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ON THE GEOMETRICAL DESCRIPTION OF SOME IDEALS

ANDREANA S. MADGUEROVA

The present article deals with the geometrical description of some ideals of algebras of complex-valued functions. A property of Max Noether's description of the polynomial ideals is given with a direct proof as remark.

Let G be an n-dimensional, connected, compact, C^∞ manifold. Let D^ν_G be the algebra of all complex-valued, $(c-v)$, functions f on G for which $f\circ\varphi^{-1}\in C^\nu_{\varphi(U)}$ for any local chart (U,φ) of G. Here $C^\nu_{\varphi(U)}$ is the space of all $c-v$ functions on $\varphi(U)$ with continuous partial derivatives up to order ν inclusive, $\nu=0, 1, \dots ; \infty$. Theorem. Each ideal $\mathcal{J}$ of D^ν_G is determined by the sets (1): $N_{\alpha(p)}=\{s: s\in G,\ Af(s)=0$ for $\forall f\in\mathcal{J},\ \forall A\in\alpha^{(p)}\},\ \forall p=(p_1,\dots,p_n)\in Z^n_+$, where $\alpha=\alpha(\mathcal{J})$ is a C-linear, differential-invariant space of linear differential operators on G of order not larger than ν, with "coefficients" whose moduli are upper semicontinuous; $\alpha^{(p)}=\{A^{(p)}: \forall A\in\alpha\}$. The space α is moreover finite-dimensionai if $\nu=N=0, 1, \dots,$ Almost inversely for each C-linear differential-invariant space α of linear differential operators on G of order not larger than μ, with "coefficients" in C^∞_G, there exists a closed ideal $\mathcal{J}$ in D^ν_G, with $\nu\geqq\mu$, which is determined by the sets (1).

Analogous results are true for the ring of the polynomials on R^n; for the algebras D^ν_K, where K is a compact, n-dimensional, connected, C^∞ manifold with a boundary: for the algebras $D(\alpha)$ of type C; for the corresponding algebras of real-valued functions.

Generalized strong A-derivatives, where A is an arbitrary linear differential operator on G with "coefficients" in C^∞_G, are introduced and are used in the present article (Lemma 4). Such A-derivatives are also of independent interest. The equation $Af=u$, where u is continuous, has a continuous solution f iff there exists the generalized strong derivative Af, equal to u.

The present article deals with a geometrical description of some ideals of algebras of complex-valued functions. A property of Max Noether's description of the polynomial ideals [1-3] is given as remark with a direct proof.

Preliminary. Let G be an n-dimensional, connected, C^∞ manifold. Let D^ν_G be the algebra of all complex-valued, $(c-v)$, functions f on G with compact supports for which $f\circ\varphi^{-1}\in C^\nu_{\varphi(U)}$ for any local chart (U,φ) of G. Here $C^\nu_{\varphi(U)}$ is the space of all $c-v$ functions on $\varphi(U)$ with continuous derivatives up to order ν inclusive; $\nu=0, 1, \dots; \infty$. $D^\nu_{R^n}$ will be denoted by D^ν and $C^\nu_{R^n}$ by C^ν. The algebras D^ν_G are examined with their usual topologies.

Definition 1. *A linear continuous map $A: D^\nu_G\to D^0_G$ is called a linear differential operator on G if for any given local chart (U,φ) of G the transfer $A\circ\varphi^{-1}$ is a linear differential operator on $\varphi(U)$ (see [4]). If moreover the coefficients of the operator $A\circ\varphi^{-1}$ have some property $\mathfrak{R}$ on $\varphi(U)$ for each local chart (U,φ) of G, then we shall say that the coefficients of the operator A have the same property $\mathfrak{R}$ on G; The operator derivative $A^{(p)}$, $p=(p_1,\dots,p_n)\in Z^n_+$, of A is called the linear differential operator on G whose transfer $A^{(p)}\circ\varphi^{-1}$ in every local chart (U,φ) of G is the p-operator derivative of the transfer of A in this chart, where the operator* *derivative* $B^{(p)}$ *of the linear differential operator* $B=\Sigma b_k D^k$ *on* R^n *is* $\Sigma\binom{k}{p}p!\,b_k D^{k-p}$ *with* $D^r=\partial^{|r|}/\partial x^r$, $x=(x_1,\ldots x_n)$, $r=(r_1,\ldots,r_n)$.

Definition 2. *The C-linear space* α *of linear differential operators on G is called differential-invariant if* $A\in\alpha$ *implies that all operator derivatives* $A^{(p)}$ *of the operator A also belong to* α, $p=(p_1,\ldots,p_n)$, $|p|=p_1+\cdots+p_n\geq 0$; *In the case* $G\subset R^n$, *if moreover all operators of* α *are with constant coefficients, then* α *is called still and homogeneous too.*

Definition 3. (*Cf.* [5]). *An ideal* $\mathcal{J}$ *of an algebra R is called a primary ideal of R if* $\mathcal{J}$ *is contained in an unique maximal ideal M of R. If R is an algebra of c—v functions on a set G and if M consists of all functions* $f\in R$ *with* $f(s_0)=0$ *for some fixed* $s_0\in G$, *then we shall denote M by* $M(s_0)$ *and* $\mathcal{J}$ *by* $\mathcal{J}(s_0)$ *and shall say that M is the maximal ideal at the point* s_0 *and* $\mathcal{J}$ *is a primary ideal at the point* s_0.

G. E. Shilov proves in [5] that if R is a regular Banach algebra of $c-v$ functions on a compact G without radical (i. e., the intersection of all maximal ideals of R consists only of the zero element of R), then there exists a minimal closed primary ideal $\mathcal{J}(s_0)$ at the points s_0 for any fixed $s_0\in G$. (An algebra R of $c-v$ functions on the topological space G is called regular if for each compact $F\subset G$ and each $s_0\in G$ with $s_0\notin F$ there exists a function $f\in R$ such that $f(s_0)\neq 0$ and $f(s)=0$ for $\forall s\in F$)

Formulation of the results.

Further let G be a compact connected n-dimensional C^∞ manifold.

