Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

CONDITIONS OF ANGELIC TYPE IN FUNCTION SPACES

MIGUEL BARCELÓ, MIGUEL A. CANELA

This paper deals with a class of topological spaces in which α-compactness and compactness coincide and the tightness of a compact subset is less or equal than α, α being an infinite cardinal number. This class is a natural extension of the class of strictly angelic spaces, introduced by W. Govaerts. Sufficient conditions are given for a space of continuous function to belong to this class, and some results on locally convex spaces are obtained as an application.

1. Introduction. The aim of this paper is to give a description of a class of topological spaces in which α-compactness and compactness coincide, α being an infinite cardinal number, which will remain fixed throughout this paper. This class is α-productive (closed by products of families of cardinality $\leq\alpha$).

The class described will be denoted by A_α, where the "A" stands for **angelic.** Indeed, for $\alpha=\aleph_0$, our class coincides with the class of **strictly angelic spaces** introduced by W. Govaerts [2].

In Section 2 we recall some topological notions which will be used throughout this work. In Section 3 we introduce the class A_α, proving some properties. The main result is the stability for products of cardinality $\leq\alpha$, and the key for this result is a theorem of V. I. Malyhin [4]. In Section 4 we give sufficient conditions for a space of continuous functions to belong to A_α, endowed with the topology of pointwise convergence. In Section 5 we restrict ourselves fo locally convex spaces, obtaining with the tools given in the previous Sections some results appeared in a former paper by M. Valdivia [5].

2. Some topological notions. α will denote a fixed infinite cardinal number. A subset S of a topological space X (all the spaces involved are Hausdorff) is **α-compact** if every net $(x_i)_{i\in I}$ contained in S, with $|I|\leq\alpha$ has a cluster point $x\in S$. S is **relatively α-compact** if every net $(x_i)_{i\in I}$, with $|I|\leq\alpha$ has a cluster point $x\in X$. Every (relatively) compact subset is (relatively) α-compact, but not conversely. A counterexample can be obtained modifying the usual example of a sequentially compact space which is not compact [1, 1.2(7)].

If X is a topological space, the **tightness** of X, $t(X)$, is the minimal cardinal number ν with the following property: if S is a subset of X and $x\in\mathrm{cl}(S)$ is a closure point of S, there is a subset $M\subset S$ with $|M|\leq\nu$ and $x\in\mathrm{cl}(M)$. The **density character** of X, $d(X)$, is the minimal cardinality of a dense subset of X. The **weight** of X, $w(X)$, is the minimal cardinality of a basis of open subsets for the topology of X. The **weight of X at the point** $x\in X$, $w_x(X)$, is the minimal cardinality of a basis of neighbourhoods of x. Clearly $w_x(X)\leq w(X)$, but not conversely. For these and other cardinal functions of the General Topology, [3] can be used as a standard reference.

3. The class A_α. The class A_α will be the class of all topological spaces X satisfying the following conditions:

(i) Every relatively α-compact of X is relatively compact.
(ii) Every compact subset of X has tightness $\leq\alpha$.
(iii) If a subset $S\subset X$ is compact and $d(S)<\alpha$, then $w_x(S)\leq\alpha$ for every $x\in S$.

We give next the properties of this class.

3.1. Proposition: *The condition (ii) of the preceding definition can be replaced by:*
(ii)′ *If $S\subset X$ is relatively compact and $x\in \mathrm{cl}(S)$, there is a net $(x_i)_{i\in I}$ contained in S, with $|I|\leq\alpha$ and $\lim x_i=x$, obtaining an equivalent definition.*

Proof: (ii)′ implies (ii). Conversely, if $t(S)\leq\alpha$ for a compact subset $S\subset X$, we take $M\subset S$ with $x\in \mathrm{cl}(M)$ and $|M|\leq\alpha$. Then $\mathrm{cl}(M)$ is compact and has density character $\leq\alpha$, and, by (iii) the weight at x is $\leq\alpha$. Therefore, we can construct a net in M satisfying the prerequisites of (ii)′. Q.E.D.

3.2. Corollary: *Every subset of a space of the class A_α is in A_α (with the induced topology).*

Proof: Direct from 3.1. Q.E.D.

3.3. Proposition: *Let τ, τ', be topologies on a set X, τ' being regular and finer than τ. If (X, τ) belongs to A_α, (X, τ') also belongs to A_α.*

Proof. The proof is a slight modification of the proof of the "angelic lemma" [1,3.1]. Suppose that S is a τ'-relatively α-compact subset of X, and $D\subset S$. Then $\mathrm{cl}_{\tau'}(D)\subset \mathrm{cl}_\tau(D)$. Conversely, if $x\in \mathrm{cl}_\tau(D)$, there is a net $(x_i)_{i\in I}$ contained in D, with $|I|\leq\alpha$ and τ-$\lim x_i=x$. But $(x_i)_{i\in I}$ has a τ'-cluster point, and hence τ'-$\lim x_i=x$. Therefore, $cl_\tau(D)=cl_{\tau'}(D)$. In particular $cl_\tau(S)=cl_{\tau'}(S)$.

