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## ON ISOMETRY GROUP OF SOME PSEUDORIEMANNIAN MANIFOLD

HALINA FELIŃSKA

Let  $(M, g)$  be an  $n$ -dimensional ( $n \geq 2$ ) pseudoriemannian manifold. We denote by  $TM$  the tangent bundle of  $M$ . We will consider  $TM$  as a pseudoriemannian manifold with the metric  $g^c$  (a complete lift of  $g$ ). It is known that the dimension of the isometry group of  $TM$  is at most  $n(2n+1)$ , (see [1], Th. 3.3, p. 238). In this paper we give an exact dimension of the isometry group of  $(TM, g^c)$  provided that  $(M, g)$  is a riemannian manifold of constant curvature.

Let  $(M, g)$  be an  $n$ -dimensional ( $n \geq 2$ ) pseudoriemannian manifold,  $F^1(M)$  — the Lie algebra of vector fields on  $M$ ,  $\nabla$  — a riemannian connection on  $M$ , and  $R$  — the curvature tensor of  $\nabla$ .

Similarly as for riemannian manifolds ([1], p. 42) we can prove

**Lemma 1.** *If  $LG$  is the Lie algebra of the isometry group  $G$  of  $M$  and  $F^1_{KZ}(M)$  is the Lie algebra of complete Killing vector fields on  $M$ , then  $LG$  is isomorphic to  $F^1_{KZ}(M)$ .*

Let  $TM$  be a manifold of the tangent bundle to  $(M, g)$ .

We introduce on  $TM$  the metric  $g^c$ , which is a complete lift of  $g$ . It is known [2] that, if a matrix of the metric tensor  $g$  in a local coordinate system  $(U, (x^i))$  on  $M$  is of the form  $(g_{ij})_{i,j=1,\dots,n}$ , then

$$(g^c_{AB}) = \begin{pmatrix} y^k \partial_k g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}, \quad A, B = 1, \dots, 2n$$

is a matrix of the tensor  $g^c$  in the associated coordinate system  $(\pi^{-1}(U), (x^i, y^i))$

In the sequel, we will consider a linear connection  $\nabla^c$  on pseudoriemannian manifold  $(TM, g^c)$ , which is a complete lift of Levi-Civita connection  $\nabla$  on  $M$  to  $TM$ , [2]  $\nabla^c$  is a Levi-Civita connection, too. Moreover, we will consider the isometry groups or equivalently Killing vector fields. It is well known, non-zero Killing vector fields do not exist on each pseudoriemannian manifold. It can be proved ([2], p. 79) that each Killing vector field on  $TM$  which preserves fibres is of the form  $X^c + Y^v$ , where  $X^c, Y^v$  are complete and vertical lifts of Killing vector fields  $X$  and  $Y$  on  $M$  to the tangent bundle  $TM$ , respectively.

We prove

**Lemma 2.** *Let  $(M, g)$  be a riemannian manifold of the constant curvature  $k \neq 0$  and  $A$  be a tensor field on  $M$  of the type  $(1, 1)$ . The condition a)  $R(A(Z), Y)W = 0$  for arbitrary  $Z, Y, W \in F^1(M)$ , b)  $A = 0$ , are equivalent.*

**Proof.** In arbitrary coordinate system  $(U, x)$  on  $M$  the curvature tensor  $R$  satisfies the identities  $R^l_{ijh} = k(g_{jh}\delta^l_i - g_{ih}\delta^l_j)$ ,  $A^s_i R^l_{s/jh} = 0$ . Hence we get  $A^s_l (g_{sh}\delta^l_j - g_{jh}\delta^l_s) = 0$ . Contracting it with  $g^{jk}$ , we get  $A^l_i = 0$  for all  $i, l = 1, \dots, n$ . Now we are able to prove the following theorem:

**Theorem 3.** *If  $(M, g)$  is  $n$ -dimensional riemannian manifold of constant curvature  $k \neq 0$ , then the isometry group of pseudoriemannian manifold  $(TM, g^c)$  is  $n(n+1)$ -parameter group.*

**Proof.** Let  $(M, g)$  be  $n$ -dimensional riemannian manifold of constant curvature  $k \neq 0$ . Thus the dimension of the isometry group  $G$  of  $M$  is equal to  $m = \frac{1}{2}n(n+1)$ . Lemma 1 implies that the Lie algebra  $LG$  of  $G$  is isomorphic to the Lie algebra  $F_{KZ}^1(M)$  of complete Killing vector fields on  $M$ . Let vector fields  $X_1, \dots, X_m$  forms the basis of  $R$ -algebra  $F_{KZ}^1(M)$ . Then, the vector fields  $X_1^v, \dots, X_m^v, X_1^c, \dots, X_m^c$  are  $R$ -linearly independent complete Killing vector fields on  $TM$ , [2]. Now, (by virtue of Th 12.16, p. 79 [2]) it is sufficient to show that there exist no a non-zero (1,1) tensor field  $A$  on  $M$ , which satisfies the identities

$$R(A(Z), Y)W = A(R(Z, Y), W) = R(Z, A(Y))W = R(Z, Y)A(W) = 0$$

for arbitrary  $Z, Y, W \in F^1(M)$ . Lemma 2 implies that such  $A$  has to be equal to zero.

**Corollary.** *Let  $(M, g)$  be  $n$ -dimensional riemannian manifold of constant curvature  $k \neq 0$ . Then the equation  $L_x g^c = 0$  on pseudoriemannian manifold  $(TM, g^c)$  has exactly  $n(n+1)$   $R$ -linearly independent solutions.*

#### REFERENCES

1. S. Kobayashi, K. Nomizu. Foundations of differential geometry. I. London, 1963.
2. K. Yano, S. Ishihara. Tangent and cotangent bundles. New York, 1973.

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