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WEAK STABILITY OF A CERTAIN CLASS OF MARKOV PROCESSES AND APPLICATIONS TO NONSINGULAR STOCHASTIC DIFFERENTIAL EQUATIONS

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In this paper we deal mainly with Lyapunov type stability of evolution operators generated by certain Markov processes in a space of probabilistic measures with weak topology. Also, weak stability in the compactified state space is investigated. The results are applied to nonsingular stochastic differential equations. The paper is related to the previous one by the author [6] where strong stability was investigated by the method of lower measure developed by A. Lasota [5]. It also can be viewed as a continuation of some earlier results by R. Khasminskii [1], G. Maruyama and H. Tanaka [2] and others.

I. Introduction and notations. Let $(X, \rho, \mathscr{B})$ be a σ-compact complete separable metric space with the σ-algebra $\mathscr{B}$ of Borel sets and let $P(s, x, t, A)$, $0 \leq s \leq t < \infty$, $x \in X$, $A \in \mathscr{B}$ be a transition probability corresponding to an X-valued Markov process. Denote by $\mathscr{P}$ the set of probability measures defined on $\mathscr{B}$ and let d be a metric on $\mathscr{P}$ realizing the weak convergence of measures and satisfying

$$(1.1) \qquad (\mu_n), (\nu_n) \in \mathscr{P}, \quad \mu_n - \nu_n \rightharpoonup 0 \Rightarrow d(\mu_n, \nu_n) \to 0,$$

where $\rightharpoonup$ stands for the weak convergence of measures (for instance, we can take

$$d(\mu, \nu) = \sum_{i=0}^{\infty} 2^{-i} \left| \int_X \varphi_i d\mu - \int_X \varphi_i d\nu \right|,$$

where $\{\varphi_i\}$ is a countable dense subset of the open unit ball of the space of uniformly continuous real functions on X. For $0 \leq s \leq t < \infty$ we define

$$S_{s,t}: \mathscr{P} \to \mathscr{P}, \quad S_{s,t}\mu(A) = \int_X P(s, x, t, A)\mu(dx), \quad A \in \mathscr{B},$$

and

$$T_{s,t}: \mathsf{C} \cap \mathsf{M} \to \mathsf{M}, \quad T_{s,t} f(x) = \int_X f(y) P(s, x, t, dy), \quad x \in X,$$

where C and M are the sets of continuous and bounded functions on X, respectively. When the corresponding Markov process is time-homogeneous we write $P(t, x, A)$, S_t, T_t, etc. Furthermore, let $X^* = X \cup \{\infty\}$ be the one-point compactification of the space X and let the symbols $\mathscr{B}^*$, $\mathscr{P}^*$, d^*, $\overset{X^*}{\rightharpoonup}$, C^*, M^* have the same meaning in X^* as the respective ones introduced above in X. For $0 \leq s \leq t$ we define

$$\varphi_{s,t}: \mathscr{P}^* \to \mathscr{P}^*, \quad \varphi_{s,t}\mu = \mu_\infty \delta_\infty + (1 - \mu_\infty) S_{s,t} \mu^X, \quad \mu \in \mathscr{P}^* \setminus \{\delta_\infty\},$$

where $\mu_\infty = \mu(\{\infty\})$, δ_∞ is the Dirac measure at the point $\{\infty\}$ and

$$\mu^X = (1 - \mu_\infty)^{-1} (\mu - \mu_\infty \delta_\infty) \mid \mathscr{B};$$

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set $\varphi_{s,t}\delta_\infty = \delta_\infty$. Furthermore, set

$$\tau_{s,t}\colon \mathsf{C}^* \to \mathsf{M}^*, \quad \tau_{s,t} f(x) = T_{s,t} f(x), \quad x \in X,$$
$$\tau_{s,t} f(\{\infty\}) = f(\{\infty\}) = f_\infty.$$

$\tau_{s,t}$ and $\varphi_{s,t}$ are the natural extensions of $T_{s,t}$ and $S_{s,t}$ on the spaces X^* and $\mathscr{P}^*$, respectively. It is easily seen that we can write

$$\int_{X^*} f(x) d[\varphi_{s,t}\mu] = \int_{X^*} \tau_{s,t} f d\mu.$$

The paper is divided into four sections. In Section 2 the general case is treated — sufficient and, in some cases, necessary and sufficient conditions are established under which the systems $\{S_{s,t}\}$ and $\{\varphi_{s,t}\}$ are continuous (or Lyapunov stable) in the spaces $(\mathscr{P}, d)$ and $(\mathscr{P}^*, d^*)$, respectively. Section 3 contains some applications of the results of Section 2 to the case of nonsingular stochastic differential equations of Itô's type. Some examples are given in Section 4.