Proposition 1. *Let* $\mathcal{C}$ *be one the spaces* D^ν_G, $\nu=0,1,\ldots;\infty$. *Each close primary ideal* $\mathcal{J}$ *of* $\mathcal{C}$ *is of the kind*

(1) $\mathcal{J}=\{f: f\in\mathcal{C},\ Af(s^*)=0 \text{ for } \forall A\in\alpha\}$ *for some fixed* $s^*=s^*(\mathcal{J})\in G$,

where α *is a G-linear differential-invariant space of linear differential operators on G of order not larger than* ν *with coefficients in* C^∞_G, *and* $I\in\alpha$, ($Ig=g$). *If* $\mathcal{J}$ *is contained in the maximal ideal* $M=M(s_0)$ *then* $s^*=s_0$; *The space* α *is moreover finite-dimensional if* $\nu=N=0,1,\ldots$. *Inversely, there exists a primary closed ideal* $\mathcal{J}$ *in* D^ν_G, *determined by* (*1*), *for each fixed point* $s^*\in G$ *and for each fixed C-linear differential-invariant space* α *of linear differential operators on G, with coefficients in* C^∞_G, *of order not larger than* μ, $\mu\leq\nu$, *and* $I\in\alpha$, *where* $Ig=g$.

Remark 1. Let K be an n-dimensional, connected C^∞ manifold with a boundary. Let D^ν_K be the algebra of all $c-v$ functions f with compact supports for which $f^*\circ\varphi^{*-1}\in C_{\varphi^*(U^*)}$ for any local chart (U,φ) of K. Here (U^*,φ^*) and f^* are the corresponding extensions (if necessary) of the chart (U,φ) of this manifold K with a boundary and of the function f on U^*, $\nu=0,1,\ldots;\infty$.

If K is moreover compact, then the results of Proposition 1 and Proposition 2 are literally extended, so that we can deem $\mathcal{C}$ equal to D^ν_K. The proofs need only some evident changes. In Proposition 1, the space α is moreover also homogeneous if $K\subset R^n$.

In the case $\nu=1$ and $\mathcal{C}=D^1_K$, $K\subset R^n$, the direct part of Proposition 1 is obtained by Shnol' [7]; In the case $n=1$, $\mathcal{C}=D^N_K$, $K\subset R^1$, the direct result of Proposition 1 is received by G. E. Shilov in [8]. The direct parts of Propositions 1 and 2 are very near to the Theorem of Whitney on ideals [9].

Proposition 2. *Each closed ideal* $\mathcal{J}$ *of* $\mathcal{C}$ *is determined by the sets*

(2) $N_{\alpha(p)}=\{s: s\in G,\ Af(s)=0 \text{ for } \forall f\in\mathcal{J},\ \forall A\in\alpha^{(p)}\}$, $\forall p=(p_1,\ldots,p_n)$, $p\geq 0$,

where $\alpha=\alpha(\mathcal{J})$ *is a C-linear differential-invariant space of linear differential operators on G of order not larger than* ν, *with "coefficients" whose absolute values are upper semicontinuous;* $\alpha^{(p)}=\{A^{(p)}: \forall A\in\alpha\}$. *The space* α *is moreover finite-dimensio-*

nal if $\nu = N, 0, 1 \ldots$. *Almost inversely, for each C-linear differential-invariant space* α *of linear differential operators on G, of order not larger than* μ, *with coefficients in* C_G^∞, *there exists a close ideal* $\mathcal{I}$ *in* $\mathcal{C}$ *with* $\nu \geqq \mu$, *which is determined by the sets (2).*

It follows from Propositions 1 and 2 that:

Corollary. Every closed ideal in $\mathcal{C}$ is an intersection of closed primary ideals.

In the case $K \subset R^n$, $\nu = 0$, Proposition 2 is the well known theorem of Stown; The direct part of Proposition 2 (without the property of upper semicontinuity), in the case $n=1$, $\nu = N$, $K = [a, b] \subset R^1$, is a theorem of G. E. Shilov [8]. Also confer [10].

Proposition 3. *Let* $n=1$. *Let* $M^0 \supset M^1 \supset \cdots \supset M^N$ *be closed sets in* R^1. *There exists a closed ideal* $\mathcal{I}$ *of* D^N *with*

$$M^j = \{x : D^s f(x) = 0 \text{ for } \forall s \leqq j, \forall f \in \mathcal{I}\}, \quad j = 0, 1, \ldots, N, \tag{3}$$

if and only if each boundary point a of the arbitrary fixed set M^{j_0} *belongs also to all other sets* M^j.

Definition 4. *Let* α *be a C-linear finite-dimensional differential-invariant space of linear differential operators on G with coefficients in* C_G^∞, *and* $I \in \alpha$, *where* $Ig = g$. *Let* $D(\alpha)$ *be the completion of* D_G^∞ *by the norm q,*

$$qf = \sum_{A \in \mathfrak{B}(\alpha)} \sup_{s \in G} |Af(s)|, \text{ where } \mathfrak{B}(\alpha) \text{ is a finite basis of } \alpha. \tag{4}$$

As was proved in [12], any such $D(\alpha)$ is an algebra of $c-v$ functions on G of type C up to a natural isomorphism γ which maps to every sequence $\{f_m\}$, with $f_m \in D_G^\infty$ and $\{Af_m\}$ — Cauchy sequences in the norm (4) on G for $\forall A \in \alpha$, maps: $\gamma(\{f_m\}) = \lim \{f_m\} = f$.

The functions f of $D(\alpha)$ have A-generalized strong derivatives Af of Laurent — Schwartz — Sobolev type for $\forall A \in \alpha$. We need the following Lemma 4 for the introduction of these derivatives:

Lemma 4. ([12]). *Let A be a linear differential operator on G with coefficients in* C_G^∞. *Let there exists such a sequence* $\{\varphi_m\}$, $\varphi_m \in D_G^\infty$, *for the function* $h \in D_G^0$ *that*

$$\{\varphi_m\} \xrightarrow{\text{uniformly on } G} h \quad and \quad \{A\varphi_m\} \xrightarrow{\text{uniformly on } G} H.$$

If we have

$$\{\psi_m\} \xrightarrow{\text{uniformly on } G} h \quad and \quad \{A\psi_m\} \xrightarrow{\text{uniformly on } G} M$$

for another sequence $\{\psi_m\}$, $\psi_m \in D_G^\infty$, *then it follows that* $H \equiv M$ *on G.*

The unique function $H \in D_G^0$ will be denoted by Ah and will be called a generalized strong A-derivative of the function h.

