu(X)Y-\nu(Y)X+\mu(JX)JY-\mu(JY)JX \tag{1}$$

$$H(X,Y):=(\nabla_X J)Y=\lambda(JY)X-\lambda(Y)JX \tag{2}$$

where ν, λ, μ are 1-forms on M.

For a symmetric J-connection we have $T=H=0$.

Theorem A [1]. *On an almost complex manifold (M, J) there exists a C-semi-symmetric W-complex connection iff (M, J) is a complex manifold.*

Let $R, \rho, \tilde{\rho}, \sigma, \tilde{\sigma}$ be the curvature tensor, the Ricci tensor, the associated Ricci tensor, the Shouten tensor and the associated Shouten tensor respectively, i. e.

$$\begin{aligned}&\rho(X, Y)=tr(Z\to R(Z, X)Y);\ \tilde{\rho}(X, Y)=tr(Z\to R(JZ, X)Y);\\ &\sigma(X, Y)=tr(Z\to R(X, Y)Z);\ \tilde{\sigma}(X, Y)=tr(Z\to R(X, Y)JZ)\end{aligned} \tag{3}$$

Let W_p, $p\in M$ be the space of all curvature tensors of type (1, 3) over T_pM, $p\in M$, i. e $W_p=\{R\in(1, 3)/R(X, Y)Z=-R(Y, X)Z\}$. The general complex linear group $GL(n, C)$ acts on W_p, $p\in M$ in the usual way: $(aR)(X, Y)Z=aR(a^{-1}X, a^{-1}Y)a^{-1}Z$. The space W_p is decomposed into two invariant components [1]: $W_p=W_1\oplus W_2$; $W_2=\{R\in W_p/\rho=\tilde{\rho}=\sigma=\tilde{\sigma}=0\}$; $W_1=\{R\in W_p/R(X, Y)Z\in \text{span}\{X, Y, Z, JX, JY, JZ\}\}$. The projection R_2 of R on W_2 is the Weyl component of R. The Weyl holomorphic tensor $WHP(R)$ is defined as the Weyl component R_2 of R [1], i. e. $WHP(R):=R_2$. The Weyl holomorphic tensor is described in the following way:

$$\begin{aligned}WHP(R)(X, Y)Z=&R(X, Y)Z+M(Y, Z)X-M(X, Z)Y-P(X, Y)Z-L(Y, JZ)JX\\ &+L(X, JZ)JY+Q(X, JY)JZ\end{aligned} \tag{4}$$

where M, P, L, Q are tensors of type (0, 2). These tensors are determined uniquely by

ρ, $\tilde{\rho}$, σ, $\tilde{\sigma}$. The H-projective curvature tensor $HP(R)$ of a symmetric J-connection introduced by Y. Tashiro [4] is determined from (4) by the additional conditions: $M=P=L=Q$ where

$$(5)\qquad M(X,Y)=-\frac{1}{2n+2}\Big(\rho(X,Y)+\frac{1}{2n-2}(\rho(X,Y)+\rho(Y,X)-\rho(JX,JY)-\rho(JY,JX)\Big).$$

Theorem B [1]. *The Weyl holomorphic tensor $WHP(R)$ of a symmetric J-connection is equal to the H-projective curvature tensor $HP(R)$.*

Two linear connections ∇ and ∇'on (M,J) are said to be HS-projectively equivalent if

$$(6)\qquad \overset{\bullet}{\nabla}_X Y=\nabla_X Y+\alpha(X)Y+\beta(Y)X-\gamma(JX)JY-\delta(JY)JX$$

holds for arbitrary 1-forms $\alpha, \beta, \gamma, \delta$ on M[1].
The H-projective group introduced by Y. Tashiro [4] is determined by (6) and the additional conditions: $\alpha=\beta=\gamma=\delta$.

Theorem C [4]. *A symmetric J-connection ∇ on a 2n-dimensional ($2n\geqq 6$) complex manifold (M,J) is H-projectively flat iff the H-projective tensor $HP(R)$ of ∇ vanishes.*

Theorem D [1]. *The Weyl holomorphic tensor $WHP(R)$ of a C-semi-symmetric W-complex connection is an invariant of the HS-projective group.*

Theorem F [1]. *A C-semi-symmetric W-complex connection ∇ on a 2n-dimensional ($2n\geqq 6$) complex manifold (M,J) is HS-projectively equivalent to a flat symmetric J-connection iff the Weyl holomorphic tensor $WHP(R)$ of ∇ vanishes.*

A H-projectively flat Kaehler manifold has been characterized in [5] as follows:

Theorem G [5]. *Let (M,g,J) ($\dim M=2n\geq 6$) be a Kaehler manifold. The following conditions are equivalent:*

a) *M is H-projectively flat;*

b) *M is of constant holomorphic sectional curvature;*

A H-projectively flat Kaehler manifold (M,g,J) with B-metic ($g(JX,JY)=-g(X,Y)$) has been characterized in [2] as follows:

Therem H [2]. *Let (M,g,J) ($\dim M=2n\geqq 6$) be a connected Kaehler manifold with B-metric. The following conditions are equivalent:*

a) *M is H-projectively flat;*

b) *M is of constant totally real sectional curvatures.*

2. Pure C-semi-symmetric J-connections. Let α be a 1-form on an almost complex manifold (M,J) and $\tilde{\alpha}$ be a 1-form associated to α, i. e $\tilde{\alpha}(X):=\alpha(JX)$. A 1-form α is said to be H-closed if the associated 1-form $\tilde{\alpha}$ to α is closed, i. e. $d(\alpha_0 J)=0$.

Definition. *A linear connection ∇ on an almost complex manifold (M,J) is said to be pure C-semi-symmetric J-connection if ∇ is a J-connection and the torsion tensor T of ∇ is given by:*

$$(7)\qquad T(X,Y)=\tau(X)Y-\tau(Y)X-\tau(JX)JY+\tau(JY)JX,$$

where τ is an 1-form on M. If τ is closed and H-closed ∇ is said to be a special pure C-semi-symmetric J-connection. A pure C-semi-symmetric J-connection is C-semi-symmetric W-complex connection. Applying theorem A we have

Theorem 1. *On an almost complex manifold (M,J) there exists a pure C-semi-symmetric J-connections iff M is a complex manifold.*

Two linear connections ∇ and ∇' on (M,J) are called pure HS-projectively equivalent if

$$(8)\qquad \nabla'_X Y=\nabla_X Y+\alpha(X)Y+\beta(Y)X-\alpha(JX)JY-\beta(JY)JX$$

holds for arbitrary 1-forms α, β on M. If α and β are closed and H-closed we have a special pure HS-projective transformations. It is easy to verify that the pure HS-projective (special pure HS-projective) transformations form a subgroup of the HS-projective group.

Theorem 2. a) *A linear connection ∇ on an complex space (M, J) is pure C-semi-symmetric (special pure C-semi-symmetric) J-connecton iff it is pure HS-projectively (special pure HS-projectively) equivalent to a symmetric J-connection.*

b) *The class of pure C-semi-symmetric (special pure C-semi-symmetric) J-connections is an invariant of the pure HS-projective (special pure HS-projective) group.*

Proof: Let ∇ is a pure C-semi-symmetric (special pure C-semi-symmetric) J-connection with torsion tensor T given by (7). The connection $\nabla' = \nabla - 1/2T$ is symmetric J-connection. The inverse follows by the simple verification

The statement b) follows by a straightforward calculation.

Now we characterized the subclass of special pure C-semi-symmetric J-connections.

Theorem 3. *Let ∇ be a pure C-semi-symmetric J-connection on an 2n-dimensional $(2n \geqq 6)$ complex space (M, J) and R be the curvature tensor of ∇. The following conditions are equivalent:*

a) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$;

b) *∇ is special pure C-semi-symmetric J-connection.*

Proof. First we prove the following lemma:

Lemma 1. *Let ∇ be a pure C-semi-symmetric J-connection with torsion tensor T given by (7). Then the following conditions are equivalent:*

a) *The 1-form τ is (7) in closed and H-closed;*

b) $(\nabla_X \tau)Y - (\nabla_Y \tau)X = 0$; $(\nabla_X \tilde{\tau})Y - (\nabla_Y \tilde{\tau})X = 0$; $\tilde{\tau} = \tau_0 J$.

Proof of the Lemma 1. For arbitrary 1-form α on M we have:

$$d\alpha(X, Y) = (\nabla_X \alpha)Y - (\nabla_Y \alpha)X - \alpha(T(X, Y)); \tag{9}$$

Specially for τ and $\tilde{\tau}$ from (7) we calculate:

$$\tau(T(X, Y)) = 0; \quad \tilde{\tau}(T(X, Y)) = 0. \tag{10}$$

Let $d\tilde{\tau} = d\tau = 0$. Using (10) from (9) we have the condition b). The inverse follows similarly from (9) and (10).