Some of the proofs are rather sketchy, especially in Section 2; they can be found in full detail in [7] (some of them are straightforward). The fundamental theory of stochastic differential equations in a self-contained form can be found, for instance in [3].

For $\varepsilon > 0$ we denote $U_\varepsilon(x) = \{y \in X,\ \rho(x, y) < \varepsilon\}$ and $U_\varepsilon = U_\varepsilon(0)$, if X is a linear space. The symbol $\int_Y f d\mu$ can be also written as $\int_Y f(x)\mu(dx)$ or $\int_Y f d[\mu]$ and the integration domain Y can be omitted if we integrate over the whole space considered. Furthermore, $\delta_x\ (x \in X)$ stands for the Dirac measure at the point x and the symbols C_1, $\mathsf{C}_{1,2}$, $\mathbf{E}_{s,x}$, $\mathbf{E}_x$, $\mathbf{P}_{s,x}$, $\mathbf{P}_x$ are used in the obvious sense (see, e. g., [3]).

II. Weak stability — the general case. Assume that the following assumptions are fulfilled:

(2.1) For every $\varepsilon > 0$, $s \geq 0$ we have

$$P(s, x, t, U_\varepsilon(x)) \to 1 \quad \text{for } t \to s+,$$
$$P(t, x, s, U_\varepsilon(x)) \to 1 \quad \text{for } t \to s-$$

locally uniformly at $x \in X$ and the system of measures $\{P(s, x, t, .)\}_{t \in [s, T]}$ is relatively weakly compact for all $T > s$.

(2.2) The system $\{T_{s,t} f(\,.\,)|_K\}$ is equicontinuous with respect to $t > t_0 + s$ for all $s \geq 0$, $t_0 > 0$, $f \in \mathsf{C} \cap \mathsf{M}$, $K \subset X$, K compact.

Theorem 2.1. *Let (2.1), (2.2) hold and let $\mu \in \mathscr{P}$ be arbitrary. Then, for every $\varepsilon > 0$ and $s \geq 0$, there exists $\delta > 0$ such that $d(\mu, \tilde{\mu}) < \delta$, $\tilde{\mu} \in \mathscr{P}$ implies the relation*

$$\sup_{t \geq s} d(S_{s,t}\mu,\ S_{s,t}\tilde{\mu}) < \varepsilon$$

(i. e., stability in the space X holds).

Proof. First we show that the mappings

$$\Phi_s\colon (\mathscr{P}, d) \times \langle s, \infty) \to (\mathscr{P}, d); \quad [\nu, t] \to S_{s,t}\nu \tag{2.3}$$

are continuous for all $s \geq 0$. Assume $t_n \geq s$, $t_n \to t_0$, $\nu^n \in \mathscr{P}$, $\nu^n \to \nu \in \mathscr{P}$. We show that

$$S_{s,t_n}\nu^n \to S_{s,t_0}\nu \tag{2.4}$$

holds. For $f \in \mathrm{C} \cap \mathrm{M}$ we can write

$$(2.5) \qquad |\int fd[S_{s,t_n}\nu^n]-\int fd[S_{s,t_0}\nu]|=|\int T_{s,t_n}fd\nu^n-\int T_{s,t_0}fd\nu|\leq \int|T_{s,t_n}f - T_{s,t_0}f|\,d\nu^n+|\int T_{s,t_0}fd\nu^n-\int T_{s,t_0}fd\nu|.$$

It can be easily seen that (2.1), (2.2) imply $T_{s,t_n}f\to T_{s,t_0}f$ locally uniformly in X, (see e. g. [7]) and thus both terms on the right-hand side of (2.5) are small for n sufficiently great. It follows that (2.4) holds. Assume that the assertion of Theorem **2.1** is false. By (1.1) and continuity of Φ_s we get

$$(2.6) \qquad |\int fd\,[S_{s,t_n}\mu^n]-\int fd\,[S_{s,t_n}\mu]|\geq\varepsilon_0$$

for some $\varepsilon_0>0$, $s\geq 0$, $f\in\mathrm{C}\cap\mathrm{M}$, $t_n\to\infty$, $\mu^n\in\mathscr{P}$, $\mu^n\to\mu\in\mathscr{P}$. The condition (2.2) implies $T_{s,t_{n_k}}f\to\psi$ for some $\psi\in\mathrm{C}\cap\mathrm{M}$ and a subsequence $(t_{n_k})\subset(t_n)$, locally uniformly in X. Simple considerations (cf. [7]) show that

$$|\int fd\,[S_{s,t_{n_k}}\mu^{n_k}]-\int fd\,[S_{s,t_{n_k}}\mu]|=|\int T_{s,t_{n_k}}fd\mu^{n_k}-\int T_{s,t_{n_k}}fd\mu|\to 0,\ k\to\infty,$$

which is a contradiction. Thus the assertion of Theorem 2.1 is valid.