It follows from the construction of $D(\alpha)$ as a completion that each element f of $D(\alpha)$ is determined by a sequence (φ_m), $\varphi_m \in D_G^\infty$, with

$$(\varphi_m) \xrightarrow{\text{uniformly on } G} f, \quad (A\varphi_m) \xrightarrow{\text{uniformly on } G} g_A \text{ for } \forall A \in \alpha,$$

where the $c-v$ functions f, $g_A \in D_G^0$ for $\forall A \in \alpha$. Lemma 4 yields that each g_A is determined uniquely by the function f. Therefore this arbitrary element of $D(\alpha)$ can be treated as the $c-v$ function $f \in D_G^0$. Thus the elements of $D(\alpha)$ can be treated as all these functions $f \in D_G^0$ for which there exist sequences

$$(\varphi_m), \ \varphi_m \in D_G^\infty, \text{ with } (\varphi_m) \xrightarrow{\text{uniformly on } G} f, \quad (A\varphi_m) \xrightarrow{\text{uniformly on } G} g_A,$$

$\forall A \in \alpha$. The functions $g_A \in D_G^0$, uniquely determined by the function f, will be denoted by Af. $D(\alpha)$ will be examined with the norm q, (4).

Proof of Lemma 4. Let the linear differential operator B be the conjugate of the operator A, i. e., the linear differential operator $B: D_G^\infty \to D_G^0$ is such that its transfer $B \circ \varphi^{-1}$ is the conjugate operator of the transfer $A \circ \varphi^{-1}$ of A for any local chart (U, φ) of G, i. e., $\int [(A \circ \varphi^{-1}) \Phi] \Psi = \int \Phi (B \circ \varphi^{-1}) \Psi$ for each $\Phi, \Psi \in D_{\varphi(U)}^\infty$, ($D_{\varphi(U)}^\infty$ is the space of all infinitely differentiable $c-v$ functions with compact supports on $\varphi(U)$). Such an operator B on G exists, moreover, the coefficients of B are also in C_G^∞. Since

$$\int \{(A \circ \varphi^{-1}) [(\varphi_m - \psi_m) \circ \varphi^{-1}]\} \Phi = \int [(\varphi_m - \psi_m) \circ \varphi^{-1}] (B \circ \varphi^{-1}) \Phi \to 0$$

as $m \to \infty$ for $\forall \Phi \in D_{\varphi(U)}^\infty$, hence $\lim_m (A \circ \varphi^{-1})(\varphi_m \circ \varphi^{-1})(x) = \lim_m (A \circ \varphi^{-1})(\psi_m \circ \varphi^{-1})(x)$ at $\forall x \in \varphi(U)$ and on local chart (U, φ) of G. Therefore $H \equiv M$ on G.

Analogous results as Propositions 1, 2 are true in the algebras $D(\alpha)$:

Proposition 5. *Each closed primary ideal of the algebra $D(\alpha)$ is of the kind*

(5) $\mathcal{I} = \{f: f \in D(\alpha), Bf(s^*) = 0$ for some fixed $s^* = s^*(\mathcal{I}) \in G$ and $\forall B \in \beta\}$,

where β is a C-linear differential-invariant finite-dimensional subspace ot the space α with $I \in \beta$. If the ideal $\mathcal{I}$ is contained in the maximal ideal $M = M(s^0)$, then $s^* = s^0$. (Every maximal ideal M of $D(\alpha)$ is of the kind $M(s^0)$). Inversely, a closed primary ideal of $D(\alpha)$, determined by (5), corresponds to each such fixed subspace β and to each fixed $s^* \in G$.

Proposition 6. *Every closed ideal $\mathcal{I}$ in $D(\alpha)$ is determined by the sets*

(6) $N_{\beta(p)} = \{s: s \in G, Bf(s) = 0$ *for* $\forall f \in \mathcal{I}, \forall B \in \beta^{(p)}\}, \forall p, p \geqq 0$,

where *β is a $\mathcal{P}$-linear differential-invariant finite-dimensional subspace of α; $\beta^{(p)} = \{B^{(p)}$ $\forall B \in \beta\}$; $\mathcal{P}$ is the set of all $c-v$ functions g on G with $|g|$ upper semicontinuous. Almost inversely, if β is a finite-dimensional differential-invariant C-linear subspace of α, then there exists a closed ideal $\mathcal{I}$ in $D(\alpha)$ with determing sets $N_{\beta(p)}$ by (6).*

Corollary. *Each closed ideal in $D(\alpha)$ is an intersection of closed primary ideals.*

This corollary and the direct part of Proposition 5 are proved: I. by Shnol' in the case $n=1$, on intervals of R^1, when $D^1 \subset D(\alpha) \subset D^0$ in [13]; II. by Grushin in the case $n=2$, on the torus, when $D^1 \subset D(\alpha) \subset D^0$ in [14]; Confer also [10].

Proposition 7. Max Noether [1-3, 10]). *Each primary ideal $\mathcal{I}$ (i. e., a zero-dimensional ideal which is contained only in one maximal ideal, after the deinition here and in* [5-8, 10, 14]) *in* $C[x] = C[x_1, \ldots, x_n]$ *has the kind*

(7) $\mathcal{I} = \{p: p \in C[x], (Ap)(x^*) = 0$ *for some fixed* $x^* = x^*(\mathcal{I})$ *and for* $\forall A \in \alpha\}$,

where α is some C-linear finite-dimensional homogeneous differential-invariant space with $I \in \alpha$ (where $Ip = p$). Inversely, the set (7) is a zero-dimensional and moreover primary ideal in $C[x]$ for each such space α and each point $x^* \in R^n$. ($C[x]$ is as usual the set of all polynomials in $x \in R^n$ with coefficients in C).

So, only the differential-invariance of α and the inversity are new here.

It is well known that each ideal in $C[x]$ is an intersection of "algebraically primary" ideals (Theorem of Lasker [10, 2]). Each such "primary" ideal in $C[x]$ is represented by zero-dimensional ideals, hence by intersections of finite number of primary ideals. Thus E. Lasker [15], Gentzel [2, 3], V. Palamodov [3] — in a modified way, prove that the analogous result is true for each ideal in $C[x]$. Properties of the corresponding spaces α are new here, as well as the sets $N_{\alpha(p)}$.