It suffices now to show that τ *y* τ' coincide on $cl(S)$. We can suppose $X=cl(S)$ If $C\subset X$ is τ-closed, let $\mathscr{B}$ be the family of the τ'-open subsets of X which contain C. $S\cap V$ is τ'-dense in V for $V\in\mathscr{B}$, and, because of the regularity of τ':

$$C=\bigcap_{V\in\mathscr{B}} cl_{\tau'}(V)=\bigcap_{V\in\mathscr{B}} cl_{\tau'}(V\cap S)=\bigcap_{V\in\mathscr{B}} cl_\tau(V\cap S),$$

and C is τ-closed. Q.E.D.

We will prove, next, that A_α is α-productive.

3.4. Proposition: *Let $(X_i)_{i\in I}$ be a family of spaces of A_α, with $|I|\leq\alpha$. Then $\Pi_{i\in I}X_i$ belongs to A_α.*

Proof: We denote by π_i the projection onto X_i, for $i\in I$, and by π_J the projection onto $\Pi_{i\in J}X_i$, for $J\subset I$. If $S\subset\Pi_{i\in I}X_i$ is relatively α-compact, $\pi_i(S)$ is relatively α-compact in X_i, and hence relatively compact. We have (i).

Suppose that $S\subset\Pi_{i\in I}X_i$ is compact, and we are going to show that $t(S)\leq\alpha$. We can suppose, without loss of generality, $S=\Pi_{i\in I}\pi_i(S)$. According to a result of V. I. Malyhin [4, Theorem 4], the finite product of compact spaces with tightness $\leq\alpha$ has tightness $\leq\alpha$. If x is a closure point of a subset $D\subset S$, we can find, for each $J\subset I$ finite, a subset $M'_J\subset\pi_J(D)$ with $|M'_J|\leq\alpha$ and $\pi_J(x)\in cl(M'_J)$. We can choose now $M_J\subset D$ with $\pi_J(M_J)=M'_J$ and $|M_J|\leq\alpha$. Actually, $M=\bigcup_J M_J$ has cardinality $\leq\alpha$ and $x\in cl(M)$. We have (ii).

Finally, is $S\subset\Pi_{i\in I}X_i$ is compact and $d(S)\leq\alpha$, then $d(\pi_i(S))\leq\alpha$, and thus $\pi_i(S)$ has weight $\leq\alpha$ in every point. Keeping in mind the construction of a basis of neighbourhoods for the product topology, and recalling $|I|\leq\alpha$, it is easy to see that $\Pi_{i\in I}\pi_i(S)$ has weight $\leq\alpha$ at every point. Q.E.D.

4. Spaces of continuous functions. We are going to see that certain assumptions on a topological space X imply that the space $C(X)=C(X, \mathbb{R})$ of continuous real functions belongs to the class A_α, endowed with the topology of pointwise convergence ω_X. Indeed we obtain results analogous to those obtained by W. Govaerts [2] for the case $\alpha=\aleph_0$.

4.1. Proposition: *Let X be a compact space. Then $(C(X), \omega_X)$ belongs to A_α.*

Proof: It is well known that every subset $S\subset C(X)$, ω_X-relatively countably compact, is ω_X-relatively compact and countable tightness (e. g. [1]). To check (iii), we can, replacing if necessary X by a suitable quotient, suppose that S separates the points of X. If S is ω_X-compact and $D \subset S$ is dense, with $|D| \leq \alpha$, X admits a basis of uniformity of cardinality $\leq \alpha$, and hence $d(X) \lneq \alpha$. Reversing the argument, we have a set of continuous real functions, of cardinality $\leq \alpha$, on S, which separates the points of S, and therefore $w(S) \leq \alpha$. Q.E.D.

4.2. Proposition: *Let X be a topological space with a dense subset $D \subset X$ which is retatively α-compact. Then $(C(X), \omega_X)$ belongs to A_α.*

Proof: Conditions (i) and (ii) can be obtained as in 4.1 from known results (see [1]). To check (iii), we remark that, if $S \subset C(X)$ is ω_X-compact, we can consider the mapping $\varphi_1 : X \to C(S)$ defined by $\varphi_1(x)(f) = f(x)$, which is continuous with respect to the topology ω_S. Then $\varphi_1(D)$ is relatively α-compact in $(C(S), \omega_S)$, and its closure H is a compact subset of $(C(S), \omega_S)$. Now $\varphi_2 : S \to C(H)$, defined in an analogous way, is injective and continuous, and therefore a homeomorphism, and, using 4.1, we are done. Q.E.D.