Remark. The assumption (2.1) is weak and it is fulfilled in most examples. So the "nondegeneracy condition" (2.2) can be viewed as the essential one. It is stronger than the strong Feller property, but not than the strong Feller property in the narrow sense introduced by Girsanov [8] (cf. Corollary 2.2). Thus the condition (2.2) can be replaced by (2.7) in all theorems in this section.

Corollary 2.2. *Let (2.1) be fulfilled. Assume that*

$$(2.7) \qquad \mathrm{Var}\,[P(s,x,t,.)-P(s,x_0,t,.)]\to 0,\quad x\to x_0,$$

holds for all $0\leq s\leq t<\infty$, $x_0\in X$ *(the strong Feller property in the narrow sense;* Var ν *stands for the total variation of the measure* ν*). Then the assertion of Theorem 2.1 (stability in X) is valid.*

Proof. First, we show (2.7)⇒(2.2). For $t>t_0>s$, $f\in\mathrm{C}\cap\mathrm{M}$, $x_1\in X$, $x_2\in X$ we have

$$|T_{s,t}f(x_1)-T_{s,t}f(x_2)|\leq\sup_X|T_{t_0,t}f|\,\mathrm{Var}\,[P(s,x_1,t_0,.)-P(s,x_2,t_0,.)]$$
$$\leq\sup_X|f|\,\mathrm{Var}\,[P(s,x_1,t_0,.)-P(s,x_2,t_0,.)]$$

which implies (2.2).

In the rest of this section we deal with the continuity and stability of the system $\{\varphi_{s,t}\}$. We impose the following two conditions:

(2.8) For every $\varepsilon>0$, $0\leq s<T$, $K\subset X$, K compact, there exists a compact set $M\subset X$ such that the inequality $P(s,x,t,K)<\varepsilon$ holds for all $x\in X\setminus M$, $t\in[s,T]$.

(2.9) For every $\varepsilon>0$, $s\geq 0$, $K\subset X$, K compact, there exists a compact set $M\subset X$ such that $P(s,x,t,K)<\varepsilon$ holds for all $x\in X\setminus M$, $t\geq s$.

Roughly speaking, in the following two theorems we state that (2.8) is equivalent to the continuity of the system $\{\varphi_{s,t}\}$ while the stronger condition (2.9) is equivalent to its stability.

Theorem 2.3. *Assume that* (2.1), (2.2) *are satisfied. Then:*

A. *If* (2.3) *is fulfilled, then the mappings*

(2.10) $$\Phi_s^*\colon\ [s, \infty)\times\mathscr{P}^*\to\mathscr{P}^*,\ \ (t, \nu)\longmapsto\varphi_{s,t}\nu$$

are continuous for all $s\geq 0$ *(i. e., continuity in* X^* *holds).*
B. *If* X *is a locally compact space and the mappings* Φ_s^* *are continuous for all* $s\geq 0$, *then* (2.8) *is fulfilled.*

Proof. To prove part A we must show that

$$t_n\to t_0\geq s,\quad \mu_n\xrightarrow{X^*}\mu\in\mathscr{P}^*,$$

implies

(2.11) $$\varphi_{s,t_n}\mu_n\xrightarrow{X^*}\varphi_{s,t_0}\mu.$$

For $f\in\mathbf{C}^*$ we can write

(2.12) $$\left|\int_{X^*} fd\,[\varphi_{s,t_n}\mu_n]-\int_{X^*} fd\,[\varphi_{s,t_0}\mu]\right|=\left|\int_{X^*}\tau_{s,t_n}fd\mu_n-\int_{X^*}\tau_{s,t_0}fd\mu\right|$$
$$\leq\int_{X^*\setminus M}(|\tau_{s,t_n}f-f_\infty|+|\tau_{s,t_0}f-f_\infty|)d\mu_n+\int_M|\tau_{s,t_n}f-\tau_{s,t_0}f|\,d\mu_n$$
$$+\left|\int_{X^*}\tau_{s,t_0}fd\mu_n-\int_{X^*}\tau_{s,t_0}fd\mu\right|,$$

where $M\subset X$ is a compact set. From (2.8) it follows that $\tau_{s,t_n}f,\ \tau_{s,t_0}f\in\mathbf{C}^*$ (cf. [7]) and

$$|\tau_{s,t_n}f(x)-f_\infty|\to 0\quad\text{for}\quad x\to\{\infty\}$$

holds uniformly with respect to $n\in\{0\}\cup\mathbf{N}$. Moreover, as we have shown in the proo of Theorem 2.1, $\tau_{s,t_n}f\to\tau_{s,t_0}f$, $n\to\infty$, holds locally uniformly in X. Hence by (2.12 we obtain (2.11).