Proposition 8. (Noether, Lasker, Gentzel, cf. [1-3, 15, 10]). *Each ideal* $\mathcal{J}$ *of* $C[x]$, $\mathcal{J} \neq C[x]$, *is determined by the analytic varieties*

(8) $N_{\alpha(p)} = \{x: Af(x)=0$ *for* $\forall f \in \mathcal{J}, \forall A \in \alpha^{(p)}\}$, $\forall p=(p_1, \ldots, p_n)$, $p \geqq 0$,

where $\alpha = \alpha(\mathcal{J})$ *is a C-linear, finite-dimensional differential-invariant space of linear differential operators with coefficients whose absolute values are upper semicontinuous.*

Let Φ be a space of $c-v$ functions on $S \subset R^n$, containing $C[x]$. Let $\mathcal{J}$ be an ideal in $C[x]$ with a basis of the polynomials $q_1, \ldots, q_m$. A polynomial submodul $\mathcal{J}\Phi$ corresponding to the ideal $\mathcal{J}$, is called the following subset of Φ: $\mathcal{J}\Phi = \{g: g=q_1 f_1 + \cdots + q_m f_m$ for $\forall f_1, \ldots, f_m \in \Phi\}$. It is almost evident that analogous results as Propositions 7 and 8 are true for the polynomial moduls $\mathcal{J}\Phi$ in many functional spaces Φ. For instance, Φ may be the space of the entire functions. It is well known that some problems of differential equations are reduced by Fourier — Laplace transform to multiplication by the corresponding polynomials of the eaxmined linear differential operators. That is why the new properties of differential-invariance and others introduce some supplements in the expressions and properties of the general kind of the solutions of some differential equations. (For instance, there are some supplements of the properties of the set of the differential operators $d^i_\mu(s, D)$ in [16]).

Remark 2. Let the sets $Q \subset R^n$, or $Q \subset G$, be scrutinized with the topologies, generated by the topology of R^n, respectively by the topology of G. Then the corresponding spaces α in Propositions 2 and 8 can be chosen so that on the sets $\{N_{\alpha(p^0)} - \bigcup_{p'<p^0} N_{\alpha(p')}\}$ the coefficients of the operator of α are continuous for the case of the closed ideals in $\mathscr{C}$ (Proposition 2, Remark 1) and are moreover rational functions in x in the case of the ideals in $C[x]$ (Proposition 8).

Proofs. The following lemmas will be used:

Lemma 9. $\mathscr{C}/\mathcal{J}(a) \cong K^\nu$, *where* $\mathcal{J}(a)$ *is the minimal closed primary ideal in the space* $\mathscr{C}$ *at the point* $a \in G$; $K^\nu = \begin{cases} \mathbf{C}[X]/m^{\nu+1} & if\ \nu=0, 1, \ldots \\ \mathbf{C}[[X]] & if\ \nu=\infty; \end{cases}$ $\mathbf{C}[X] = \mathbf{C}[X_1, \ldots, X_n]$ *is the formal algebra of multinomials over the field* $\mathbf{C}$; m *is its maximal ideals;* $\mathscr{C} = D^\nu_G$ (*or* D^ν_K *with* $a \in K$).

Proof. The algebras D^ν_G on G and D^ν on R^n are locally isomorphic at each point $a \in G$. Therefore $D^\nu_G/\mathcal{J}^\nu_G(a)$ is isomorphic to $D^\nu/\mathcal{J}^\nu(\varphi(a))$ and hence to $D^\nu/\mathcal{J}^\nu(0)$ for $\forall a \in G$, where $\mathcal{J}^\nu_G(a)$, $\mathcal{J}^\nu(\varphi(a))$, $\mathcal{J}^\nu(0)$ are correspondingly the minimal closed primary ideals at the point a, at the point $\varphi(a)$, ((U, φ) is any local chart of G with $a \in U$), at the point 0 of the algebras D^ν_G and D^ν correspondingly.

According to the Weierstrass's Theorem, the polynomials in $x=(x_1, \ldots, x_n)$ are everywhere dense set in D^ν on any compact. This implies that the images of the polynomials are dense in the canonical homomorphism $D^\nu \to D^\nu/\mathcal{J}^\nu(0)$. Since $\mathcal{J}^\nu(0) = \{f: f \in D^\nu, D^k f(0)=0, |k| \leq \nu\}$ (cf. for instance [10-12]), hence x^k, if $|k| > \nu = N = 0, 1, \ldots$, belongs of $\mathcal{J}^N(0)$. Let denote $D^\nu/\mathcal{J}^\nu(0)$ by $\mathfrak{R}^\nu$. Then the image of x^k, $|k| > N$, in $\mathfrak{R}^N$ is equal to the zero of $\mathfrak{R}^N$. Thus if $\nu = N$, $\mathfrak{R}^N$ is finite dimensional and $\mathfrak{R}^N$ is isomorphic to $K^N = \mathbf{C}[X]/m^{N+1}$. If $\nu=\infty$, evidently $\mathfrak{R}^\infty$ is isomorphic to $\mathbf{C}[[X]]$. (In the cases $\mathscr{C} = D^\nu_K$, the necessary changes of the proof are obvious.)

Lemma 10. *Let* $\mathcal{J} = \mathcal{J}(\alpha, x^*) = \{f: f \in \mathscr{C}, Af(x^*)=0$ *for* $\forall A \in \alpha\}$, *where* α *is a* $\mathbf{C}$*-linear space of all linear differential operators on* G *with this property (resp. on* K), $x^* \in G$ (*resp.* $x^* \in K$). *The set* $\mathcal{J}$ *is a closed ideal of* $\mathscr{C}$ *if and only if the space* α *is moreover and differential-invariant.*

Proof. Let $g \in \mathscr{C}$, $f \in \mathcal{J}$. We have

(9) $A(fg)\circ\varphi^{-1}=\sum_p D^p(g\circ\varphi^{-1}).\ (A\circ\varphi^{-1})^{(p)}(f\circ\varphi^{-1})$ in any local chart (U, φ) of G.

Thus, if α is differential-invariant, then $\mathscr{I}$ is an ideal, moreover $\mathscr{I}$ is closed. If $\mathscr{I}$ is a closed ideal, as $g\in\mathscr{C}$ are sufficiently many, then from (9) it follows that $(A\circ\varphi^{-1})^{(p)}.(f\circ\varphi^{-1})(\varphi(x^*))=0$ for $\forall f\in\mathscr{I}$, and $\forall p$, in any local chart (U, φ) of G with $x*\in U$. Thus $A^{(p)}f(x^*)=0$ for $\forall f\in\mathscr{I}$, $\forall p$. That is why α must be differential-invariant.

Proof of the Proposition 1. In $\mathscr{C}$ all maximal ideals M are of the kind $M=M(s^*)$ for some corresponding $s^*\in G$ (cf. [5, 12]). Let the primary ideal $\mathscr{I}$ be contained in the maximal ideal $M_0=M_0(s^*)$, $s^*\in G$.

