# Serdica
## Bulgariacae mathematicae publicationes

# Сердика
## Българско математическо списание

# ON ISOMETRY GROUP OF SOME PSEUDORIEMANNIAN MANIFOLD

HALINA FELIŃSKA

Let $(M, g)$ be an $n$-dimensional $(n \geqq 2)$ pseudoriemannian manifold. We denote by $TM$ the tangent bundle of $M$. We will consider $TM$ as a pseudoriemannian manifold with the metric $g^c$ (a complete lift of $g$). It is known that the dimension of the isometry group of $TM$ is at most $n(2n+1)$, (see [1], Th. 3.3, p. 238). In this paper we give an exact dimension of the isometry group of $(TM, g^c)$ provided that $(M, g)$ is a riemannian manifold of constant curvature.

Let $(M, g)$ be an $n$-dimensional $(n \geqq 2)$ pseudoriemannian manifold, $F^1(M)$ — the Lie algebra of vector fields on $M$, $\nabla$ — a riemannian connection on $M$, and $R$ — the curvature tensor of $\nabla$.

Similarly as for riemannian manifolds ([1], p. 42) we can prove

L e m m a 1. *If $LG$ is the Lie algebra of the isometry group $G$ of $M$ and $F^1_{KZ}(M)$ is the Lie algebra of complete Killing vector fields on $M$, then $LG$ is isomorphic to $F^1_{KZ}(M)$.*

Let $TM$ be a manifold of the tangent bundle to $(M, g)$.

We introduce on $TM$ the metric $g^c$, which is a complete lift of $g$. It is known [2] that, if a matrix of the metric tensor $g$ in a local coordinate system $(U, (x^i))$ on $M$ is of the form $(g_{ij})_{i,j=1,\dots,n}$, then

$$(g^c_{AB}) = \left( \begin{array}{c|c} y^k \partial_k g_{ij} & g_{ij} \\ \hline g_{ij} & 0 \end{array} \right), \qquad A, B = 1, \dots, 2n$$

is a matrix of the tensor $g^c$ in the associated coordinate system $(\pi^{-1}(U), (x^i, y^i))$

In the sequel, we will consider a linear connection $\nabla^c$ on pseudoriemannian manifold $(TM, g^c)$, which is a complete lift of Levi-Civita connection $\nabla$ on $M$ to $TM$, [2] $\nabla^c$ is a Levi-Civita connection, too. Moreover, we will consider the isometry groups or equivalently Killing vector fields. It is well known, non-zero Killing vector fields do not exist on each pseudoriemannian manifold. It can be proved ([2], p. 79) that each Killing vector field on $TM$ which preserves fibres is of the form $X^c + Y^v$, where $X^c$, $Y^v$ are complete and vertical lifts of Killing vector fields $X$ and $Y$ on $M$ to the tangent bundle $TM$, respectively.

We prove

L e m m a 2. *Let $(M, g)$ be a riemannian manifold of the constant curvature $k \neq 0$ and $A$ be a tensor field on $M$ of the type $(1, 1)$. The condition* a) $R(A(Z), Y)W = 0$ *for arbitrary $Z, Y, W \in F^1(M)$,* b) $A = 0$, *are equivalent.*

P r o o f. In arbitrary coordinate system $(U, x)$ on $M$ the curvature tensor $R$ satisfies the identities $R^l_{ijh} = k(g_{jh}\delta^l_i - g_{ih}\delta^l_j)$, $A^s_i R^l_{sjh} = 0$.
Hence we get $A^s_i(g_{sh}\delta^l_j - g_{jh}\delta^l_s) = 0$. Contracting it with $g^{jk}$, we get $A^l_i = 0$ for all $i, l = 1, \dots, n$. Now we are able to prove the following theorem:

SERDICA Bulgaricae mathematicae publicationes. Vol. 14, 1988, p. 44—45.

Theorem 3. *If* $(M,g)$ *is n-dimensional riemannian manifold of constant curvature* $k\neq 0$, *then the isometry group of pseudoriemannian manifold* $(TM, g^c)$ *is* $n(n+1)$*-parameter group.*

Proof. Let $(M, g)$ be $n$-dimensional riemannian manifold of constant curvature $k\neq 0$. Thus the dimension of the isometry group $G$ of $M$ is equal to $m=\frac{1}{2}n(n+1)$. Lemma 1 implies that the Lie algebra $LG$ of $G$ is isomorphic to the Lie algebra $F^1_{KZ}(M)$ of complete Killing vector fields on $M$. Let vector fields $X_1, \dots, X_m$ forms the basis of $R$-algebra $F^1_{KZ}(M)$. Then, the vector fields $X_1^v, \dots, X_m^v$, $X_1^c, \dots, X_m^c$ are $R$-linearly independent complete Killing vector fields on $TM$, [2]. Now, (by virtue of Th 12.16, p. 79 [2]) it is sufficient to show that there exist no a non-zero (1,1) tensor field $A$ on $M$, which satisfies the identities

$$R(A(Z), Y)W=A(R(Z, Y), W)=R(Z, A(Y))W=R(Z, Y)A(W)=0$$

for arbitrary $Z, Y, W\in F^1(M)$. Lemma 2 implies that such $A$ has to be equal to zero.

Corollary. *Let* $(M, g)$ *be n-dimensional riemannian manifold of constant curvature* $k\neq 0$. *Then the equation* $L_x g^c=0$ *on pseudoriemannian manifold* $(TM, g^c)$ *has exactly* $n(n+1)$ $R$*-linearly independent solutions.*

## REFERENCES

1. S. Kobayashi, K. Nomizu. Foundations of differential geometry. I. London, 1963.
2. K. Yano, S. Ishihara. Tangent and cotangent bundles. New York, 1973.

*Instytut Matematyki UMCS*
*Pl. M. C. Skłodowskiej 1*
*20-031 Lublin, Poland*

*Received 17. 4. 1986*