To prove part B consider an open covering $\{\operatorname{Int}K_x\}_{x\in X}$ of the space X such that $K_x\subset X$ is compact and $\operatorname{Int}K_x$ is a neighbourhood of the point x for all $x\subset X$.) Since X is a Lindelöf space we can find its countable subcovering $\mathscr{K}=\{\operatorname{Int}K_i\}_{i\in\mathbf{N}}\subset\{\operatorname{Int}K_x\}_{x\in X}$. Put

$$M_n=\bigcup_{i=0}^{n}K_i$$

and assume that B is false, so that we can find $\varepsilon_0>0$, $s\geq 0$, $K\subset X$ (K compact) $t_n\to t_0\geq s$ and $x_n\in X\setminus M_n$ such that $P(s, x_n, t_n, K)\geq\varepsilon_0$. If $R\subset X$ is compact, there exists a finite subcovering of R,

$$\{\operatorname{Int}K_{j_1},\dots,\operatorname{Int}K_{j_m}\}\subset\mathscr{K}.$$

Setting $j=\max(j_1,\dots,j_m)$ we have $x_n\in X\setminus M_n\subset X\setminus R$ for $n\geq j$. It follows that $x_n\to\{\infty\}$ and thus $\delta_{x_n}\xrightarrow{X^*}\delta_\infty$. On the other hand, local compactness of X implies normality of X^* and hence we can find a function $\psi\in\mathbf{C}^*$ such that

$$\psi(X^*)=[0, 1],\quad \psi|_K=1,\quad \psi_\infty=0.$$

We get

$$\varepsilon_0\leq\int_{X^*}\psi d\,[\varphi_{s,t_n}\delta_{x_n}]\to\int_{X^*}\psi d[\varphi_{s,t_0}\delta_\infty]=\psi_\infty=0$$

which is a contradiction.

Theorem 2.4. *Let (2.1), (2.2) be satisfied. Then:*
$\tilde{A}$. *If (2.9) is fulfilled, then for every* $\varepsilon>0$, $\mu\in\mathcal{P}^*$ *and* $s\geqq 0$ *such* $\delta>0$ *can be found that*

$$d^*(\mu, \tilde{\mu})<\delta, \quad \tilde{\mu}\in\mathcal{P}^*,$$

implies

$$\sup_{t\geqq s} d^*(\varphi_{s,t}\mu, \ \varphi_{s,t}\tilde{\mu})<\varepsilon$$

(i. e., stability in X^* *holds).*
$\tilde{B}$. *If X is a locally compact space and stability in* X^* *holds, then (2.9) is fulfilled.*

Proof. It can be easily checked that for $f\in C^*$, $s\geqq 0$, $t_0>0$ the condition (2.9) guarantees equicontinuity of the set $\mathcal{M}=\{\tau_{s,t}f\}_{t\geqq t_0+s}$ at the point $\{\infty\}$ and hence, by (2.2), relative compactness of $\mathcal{M}$ in the space C^* (cf. [7]). The rest of the proof of $\tilde{A}$ is very similar to that of Theorem 2.1 and therefore it is omitted while the proof of $\tilde{B}$ is almost the same as that of $\tilde{B}$ in the preceding theorem.

We shall conclude this section by two statements concerning the case of a homogeneous diffusion process. More precisely, assume that the following is fulfilled:

(2.13) For all $\varepsilon>0$ the relation $1-P(t, x, U_\varepsilon(x))=o(t)$, $t\to 0+$ holds locally uniformly with respect to $x\in X$.

(2.14) If $f\in M$ is Borel measurable, then $T_t f\in C$ for $t>0$ (the strong Feller property).

(2.15) $P(t, x, U)>0$ holds for all $t>0$, $x\in X$, $U\subset X$, U open.

The measure $\mu\in\mathcal{P}$ is called invariant, if the equality $S_t\mu=\mu$ holds for all $t\geqq 0$. The following theorem is a consequence of the result by Khasminskii [1] (cf. [7]).