Let the set $V\subset G$ and let g be a function on G (resp. let $\mathfrak{F}$ be a set of functions on G). As usual the restriction of g on V (resp. the set of the restrictions on V of all functions of $\mathfrak{F}$) will be denoted by $g|V$ (resp. by $\mathfrak{F}|V$). The algebras $\mathscr{C}$ and $D^\nu_{R^n}=D^\nu$ are locally isomorphic. Such are also the ideals $\mathscr{I}$ and $\mathscr{I}'(\varphi(s^*))=\mathscr{I}'$ for any local chart (U, φ) of G with $s^*\in U$, where $\mathscr{I}'=\{f\in D^\nu, f(\varphi(s))|U\in\mathscr{I}|U\}$. Evidently $\mathscr{I}'$ is a closed primary ideal of the algebra D^ν and $\mathscr{I}'=\mathscr{I}'$ $(\varphi(s^*))$. (Moreover without loss of the generality we may deem that $\mathscr{I}'$ belongs to the zero of R^n, i. e., $\mathscr{I}'=\mathscr{I}'(0)\subset M'(0)$.) Let I^* be the image of the ideal $\mathscr{I}$ in the canonical homomorphisms $\mathscr{C}\to\mathscr{C}/\mathscr{J}^\nu_G(s^*)\to K^\nu$, where $\mathscr{J}^\nu_G(s^*)$ is the minimal closed primary ideal of $\mathscr{C}$ at s^*. I^* is also the image of $\mathscr{I}'$ in the canonical homomorphisms $D^\nu\to D^\nu/\mathscr{J}^\nu(0)\to K^\nu$, where $\mathscr{J}^\nu(0)$ is the minimal closed primary ideal of D^ν at the point 0. The closure of the prototype of the ideal I^* in $\mathscr{C}$ containes $\mathscr{I}$. Let investigate the space $\mathscr{A}$ of all linear functionals on K^ν which annul on I^*. The correspondence $\mathscr{A}\leftrightarrows I^*$ is one-to-one. In $\mathscr{C}$ these are the linear continuous functionals concentrated at the point s^*, which are zero on $\mathscr{I}$, respectively in D^ν these are the linear continuous functionals on D^ν, concentrated at the point $\varphi(s^*)$, (without loss of the generality we may assume $\varphi(s^*)=0$), which are zero on $\mathscr{I}'$. It is well known (cf. [17-18]) that the general kind of the linear continuous functionals on D^ν, concentrated at the zero is $\sum_k a_k D^k_x\delta(x)$, where δ is the Dirac's function and $D^k_x=\partial^{|k|}/\partial x_1^{k_1}\ldots\partial x_n^{k_n}$. Let α' be the C-linear space of all linear constant-coefficient differential operators which are annulled on $\mathscr{I}'$ at $\varphi(s^*)$. Let α be the uniquely determined corresponding space of linear differential operators on G, (which transfer is α'). So we have received a correspondence $\mathscr{I}\longrightarrow\alpha$. Such a correspondence $\mathscr{I}\longrightarrow\alpha$ is received and more directly in [12] (see also the proof of Proposition 2). Also, as $\mathscr{I}=\mathscr{I}(s^*)$, hence the operator $I\in\alpha$.

Accordingly to Lemma 10, the space α is differential-invariant. If $\nu=N$, then evidently α is moreover and finite-dimensional.

Inversely, if α is a C-linear differential-invariant space of linear differential operators on G of order not larger than ν, with coefficients in C^∞_G and with $I\in\alpha$, then obviously the set $\mathscr{I}=\{f\in\mathscr{C}, Af(s^*)=0 \text{ for } \forall A\in\alpha\}$ is a closed primary ideal in $\mathscr{C}$ for each fixed $s^*\in G$.

Proof of Proposition 2. Let $\mathscr{I}$ be a closed ideal in $\mathscr{C}$. Let fix an arbitrary $s\in G$. And let us study the image I_s of the closed ideal $\mathscr{I}$ in the canonical homomorphisms ω^*_s: $\mathscr{C}\to\mathscr{C}/\mathscr{I}^\nu_G(s)\to K^\nu=K^\nu_s$ (see Lemma 9). I_s is a closed ideal after the general theory (cf. [5]). The closed ideal I_s of the algebra

$$K^\nu=K^\nu_s=\begin{cases}\mathsf{C}[{}_sX]/m^{N+1} & \text{if } \nu=N=0, 1,\ldots\\ \mathsf{C}[[{}_sX]] & \text{if } \nu=\infty\end{cases}$$

determines an unique C-linear space $\mathscr{A}_s$ of all linear functionals on $K^\nu=K^\nu_s$, annulling on I_s. (If $I_s=K^\nu_s$ then $\mathscr{A}_s=\{0\}$.) The correspondence $I_s\leftrightarrow\mathscr{A}_s$ is moreover one-to-one. Obviously the space $\mathscr{A}_s$ is finite-dimensional if $\nu=N=0, 1,\ldots$. Let (U, φ) be a local chart of G with $s \in U$. Let $\mathcal{I}'_s$ be the prototype in D^ν of I_s in the homomorphisms $D^\nu \to D^\nu/\mathcal{I}^\nu(\varphi(s)) \to K^\nu$. The space $\mathcal{A}_s$ uniquely determines a C-linear space α'_s of all constant-coefficients linear differential operators on R^n annulling on $\mathcal{I}'_s$ at the point $\varphi(s)$. This is the natural correspondence in which to X^k corresponds the differential operator $D^k = \partial^{|k|}/\partial x_1^{k_1} \dots \partial x_n^{k_n}$. The space α'_s uniquely determines a C-linear space α_s of linear differential operators on U, annulling on $\mathcal{I}$ at the point s, with coefficients in C^∞_U.

The set $\{\alpha_s\}_{s \in G}$ constructs at least one C linear space α of linear differential operators A on G, such that A at the point s is equal to some operator $B \in \alpha_s$. (Even in the case $K \subset R^n$, the coefficients of the operators of α may depend on s, as the ideals I_s depend on s.) If $\nu = N = 0, 1, \dots$, the space α is moreover finite-dimensional. As I_s, $\mathcal{I}'_s$, $\mathcal{I}$ are ideals, all spaces α_s must by moreover and differential-invariant. That is why α also must be differential-invariant. The space α determines uniquely the sets $N_{\alpha(p)}$. As the correspondences $I_s \leftrightarrow \mathcal{A}_s \leftrightarrow \alpha_s$ are one-to-one, and as to different ideals $\mathcal{I}^*$, $\mathcal{I}^{**}$ of $\mathcal{C}$ correspond different sets $\{I^*_s\}_{s \in G}$, $\{I^{**}_s\}_{s \in G}$ of ideals in $\{K^\nu_s\}_{s \in G}$, then the sets $\{N_{\alpha(p)}\}$, $\forall p$, $p \geq 0$, determine the ideal $\mathcal{I}$.