Proof of the Theorem 3. The first Bianchi identity for R is [3]:

$$\sigma\{R(X, Y)Z\} = \sigma\{T(T(X, Y)Z) + (\nabla_X T)(Y, Z)\}, \tag{11}$$

where σ denote the ciclyc sum of X, Y, Z. Using (7), we calculate:

$$\begin{aligned} T(T(X, Y), Z) = {} & (\tau(Y)\tau(Z) - \tau(JY)\tau(JZ))X - (\tau(X)\tau(Z) - \tau(JX)\tau(JZ))Y - (\tau(Y)\tau(JZ) \\ & + \tau(JY)\tau(Z))JX + (\tau(X)\tau(JZ) + \tau(JX)\tau(Z))JY. \end{aligned} \tag{12}$$

$$(\nabla_X T)(Y, Z) = (\nabla_X \tau)Y.Z - (\nabla_X \tau)Z.Y - (\nabla_X \tilde{\tau})Y.JZ + (\nabla_X \tilde{\tau})Z.JY. \tag{13}$$

Substituting (12) and (13) in (11), we find:

$$\sigma\{R(X, Y)Z\} = \sigma\{[(\nabla_X \tau)Y - (\nabla_Y \tau)X]Z - [(\nabla_X \tilde{\tau})Y - (\nabla_Y \tilde{\tau})X]JZ\}. \tag{14}$$

Let $d\tau = d\tilde{\tau} = 0$. Then (14) and Lemma 1 imply $\sigma\{R(X, Y)Z\} = 0$. Let $\sigma\{R(X, Y)Z\} = 0$. Denoting $(\nabla_X \tau)Y := A(X, Y)$ and using $\nabla J = 0$, we have $(\nabla_X \tilde{\tau})Y = A(X, JY)$. Contracting in (14), we obtain:

$$(2n-3)(A(X,Y)-A(Y,X))+(A(JX,JY)-A(JY,JX))=0.$$

In view in the fact $2n\geqq 6$ we can conclude $A(X,Y)-A(Y,X)=0$; $A(JX,JY)-A(JY,JX)=0$ which imples τ is closed and H-closed by Lemma.

Using Lemma 1 for a symmetric J-connection we have

Theorem 4. *Let ∇ be a symmetric J-connection. Then the curvature tensor R of ∇ is an invariant under the following transformations*: $\nabla'_X Y=\nabla_X Y+\alpha(X)Y-\beta(JX)JY$, *where α is closed and β is H-closed 1-forms.*

Proof: For the curvature tensors R' of ∇' and R of ∇ we calculate:

$$R'(X,Y)Z=R(X,Y)Z+[(\nabla_X\alpha)Y-(\nabla_Y\alpha)X]Z-[(\nabla_X\tilde{\beta})Y-(\nabla_Y\tilde{\beta})X]JZ;\tilde{\beta}=\beta_0 J.$$

Applying Lemma 1 we obtain $R'=R$.

From the Theorem 3 we have

Theorem 5. *If on a complex space ($2n\geqq 6$) there exists a flat pure C-semi-symmmetric J-connection ∇ then ∇ is special pure C-semi-symmetric J-connection.*

Theorem 6. *Let ∇ be a special pure C-semi-symmetric J-connection on a complex space (M,J) ($\dim M=2n\geqq 6$) with curvature tensor R. Then the Weyl holomorphic tensor $WHP(R)$ of ∇ is equal to the H-projective tensor $HP(R)$ of ∇.*

Proof: For the Weyl component $R_2=WHP(R)$ of R we have

$$WHP(R)=R_2=R-R_1;\ R_1(X,Y)Z\in \operatorname{span}\{X,Y,Z,JX,JY,JZ\}. \tag{15}$$

For a J-connection it is well-known $RJ=JR$. Using theorem 3 we find:

$$R_1(X,Y)Z=-L(Y,Z)X+L(X,Z)Y+(L(X,Y)-L(Y,X))Z+L(Y,JZ)JX-L(X,JZ)JY$$
$$-(L(X,JY)-L(Y,JX))JZ. \tag{16}$$

Contracting in (16) after some calculations, we obtain:

$$L(X,Y)=-\frac{1}{2n+2}(\rho(X,Y)+\frac{1}{2n-2}(\rho(X,Y)+\rho(Y,X)-\rho(JX,JY)-\rho(JY,JX)). \tag{17}$$

Substituting (17) and (16) in (15), we get $WHP(R)=HP(R)$.

Theorem 7. *The H-projective tensor of a special pure C-semi-symmetric J-connection is an invariant of the special pure HS-projective group.*

Proof: It follows immediately from Theorem 2, Theorem D, Theorem 6.

Theorem 8. *A special pure C-semi-symmetric J-connection on a complex space (M,J) ($\dim M=2n\geqq 6$) is pure HS-projectively equivalent to a flat symmetric J-connection iff the H-projective tensor $HP(R)$ of ∇ vanishes.*

Proof: Let ∇ is pure HS-projectively equivalent to a flat symmetric J-connection $\overset{*}{\Delta}'$. Theorem 6 and Theorem D imply $HP(R)=HP(R')=0$. Let $HP(R)=0$. Theorem 2 states ∇ is special pure HS-projectively equivalent to a symmetric J-connection. $\bar{\nabla}$ and Theorem 7 imply $HP(\bar{R})=HP(R)=0$. Using Theorem C $\bar{\nabla}$ is H-projectively egui-valent to a flat symmetric J-connection $\overset{\bullet}{\nabla}$. Hence, ∇ is pure HS-projectively equiva-lent to $\overset{\bullet}{\nabla}$.