Theorem 2.5. *Assume that (2.13)—(2.15) are satisfied. Then one of the following two possibilities occurs:*
α) *If there exists an invariant measure* $\mu\in\mathcal{P}$, *then*

$$S_t\nu\to\mu, \ t\to\infty, \text{ holds for all } \nu\in\mathcal{P}.$$

β) *Otherwise,* $\varphi_t\mu\overset{X^*}{\to}\delta_\infty$, $t\to\infty$ *holds for all* $\mu\in\mathcal{P}^*$.

Corollary 2.6. *If (2.1), (2.2), (2.13)—(2.15) and (2.8) are fulfilled, then* $\{\varphi_t\}_{t\geqq 0}$ *is a semidynamical system defined on the space* $(\mathcal{P}^*, d^*)$ *with a stationary point* δ_∞. *If, moreover, (2.9) holds, then the stationary point* d_∞ *is globally asymptotically stable (in the Lyapunov sense).*

III. Applications to nonsingular stochastic differential equations. In this section we put $X=R_n$ and assume that the transition probability P is associated with the solution of the stochastic differential equation

$$d\zeta_t=b(t, \zeta_t)dt+\sigma(t, \zeta_t)dw_t. \tag{3.1}$$

Here w_t is an l-dimensional Wiener process, b and σ are an n-dimensional vector and an $n\times 1$ matrix, respectively, b and σ both defined on $R_+\times R_n$, Borel measurable and satisfying

$$|b(t, x_1)-b(t, x_2)|+|\sigma(t, x_1)-\sigma(t, x_2)|\leqq K_N|x_1-x_2|, \ K_N>0 \tag{3.2}$$

for all $N>0$ and $|x_1|+|x_2|+t\leqq N$. Let L be the infinitesimal operator corresponding to the equation (3.1), i. e.

$$LV(t, x)=[\frac{\partial V}{\partial t}+\sum_i b_i \frac{\partial V}{\partial x_i}+\frac{1}{2}\sum_{i,j} a_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j}](t, x)$$

for $V\in C_{1,2}(Q)$, $Q\subset R_+\times R_n$ open, where $(a_{ij}(t, x))=\sigma(t, x)\,\sigma^T(t, x)$. We assume that

$$LW(t, x)\leqq c\beta(W(t, x)), \quad (t, x)\in R_+\times R_n \tag{3.3}$$

holds for some $c>0$, where $W\in C_{1,2}(R_+\times R_n)$, $W\geqq 0$, $\beta\in C_1(R_+)$ is a concave increasing function and the conditions

$$\int_0^\infty \frac{du}{1+\beta(u)}=\infty, \quad \lim_{|x|\to\infty}\ \inf_{0\leqq t\leqq T} W(t, x)=\infty$$

are valid for all $T>0$. It is known (cf. [4]) that under the assumptions (3.2), (3.3) the equation (3.1) has a solution which is unique in the obvious sense and the corresponding transition probability P satisfies (2.1). All equations in this paper are assumed to satisfy the existence and uniqueness conditions (3.2), (3.3). Now we give conditions which will be used to guarantee (2.2). Let

$$\sum_{i,j} a_{ij}(t, x)\,\nu_i\,\nu_j\geqq m(x)\,|\,\nu\,|^2 \tag{3.4}$$

for all $(t, x)\in R_+\times R_n$, $\nu=(\nu_i)\in R_n$, where $m>0$ is a continuous function and let

$$|\,a_{ij}(t, x)\,|+|\,b_i(t, x)\,|\leqq K_N, \tag{3.5}$$

$$|\,a_{ij}(t, x)-a_{ij}(s, y)\,|+|\,b_i(t, x)-b_i(s, y)\,|\leqq K_N(\,|\,x-y\,|^\alpha+|\,t-s\,|^{\alpha/2})$$

for some $\alpha\in(0, 1)$, $K_N>0$ and all $N>0$, $t, s\in R_+$, $x, y\in R_n$, $|\,x\,|+|\,y\,|\leqq N$, and $i, j=1, 2, \ldots, n$.

Theorem 3.1. *Assume that the transition probability P corresponds to the solution of the equation (3.1) and that (3.4), (3.5) are fulfilled. Then the assertion of Theorem 2.1 (i. e. stability in the space $X=R_n$) is valid.*