It remains to prove that the space α can be chosen so that "the coefficients" of its operators have the upper semicontinuous modules on G''. (For a manifold G, this signifies that for each local chart (U, φ) of G the coefficients $a_k(x)$ of the operator $A \circ \varphi^{-1}$, $\forall A \in \alpha$, have upper semicontinuous modules $|a_k(x)|$ on $\varphi(U)$.) Let F be the closed set $F = \{s : s \in G, f(s) = 0$ for $\forall f \in \mathcal{I}\}$, (classical result is that $F = \emptyset$ if and only if $\mathcal{I} = \mathcal{C}$). Evidently $A \equiv 0$ on $G - F$ for all $A \in \alpha$. For such operators it is sufficient to prove our assertion only on F. Further, let assume that $F \neq \emptyset$ (i. e., $\mathcal{I} \neq \mathcal{C}$). Obviously for each $s \in F$, the maximal ideal $M(s)$ at the point s contains the ideal $\mathcal{I}$. Let consider all closed primary ideals at the arbitrary fixed point $s \in F$ which contain $\mathcal{I}$. Their intersection $\mathcal{I}(s)$ is also a closed primary ideal at s which contains $\mathcal{I}$. (And $\mathcal{I}(s)$ is the least ideal with all these properties). We have $M(s) \supset \mathcal{I}(s) \supset \mathcal{I}^\nu_G(s)$, where $\mathcal{I}^\nu_G(s)$ is the minimal closed primary ideal of $\mathcal{C}$ at s. Let $\omega_s : \mathcal{C} \to \mathcal{C}/\mathcal{I}(s)$ be the corresponding canonical quotient homomorphism. Let $|.|_s$ be the quotient norm in $\mathcal{C}/\mathcal{I}(s)$. Let fix an arbitrary $f \in \mathcal{C}$ and examine the function $F(s) = \begin{cases} |\omega_s(f)|_s & \text{if } s \in F \\ 0 & \text{if } s \in G - F. \end{cases}$ We shall prove that $F(s)$ is upper semicontinuous on G. Since $F(s) \geq 0$, it is sufficient to prove it only on F. This is equivalent to prove that all sets $F_A = \{s : s \in F, F(s) \geq A\}$ are closed for $\forall A \in \mathrm{R}$. Let fix an arbitrary $A \in R$, and let the arbitrary sequence $\{s_\mu\} \to s_0$, $s_\mu \in F_A$. So $F(s_\mu) \geq A$. It is sufficient to prove that also $F(s_0) \geq A$: The definition of the quotient norm $|.|_s$ is $|\omega_s(f)|_s = \inf\{\|\varphi\|, \varphi \in \mathcal{C}, (\varphi - f) \in \mathcal{I}(s)\}$. We have $\mathcal{I}(s) \supset \mathcal{I}$, but it is true $|\omega_s(f)|_s = \inf\{\|\varphi\|, \varphi \in \mathcal{C}, (f - \varphi) \in \mathcal{I}(s), (f - \varphi)|U_{s,\varphi} \in \mathcal{I}|U_{s,\varphi}$, where $U_{s,\varphi}$ is an appropriate neighbourhood of $s\}$: Since $K^\nu_s = \begin{cases} \mathrm{C}[{}_sX]/m^{N+1} & \text{if } \nu = N, \\ \mathrm{C}[[{}_sX]] & \text{if } \nu = \infty, \end{cases}$ and since any ideal I_s of K^ν_s one-to-one determines a closed primary ideal $\mathcal{I}'(\varphi(s))$ of D^ν (resp. a closed primary ideal $\mathcal{I}^*(s)$ of $\mathcal{C}$) which primary ideal $\mathcal{I}'(\varphi(s))$ (resp. $\mathcal{I}^*(s)$) is the closure of the prototype of I_s in the maps $D^\nu \to D^\nu/\mathcal{I}^\nu(\varphi(s)) \to K^\nu$ (resp. $\mathcal{C} \to \mathcal{C}/\mathcal{I}^\nu_G(s) \to K^\nu$), hence Proposition 1 yields that the ideals $\mathcal{I}'(\varphi(s))$, $\mathcal{I}^*(s) = \mathcal{I}(s)$ are defined by some corresponding spaces $\alpha_{\varphi(s)}$, α_s and for each $f \in \mathcal{I}(s)$ we have $Af(s) = 0$, $\forall A \in \alpha_s$. The functions of $\mathcal{I}$ have the same property at the point s. As the differention is a local property and as the infinimum for $|.|_s$ can be taken only on functions φ, which are equal to zero out of a neighbourhood $U = U(\varphi, f)$ of s, then we can affirm that $|\omega_s(f)|_s = \inf\{\|\varphi\|, (f - \varphi) \in \mathcal{I}(s)\} = \inf\{\|\varphi\|, (f - \varphi) \in \mathcal{I}(s), (f - \varphi)|U_{s,\varphi} \in \mathcal{I}|U_{s,\varphi}$, where $U_{s,\varphi}$ is an appropriate neighbourhood of $s\}$.

Let scrutinize

$$F(s)=|\omega_s(f)|_s=\inf\{\|\varphi\|,\ (f-\varphi)\in\mathscr{I}(s),\ (f-\varphi)\,U_{s,\varphi}\in\mathscr{I}\,|U_{s,\varphi}\}.$$

Let $\varepsilon>0$ be an arbitrary fixed. So there exists a function φ^*, with $(f-\varphi^*)\in\mathscr{I}(s_0)$, $(f-\varphi^*)\,|U_{s_0,\varphi^*}\in\mathscr{I}\,|\,U_{s_0,\varphi^*}$, such that $F(s_0)>\|\varphi^*\|-\varepsilon$. For every sufficiently large μ, $(\mu>\mu_0)$ we have $s_\mu\in U_{s_0,\varphi^*}$. For such $\mu>\mu_0$: $F(s_\mu)=\inf\{\|\varphi\|,\ (f-\varphi)\in\mathscr{I}(s_\mu),\ (f-\varphi)|U_{s_{\mu},\varphi}\in\mathscr{I}\,|\,U_{s_\mu,\varphi}\}\leq\|\varphi^*\|$. Thus $F(s_0)>\|\varphi^*\|-\varepsilon\geqq F(s_\mu)-\varepsilon\geqq A-\varepsilon$. As $\varepsilon>0$ is arbitrary fixed, then $F(s_0)\geqq A$. This finishes the proof that the function $F(s)$ is upper semicontinuous.