The Theorem 7 and Theorem 8 show that the H-projective tensor characterizes special pure C-semi-symmetric J-connections as that tensor characterizes symmetric J-connections. Using Theorem 7, Theorem 4 and Theorem C, we have

Theorem 9. *A symmetric J-connection on a complex manifold (M,J) ($\dim M=2n\geqq 6$) is H-projectively flat iff ∇ is pure HS-projectively equivalent to a flat special pure C-semi-symmetric J-connection.*

Definition. *Let* $(M,\ g,\ J,\ \overset{\circ}{\nabla})$ *be a Kaehler manifold with the Levi-Civita connection* $\overset{\circ}{\nabla}$. *A linear connection* ∇ *on a Kaehler manifold* $(M,\ g,\ J)$ *is said to be a pure C-semi-symmetric HW-metric J-connection if the torsion tensor T, the covariant derivative H of J and the covariant derivative G of the metric g satisfy:*

$$T(X,\ Y)=(\alpha-\beta)(X)Y-(\alpha-\beta)(Y)X-(\alpha-\beta)(JX)JY+(\alpha-\beta)(JY)JX;\quad H(X,\ Y)=0;$$

$$G(Y,\ Z)\ X:=(\nabla_X g)(Y,Z)=-2\alpha(X)g(Y,\ Z)-\beta(Y)g(X,\ Z)-\beta(Z)g(X,\ Y)-\beta(JY)g(X,JZ)$$

$$-\beta(JZ)g(X,\ JY),$$

where α, β *are 1-forms on M.*

By a simple verification we have

Proposition 1. A linear connection ∇ on a Kaehler manifold $(M,\ g,\ J)$ is pure C-semi-symmetric HW-metric J-connection iff ∇ is pure HS-projectively equivalent to the Levi-Civita connection $\overset{\circ}{\nabla}$.

Applying theorem 9, theorem 6 and proposition 1 to a Kaehler manifold we have

Theorem 10. *Let* $(M,\ g,\ J)$ $(dim\ M=2n\geq 6)$ *be a Kaehler manifold. Then the following conditions are equivalent:*

a) *M is of constant holomorphic sectional curvature;*

b) *There exists a flat pure C-semi-symmetric HW-metric J-connection.*

Definition. *Let* $(M,\ g,\ J,\ \overset{\circ}{\nabla})$ *be a Kaehler manifold with B-metric and* $\overset{\circ}{\nabla}$ *be the Levi-Civita connection. A linear connection* ∇ *on a Kaehler manifold with B-meric* $(M,\ g,\ J)$ *is said to be a pure C-semi-symmetric BW-metric J-connection if the orsion tensor T, the covariant derivative H of J and the covariant derivative G of he metric g satisfy:* $H(X,\ Y)=0$;

$$T(X,\ Y)=(\alpha-\beta)(X)Y-(\alpha-\beta)(Y)X-(\alpha-\beta)(JX)JY+(\alpha-\beta)(JY)JX,$$

$$G(Y,Z)X:=(\nabla_X g)(Y,\ Z)=-2\alpha(X)g(Y,\ Z)-\beta(Y)g(X,Z)-\beta(Z)g(X,Y)+2\alpha(JX)g(Y,JZ)$$

$$+\beta(JY)g(X,\ JZ)+\beta(JZ)g(X,JY),$$

We have

Proposition 2. A linear connection ∇ on a Kaehler manifold with B-metric is pure C-semi-symmetric BW-metric J-connection iff ∇ is pure HS-projectively equivalent to the Levi-Civita connection $\overset{\circ}{\nabla}$.

Applying Theorem 9, Theorem H and Proposition 2 to a Kaehler manifold with B-metric, we have

Theorem 11. *Let* $(M,\ g,\ J)$ $(\dim M=2n\geq 6)$ *be a connected Kaehler manifold with B-metric. The following conditions are equivalent:*

a) *M is of constant totally real sectional curvatures;*

b) *There exists a flat pure C-semi-symmetric BW-metric J-connnection.*

REFERENCES

1. G. Ganchev, S. Ivanov. Plane axiom on almost complex manifold with a linear connections. (To apper).
2. G. Ganchev, S. Ivanov. Holomorphically concircular invariants on conformally Kaehler manifolds with B-metric. (To appear).
3. S. Kobajashi, K. Nomizu. Foundations of Differential Geometry. vol. I — 1963, vol. II —1969, New York.
4. Y. Tashiro. On holomorphically projective correspondens in an almost complex space. *Math. J. Okajma Univ.*, **6**, 1957, No 2, 147 — 152.
5. K. Yano. Differential Geometry on Complex and Almost Complex spaces New York, 1965.

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