Proof. This statement is a consequence of Theorem 2.1. We only need to verify the assumption (2.2). Let $f\in C\cap M$, $R>0$ and for $0\leqq s<t$ put

$$u_t(s, x)=\mathbf{E}_{s,x}f(\zeta_t)=T_{s,t}\,f(x). \tag{3.6}$$

Let $V_t(s, x)$ be the solution of the problem

$$LV_t(s, x)=0 \quad \text{for} \quad (s, x)\in[0, t)\times U_{R+1},$$

$$V_t(t, x)=f(x)\ (=u_t(t, x)), \quad V_t(s, x)\,|\,[0, t)\times\partial U_{R+1}=u_t(s, x)$$

and denote by τ the exit time (after s) from the set $(s, t)\times U_{R+1}$. (It can be easily shown that $\tau<\infty$ a. s.) Using the Itô's formula we get

$$V_t(s, x)=\mathbf{E}_{s,x}V_t(\tau, \zeta_\tau),$$

and hence

$$V_t(s, x)=\mathbf{E}_{s,x}u(\tau, \zeta_\tau)=\mathbf{E}_{s,x}\mathbf{E}_{\tau,\zeta_\tau}f(\zeta_t)=\mathbf{E}_{s,x}f(\zeta_t)=u_t(s, x).$$

By Schauder's interior estimate it follows that for any $t>t_0>0$ we have

$$|\,\frac{\partial u}{\partial x_i}(s, x)\,|=|\frac{\partial V_t}{\partial x_i}(s, x)\,|\leqq K\sup|\,V_t\,|=K\sup|\,u_t\,|\leqq K\sup|\,f\,|$$

for some $K>0$ and all $(s, x)\in[0, t-t_0]\times\bar{U}_R$. Noting that the constant K does not depend on $t>t_0$, we get (2.2).

Next we give two corollaries of Theorems 2.3 and 2.4, in which we use Lyapunov functions to guarantee (2.8) and (2.9), respectively.

Corollary 3.2. *Let (3.4) and (3.5) be fulfilled and assume that there exists a positive function* $V \in C_{1,2}(\mathbf{R}_+ \times (\mathbf{R}_n \setminus U_r))$, *where* $r \geqq 0$, *such that* $\forall\, T > 0$ *we have*

$$\sup_{0 \leqq s \leqq T} V(s, x) \to 0 \quad for \quad |x| \to \infty \tag{3.7}$$

and

$$LV(s, x) \leqq cV(s, x), \quad (s, x) \in \mathbf{R}_+ \times (\mathbf{R}_n \setminus U_r) \tag{3.8}$$

hold for some $c > 0$. *Then the assertion of Theorem 2.3 (continuity in* $X^* = \mathbf{R}_n^*$*) is valid.*

Proof. To show that the assumptions of Theorem 2.3 are satisfied it only remains to prove (2.8) ((2.2) was verified in the preceding proof). Set $H(t, x) = V(t, x)e^{-ct}$. From (3.8) it follows that $LH(t, x) \leqq 0$. Take $s \geqq 0$, $R > R_0 > r$ and denote by τ_R and τ the exit times (after s) from $U_R \setminus \bar{U}_{R_0}$ and $\mathbf{R}_n \setminus \bar{U}_{R_0}$, respectively. Using the Itô's formula we get

$$\mathbf{E}_{s,x} H(\tau_R \wedge t, \zeta_{\tau_R \wedge t}) \leqq H(s, x)$$

for all $t > s$, $R > |x| > R_0$. Taking $R \to \infty$, we get by the Fatou's lemma

$$\mathbf{E}_{s,x} V(\tau \wedge t, \zeta_{\tau \wedge t}) \leqq V(s, x)e^{c(t-s)} \tag{3.9}$$

and thus

$$\mathbf{P}_{s,x}[\zeta_\lambda \in U_{R_0} \text{ for some } \lambda \in [s, t]] \leqq \frac{V(s, x)e^{c(t-s)}}{\inf\limits_{[s,t] \times \bar{U}_{R_0}} V}.$$

It follows that $P(s, x, \lambda, U_{R_0}) \to 0$ for $|x| \to \infty$ uniformly with respect to $\lambda \in [s, t]$.

Corollary 3.3 *Let (3.4) and (3.5) be fulfilled and assume that there exists a function* $u \in C_{1,2}(\mathbf{R}_+ \times (\mathbf{R}_n \setminus U_r))$, *where* $r \geqq 0$, *such that*

$$\sup_{s \in [0, \infty)} u(s, x) \to 0 \quad for \quad |x| \to \infty. \tag{3.10}$$

$$Lu(s, x) \leqq 0, \quad (s, x) \in \mathbf{R}_+ \times (\mathbf{R}_n \setminus U_r) \tag{3.11}$$

and

$$\inf_{\mathbf{R}_+ \times \partial U_R} u > 0 \tag{3.12}$$

hold for all $R > r$. *Then the assertion of Theorem 2.4 (stability in* $X^* = \mathbf{R}_n^*$*) is valid.*

The proof is quite analogous to the preceding one and, therefore, could be omitted.