Let (U,φ) be a local chart of G with $s\in U$. Let $\omega'_{\varphi(s)}$ be the canonical homomorphism $\omega'_{\varphi(s)}\colon D^\nu\to D^\nu/\mathscr{I}'(\varphi(s))\to K^\nu$. We get for $g=\sum_k\frac{(x-t)^k}{k!}\cdot D^k g(t)$ with $t=\varphi(s)$ that $\omega'_{\varphi(s)}(g)=\sum_{|k|\leq\nu}\frac{X^k}{k!}D^k g(t)$. Let assume for simplicity that $\nu=N$. Each ideal $I_s=I_{\varphi(s)}=I_t$ has a basis $\mathfrak{B}\,[I_t]=\sum_{Q<|k|\leqq N}B_k^j X^k$, $j=0,1,\dots,p$, where the elements $\sum_{Q<|k|\leqq N}B_k^j X^k$, $j=0$, $1,\dots,p$, with $B_k^0=0$, $Q\geqq0$ if $s\in F$, are linearly independent; B_k^j are constants in X but eventually not in t; $\varphi(s)=t$. Therefore we may determine $X^{k_j}=\sum_{\tau_s}A_k^j X^k \pmod{I_{\varphi(s)}}$ where $\tau_s=\{k=(k_1,\dots,k_n)\in Z_+^n\colon\ Q<|k|\leqq N,\ k\neq k_q,\ |k|\leqq|k_j|,\ q,j=0,1,\dots,p\}$; s is fixed; $A_k^0=0$, $A_k^j\in\mathbf{C}$, and if $|k|>|k_j|$ then $A_k^j=0$, $j=0,1,\dots,p$. Thus

$$K_{\varphi(s)}=K_s=K_t=K^N/I_{\varphi(s)}=\{\sum_{|k|\leqq N}a_kX^k,\ X^{k_j}=\sum_{\tau_s}A_k^j X^k,\quad j=0,1,\dots,p.$$

Let

$$K_t\ni\sum_{|k|\leqq N}a_kX^k=\sum_{|k|\leqq N,\ k\neq k_j}a_kX^k+\sum_{j=0}^{p}a_{k_j}\sum_{\substack{0\leqq|k|\leqq N\\ k\neq k_q;\ |k|\leqq|k_j|}}A_k^jX^k=\sum_{|k|\leqq Q}a_kX^k$$

$$+\sum_{\tau_s}(a_k+\sum_{j=0}^{p}a_{k_j}A_k^j)\,X^k,\quad q,\ j=0,1,\dots,p,$$

where if $|k|>|k_j|$ then $A_k^j=0$.

The norm in the finite-dimensional algebra K^N/I_t up to an equivalence is determined by each basis of the linear functionals on K^N, which are zero on I_t. The norm in K^N/I_t is determined uniquely, with precision up to equivalence by:

$$(11)\qquad |\sum_{|k|\leqq N}a_kX^k|_{\varphi(s)}=\sum_{|k|\leqq Q}b_k(t)\,|a_k|+\sum_{\tau_s}d_k(t)\,|a_k|+\sum_{j=0}^{p'}a_{k_j}A_k^j(t)\,|,$$

where $A_k^j(t)=0$ if $|k|>|k_j|$; $t=\varphi(s)=(t_1,\dots,t_n)$; $b_k(t)>0$; $d_k(t)>0$. This implies for $P(x)=\sum_k\frac{(x-t)^k}{k!}(D^kP_*)(t)$ that

$$|\omega'_t(P)|_t=\sum_{|k|\leqq Q}b_k(t)\,|D^kP_*(t)|\,/k!+\sum_{\tau_s}d_k(t)\,|\frac{D^kP(t)}{k!}+\sum_{j=0}^{p}A_k^j(t)D^{k_j}P_*(t)/k_j!\,|.$$

The ideal I_t is the canonical image of the ideal $\mathscr{I}'(t)\subset D^N$ in $D^N\to D^N/\mathscr{I}^N(t)\to K^N$ at the point $t=\varphi(s)$. Then the coefficients $k!\,B_k^j$ of its basis are k-derivatives at the point t of some functions in t. The coefficients $A_k^j(t)$ are rational functions in these derivatives, which are continuous in t. That is why: 1. If we can determine $X^{k_1},\dots,X^{k_p}$ by the others at the point t_0, then we can do so and moreover in an intersection

$F\cap U'_\theta$ of a (hyper) ball-neighbourhood U'_0 of t_0; 2. In U'_0 all A^j_k are continuous functions in t; 3. Since $|\omega'_t(f)|_t=\inf\{\|\psi\|, (f-\psi)\in\mathscr{I}'(t)\}$ and since the norm $\|.\|$ is the usual norm in D^N, and since $B^j_k(t)=0$ if $|k|\leqq Q$, then all $b_k(t)$, $|k|\leqq Q$, are nonzero constants in t. So moreover we can deem $b_k(t)=1$, $|k|\leqq Q$. 4. If all derivatives $k!\,B^j_k(t)$, $Q<|k|\leqq N$, and all $A^j_k(t)$, $Q<|k|\leqq N$, $k\neq k_q$, $|k|\leqq|k_j|$, q, $j=0, 1,\dots, p$, are not annulling at any point of U'_0, then $d_k(t)$ are also nonzero constants in U'_0. But this eventually is not true when some of the indicated B^j_k or A^j_k is zero at some point of U'_0. That is why we shall prove: (i). Let $k\neq k_q$, $Q<|k|\leqq N$. There exists a neighbourhood U^0 of t_0, $U^0\subset U'_0$, such that all $d_k(t)$ are upper semicontinuous on U_0; (ii). All $d_k(t)\,|A^j_k(t)|$ are upper semicontinuous on U^0, as immediately follows from (i) and from 2; (iii). Let S be a compact in $\varphi(U)$. Hence there exists a finite number of points $t_1,\dots,t_r$ in S, such that $\bigcup_m^r U_m\supset S$, where U_m is a neighbourhood of t_m corresponding to U^0. Then (11) determines a C-linear finite-dimensional space α_S of linear differential operators on S, which α_S is generated by the operators: $\{D^k, \forall k$ with $|k|\leqq Q$; $B_l=\sum_{Q<|k|\leqq N} B_{lk}D^k$, $l=1,\dots,r\}$; B_{lk} are received by the operators $M_k=(D^k/k!+\Sigma_j A^j_k(t)\,D^{k_j}/k_j!)$ and by a C^∞ decomposition of the unity, submised to $U_1,\dots,U_r$; That is why the "coefficients" $|B_{lk}|$ are upper semicontinuous.