4.3. Proposition: *Let X, Y be topological spaces.*

a) *If X admits a dense subset $D \subset X$ such that $(C(D), \omega_D)$ is in A_α, then $(C(X), \omega_X)$ is in A_α.*

b) *If $X = \bigcup_{i \in I} X_i$, where $(C(X_i), \omega_{X_i})$ is in A_α* and $|I| \leq \alpha$, *then $(C(X), \omega_X)$ is in A_α.*

c) *It $\psi : Y \to X$ is continuous and surjective, and $(C(Y), \omega_Y)$ is in A_α, then $(C(X), \omega_X)$ is in A_α.*

Proof: For a) Take the restriction map $C(X) \to C(D)$ and apply 3.3. For b), construct an injection $C(X) \to \prod_{i \in I} C(X_i)$ in a natural way and apply 3.4, followed by 3.3. For c), take the mapping $\psi^* : C(X) \to C(Y)$ defined by $\psi^*(f) = f \circ \psi$ and use 3.3. Q.E.D.

4.4. Corollary: *Let X be a topological space with a family $(X_i)_{i \in I}$ of relatively α-compact subsets such that its union is dense and $|I| \leq \alpha$.*
Then $(C(X), \omega_X)$ *is in* A_α.

Finally, we have the following result, which is a natural extension of a theorem of D. H. Fremlin ([1, 3.5], [2, Proposition 9]):

4.5. Proposition: *Let X be a topological space, and Z a metric space, and suppose that $(C(X), \omega_X)$ is in A_α. Then $(C(X, Z), \omega_X)$ is in A_α.*

Proof: The argument given in [2] for the countable case can be used. Q.E.D.

5. Applications to locally convex spaces. The results which have been stated here have a purely topological nature. Nevertheless, some particular cases have been proved by other methods, for instance for the case of a weak topology on a locally convex case.

If X, Y are real locally convex spaces, the space $L(X, Y)$ of continuous linear operators is a subspace of the space $C(X, Y)$ of continuous functions, closed with respect to ω_X. The topology induced by ω_X on $L(X, Y)$ is usually called **simple topology**. From 4.5 and 3.2 we obtain directly:

5.1. Proposition: *Let X, Y be locally convex spaces, X being the union of a family $(D_i)_{i \in I}$ of relatively α-compact subsets, with $|I| \leq \alpha$, and Y is metrizable. Then $L(X, Y)$, endowed with the simple topology, is in A_α.*

We can consider, in particular, the case in which $Y = \mathbb{R}$ and X is the dual E', endowed with the weak topology $\sigma(E', E)$. We obtain thus the following result:

5.2. Corrollary (M. Valdivia [5]): *Let E be a locally convex space with a total family $(D_i)_{i \in I}$ of $\sigma(E', E)$-relatively α-compact subsets of E', and $|I| \leq \alpha$. Then every weakly (relatively) α-compact subset of E is weakly (relatively) compact.*

Note: In [6], this result is stated under an extra assumption, the convexity of the D_i's, but this assumption is superfluous.

Using 5.1 and 3.3, we have:

5.3. Corollary (M. Valdivia [5]): *Let E be a vector space and τ and τ' two locally convex topologies on E, τ finer than τ'. Suppose that τ' admits a zero-neighbourhood basis of cardinality $\leq\alpha$, and denote by $E'=(E, \tau)'$. If $A\subset E$ is $\sigma(E, E')$-(relatively) α-compact, A is $\sigma(E, E')$-(relatively) compact.*

Final note: We have described a class of spaces in which compactness and α-compactness are the same, with good stability properties, which allows us to obtain the results of this Section. It can be remarked that almost all is the same if we replace condition (iii) of the definition by the following stronger condition: If $S\subset X$ is compact and $d(S)\leq\alpha$, then $W(S)\leq\alpha$. Nevertheless, the class obtained in this way will be more restricted, because there are separable, first countable compact spaces which are not metrizable (e. g. the Helly compact).

REFERENCES

1. K. Floret. Weakly compact sets. Berlin, 1980.
2. W. Govaerts. A productive class of angelic spaces. *J. London Math. Soc.*, **22**, 1980, 355-364.
3. I. Juhasz. Cardinal functions in Topology. (*Math. Centre Tracts*, **34**). Amsterdam, 1971.
4. V. I. Malyhin. The tightness and Suslin number in exp X and in a product of spaces. *Soviet Math. Dokl.*, **13**, 1972, 496-499.
5. M. Valdivia. Some criteria for weak compactness. *J. reine angew. Math.*, **225**, 1972, 165-169.

Received 15. 5. 1984

Escola Tècnica Superior d'Enginyers Industrials de Barcelona.
Avinguda Diagonal, Barcelona 20. Spain.

Department de Teoria de Funcions, Universitat de Barcelona.
Gran Via 585, Barcelona 7. Spain.