We conclude this section by a corollary concerning the homogeneous case. Its proof can be found in [1] where it is shown that (3.4) implies (2.13)—(2.15) and hence the assumptions of Theorem 2.5 are satisfied.

Corollary 3.4. *Assume that the coefficients* b, σ *are independent of* $t \geqq 0$ *and that (3.4) is fulfilled. Then the assertion of Theorem 2.5 is valid.*

IV. Examples.

Example 4.1. Assume that (3.4), (3.5) are fulfilled and

$$|b(t, x)| + |\sigma(t, x)| \leqq K(1 + |x|), \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}_n \tag{4.1}$$

holds for some $K>0$. Then the mappings Φ_s^* defined by (2.10) are continuous. In particular, if the coefficients b, σ are independent of $t\geqq 0$, then $\{\varphi_t\}_{t\geqq 0}$ is a semidynamical system on $(\mathcal{P}^*, d^*)$.

This assertion can be obtained from Corollary 3.2 if we use the function $V(x) = \frac{1}{1+|x|^2}$. Indeed, for some $\bar{K}>0$ and $c>0$ we have:

$$LV(t, x) = \sum_i \frac{-2x_i b_i(t, x)}{(1+|x|^2)^2} + \frac{1}{2}\sum_{i,j} a_{ij}(t, x)\left(\frac{-2\delta_{ij}}{(1+|x|^2)^2} + \frac{8x_i x_j}{(1+|x|^2)^3}\right)$$

$$\leqq \frac{1}{1+|x|^2}\left\{\sum_i \frac{|2x_i| K(1+|x|)}{1+|x|^2} + \frac{1}{2}\sum_{i,j} \bar{K}(1+|x|^2)\left(\frac{2\delta_{ij}}{1+|x|^2} + \frac{8|x_i x_j|}{(1+|x|^2)^2}\right)\right\} \leqq \frac{c}{1+|x|^2} = cV(x).$$

Remark 4.2. The above example shows that the continuity in $X^* = \mathrm{R}_n^*$ always occurs in the case when the classical Itô's existence and uniqueness conditions (3.2), (4.1) are satisfied. However, this is not true in a more general case. For example, consider the simple deterministic equation

$$\dot{\zeta} = -\zeta^2 \operatorname{sign} \zeta$$

which satisfies the weaker existence and uniqueness conditions (3.2), (3.3) with $W(x)=x^2$, $\beta(x)=x$. Denote by ζ_t^x the solution starting at the time $t=0$ from the point $x\in \mathrm{R}_n$. For any $x>0$ we have

$$\zeta_1^x = \frac{x}{1+x} \in [0, 1],$$

which contradicts (2.8). Hence we have no continuity in X^*.

Example 4.3. Consider the autonomous stochastic equation

$$d\zeta_t = b(\zeta_t)\,dt + \sigma(\zeta_t)\,dw_t \tag{4.2}$$

whose coefficients b, σ satisfy (3.4). Given $r_0>0$, put

$$u(x) = \mathrm{P}_x[\zeta_t \in U_{r_0} \text{ for some } t\geqq 0], \quad |x|>r_0.$$

It is well known that $Lu(x)=0$ holds for $|x|>r_0$ and by the strong maximum principle for elliptic equations we get $u>0$. Thus by Corollary 3.3, the stability in $X^*=\mathrm{R}_n^*$ holds provided $u(x)\to 0$ for $|x|\to\infty$ (note that in the time-homogeneous case (3.5) is fulfilled by (3.2) automatically).

Remark 4.4. Let P be the transition probability of the homogeneous diffusion process given by the equation (4.2) whose coefficients satisfy (3.4). From Corollary 3.4 it follows that $\varphi_t\nu \xrightarrow{X^*} \delta_\infty$ for $\nu\in\mathcal{P}^*$ (i. e., δ_∞ is globally attractive in $\mathcal{P}^*$) if and only if there exists no invariant measure in $\mathcal{P}$ which is implied by (2.9) (and thus by stability in $X^*=\mathrm{R}_n^*$). Note however that the stability in X^* is not equivalent to the attractivity of δ_∞ (see the next example).

Example 4.5. Consider the equation

$$d\zeta_t = b(\zeta_t)dt + dw_t, \tag{4.3}$$

where $b(x)=-x^2$ for $x\leqq 0$ and $b(x)=x$ otherwise (the existence and uniqueness conditions (3.2), (3.3) are satisfied with $W(x)=x^2+1$, $\beta(x)=x$). We will show that there exists no invariant measure in $\mathcal{P}$ and that (2.8) is not satisfied (in particular, δ_∞ is the globally attractive stationary point which is not Lyapunov stable; nor even continuity in X^* holds). We have

$$\int_0^{-\infty} \exp\left[-2\int_0^y z\,dz\right] > -\infty$$

and hence the solution of (4.3) is not a recurrent Markov process (cf. [3]). Consequently, there exists no invariant measure in $\mathscr{P}$ (nor even σ-finite invariant measure, cf. [1]).

Furthermore, we show:

(α) Given $x_0>0$ and $\eta>0$ there exists $M>0$ such that

$$\mathbf{P}_x[\zeta_1 \leqq -M] \leqq \eta \tag{4.4}$$

holds for $x \geqq x_0$. Indeed, taking $M>0$ such that

$$\mathbf{P}_{x_0}[\inf_{s\in[0,1]} \zeta_s \leqq -M] \leqq \eta$$

we get

$$\mathbf{P}_x[\zeta_1 \leqq -M] = \int_{u=0}^{1} P_{x_0}[\zeta_{1-u} \leqq -M] P[\tau \in du] \leqq \eta$$

(τ is the first passage time by x_0).

(β) Given $x_0>0$, $\varepsilon>0$ we can find $N>0$ such that

$$\mathbf{E}_x \zeta_1 \chi_{[\zeta_1 \leqq -N]} \geqq -\varepsilon \tag{4.5}$$

holds for $x \geqq x_0$. Indeed, we have ($N>0$, $x \geqq x_0$)

$$|\mathbf{E}_x \zeta_1 \chi_{[\zeta_1 \leqq -N]}| \leqq \int_{u=0}^{1} |\mathbf{E}_{x_0} \zeta_{1-u} \chi_{[\zeta_{1-u} \leqq -N]}| \, \mathbf{P}[\tau \in du]$$

$$\leqq \int_{u=0}^{1} \{\mathbf{P}_{x_0}[\zeta_{1-u} \leqq -N]\}^{1/2} \{\mathbf{E}_{x_0}(\zeta_{1-u})^2\}^{1/2} \mathbf{P}[\tau \in du].$$

t is easily seen that

$$\sup_{s\in[0,1]} \mathbf{E}_{x_0}(\zeta_s)^2 \leqq k^2$$

for some $k>0$ and hence

$$\mathbf{E}_x \zeta_1 \chi_{[\zeta_1 \leqq -N]} \geqq -k \sup_{s\in[0,1]} \{\mathbf{P}_{x_0}[\zeta_s \leqq -N]\}^{1/2}$$

which implies (β). Furthermore, we have

$$\mathbf{E}_x\, \zeta_t = \mathbf{E}_{x_0}\, \zeta_{t_0} + \mathbf{E}_x \int_{t_0}^{t} b(\zeta_s)\,ds$$

for $0<t_0<t$. By the Jensen's inequality we obtain

$$\mathbf{E}_x \zeta_t \leqq \mathbf{E}_x \zeta_{t_0} + \int_{t_0}^{t} b(E_x\, \zeta_s)\,ds.$$

Consequently, $\mathbf{E}_x \zeta_{t_0} \leqq 1$ holds for all $x \in \mathbf{R}$. Find $N>0$ from (β) corresponding to $x_0=1$ $\varepsilon=1$ (take $N \geqq 3$). For $x \geqq 1$ it follows that

$$1 \geqq \mathbf{E}_x \zeta_1 \geqq \mathbf{E}_x \zeta_1 \chi_{[\zeta_1 \leqq -N]} + \mathbf{E}_x \zeta_1 \chi_{[\zeta_1 \in (-N, N)]} + \mathbf{E}_x \zeta_1 \chi_{[\zeta_1 \geqq N]}$$

$$\geqq -1 - N\mathbf{P}_x[\zeta_1 \in (-N, N)] + N\mathbf{P}_x[\zeta_1 \geqq N] \geqq -1 - N(1 - \mathbf{P}_x[\zeta_1 \geqq N]$$

$$- \mathbf{P}_x[\zeta_1 \leqq -N]) + N\mathbf{P}_x[\zeta_1 \geqq N] \geqq -1 + 2N\mathbf{P}_x[\zeta_1 \geqq N] - N.$$

Hence we have

$$P_x[\zeta_1 \geqq N] \leqq \frac{1}{2N}(2+N) \leqq \frac{5}{6}.$$

Taking $M>0$ from α) corresponding to $x_0=1$ and $\eta=\frac{1}{12}$, we obtain

$$P(1, x, \langle -M, N\rangle) \geqq \frac{1}{12}$$

for all $x \geqq 1$ and thus (2.8) is not satisfied.

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