It remains to prove (i) by Induction: Let $|k'|=Q+1$, $k'\neq k_j$. Let put in (11) $P(x)=(x-t_0)^{k'}$. Then $|\omega'_{t_0}(P(x))|_{t_0}=\sum_{|k|\leqq Q}(D^kP)(t_0)+d_{k'}(t_0)$. Since $|\omega'_t(P)|_t$ is upper semicontinuous function in t, hence $d_{k'}(t)$ is also upper semicontinuous in t at least on a neighbourhood U^*_0 of t_0, $U^*_0\subset U'_0$. The intersection of all U^*_0 for $\forall k'$ with $|k'|=Q+1$, $k'\neq k_j$, is also a neighbourhood U^{**}_0 of t_0 in which all d_k, $|k|=Q+1$, $k\neq k_j$, are upper semicontinuous in t. Analogously, by inductive arguments, all $d_k(t)$, $Q<|k|\leqq N$, $k\neq k_j$, are upper semicontinuous on a ball-neighbourhood $U^0\subset U'_0$.

The case $\nu=\infty$: We can make the above construction for the closure $\mathscr{I}_N$ in the norm of D^N_G of the ideal $(\mathscr{I}\cap D^N_G)$ in D^N_G for $N=0, 1,\dots,$. So we can receive the corresponding C-linear, finite dimensional differential-invariant spaces $\alpha_0\subset\alpha_1\subset\dots\subset\alpha_N$ of linear differential operators on G with coefficients which absolute values are upper semicontinuous. The space $\alpha=\bigcup_N\alpha_N$ is with the required properties.

Remark. There are also other properties of $d_k(t)$ with which we may call a function $d_k(t)$ "accessible". So inversely,. if α is a C-linear differential-invariant space of linear differential operators on G of order not larger than ν, such that in any local chart (U, φ) of G the coefficients of their transfers are of the form of (11), where $d_k(t)$ are "accessible" functions and A^j_k are continuous and continuously differentiable till order $\nu-k$, inclusive, in any sufficiently small "ball" $U^*\subset\varphi(U)$, then there is an ideal closed $\mathscr{I}$ in D^ν_G, determined by (2).

Inversely, let α be a differential-invariant finite-dimensional C-linear space of linear differential operators on G of order not larger than ν with coefficients in C^∞_G. Then there exists (cf. [12]) an algebra R of $c-v$ functions on G of type C, which R is a completion of D^∞_G by the norm (4) and moreover R is with the norm (4). Let denote the minimal closed primary ideal of R at the point $s\in G$ by ${}_R\mathscr{I}(s)$ (such an ideal exists according to the general theory (cf. [5]). Let denote by $\mathscr{I}_s$ the intersection ${}_R\mathscr{I}(s)\cap\mathscr{C}$. After the construction of R (cf. [12]). $\mathscr{I}_s=\{f\colon f\in\mathscr{C}, Af(s)=0, \forall A\in\alpha\}$. Let denote by $\mathscr{I}$ the closed ideal in $\mathscr{C}$ equal to $\bigcap_{s\in G}\mathscr{I}_s$. After the construction of R, and as every function of $\mathscr{C}$ which belongs locally to some closed ideal, belongs to it ([5]) (or after Theorem of Whithey of ideals), we have, that the ideal $\mathcal{I}$ is determined by the sets $N_{\alpha(p)}$, $\forall p \in Z^n_+$.

If α is not finite-dimensional, then we must construct a projective limit R^* of algebras of type C (cf. [12]) and by R^* also can be proved the existence of a closed ideal $\mathcal{I}$ in $\mathcal{C}$, determined by the sets $N_{\alpha(p)}$, $\forall p$.

Proof of the Proposition 3. The necessity is obtained immediately by the Roll's theorem. To prove the sufficiency, let investigate all closed ideals I of D^ν for which its corresponding sets

(12) $$\mathcal{M}_s = \{x: x \in R^1, f^{(r)}(x)=0 \text{ for } \forall f \in I, \forall r \leqq s\} \text{ contain } M_s.$$

Obviously there exist such closed ideals I. The intersection of all these ideals I let denote by $\mathcal{I}$. We shall show that this closed ideal $\mathcal{I}$ satisfies the requirements of Proposition 3. Let $\mathcal{M}^s = \{x: x \in R^1$ and $f^{(r)}(x)=0$ for $\forall f \in \mathcal{I}, \forall r \leqq s\}$. Evidently $M^0 = \mathcal{M}^0$ and $M^s \subset \mathcal{M}^s$ for $s=1,\dots,N$. We want to prove moreover that $M^s = \mathcal{M}^s$ for all $s=1, \dots, N$. Suppose the contrary — that for some s, $0 < s \leqq N$, there exists a point $a \in \mathcal{M}^s$ with $a \notin M^s$. So, $a \in \mathcal{M}^s \subset \mathcal{M}^{s-1} \subset \cdots \subset \mathcal{M}^0 = \mathcal{M}^0$. Let j be the largest integer such that $a \in M^j$ and $a \notin M^{j+1}$. From the data (requirements) of Proposition 3 it follows that such a point must be an isolated point for all M^k, $k=0, 1, \dots, j$. Then evidently we can construct such an ideal I_0 with the property (12), that does not belong to the set $\mathcal{M}^{j+1}_0$ corresponding to this ideal I_0. That is why the point cannot belong to the sets $\mathcal{M}^{j+1} \supset \cdots \supset \mathcal{M}^s$. The obtained contradiction proves that $\mathcal{M}^s = M^s$, $s=1,\dots,N$.

(Another proof of the sufficiency can be received by a more algebraical way, or also by a decomposition of the unity, constructing a function F for which $M^s = \{x: x \in R^1, F^{(r)}(x)=0$ for $\forall r \leqq s\}$, $s=0, 1, \dots, N$.)

Also it is clear that the sufficiency of the requirements of Proposition 3 can be extended not only for the dimension $n=1$ (exiging for the closed set $N_{\alpha(p)}$ (see (2)) 1. $N_{\alpha(p')} \supset N_{\alpha(p'')}$ if $p' \leqq p''$, $\forall p, p', p'' \in Z^n_+$, and 2. if $a \in N_{\alpha(p^0)}$, then $a \in N_{\alpha(p)}$ for $\forall p$ which is compared with p^0 (i. e., either $p^0 \leqq p$ or $p \leqq p^0$.))

Remark. Since the algebra $D(\alpha)$ is examined with the norm q, (4), hence further we can and we shall consider as a maximal such a space, i. e., containing all linear differential operators A on G with coefficients in C^∞_G, which A are dominated by α in the suprem norm (see [19, 20]).

Proof of Proposition 5. According to the definition, every primary ideal $\mathcal{I}$ of $D(\alpha)$ is contained in some maximal ideal M of $D(\alpha)$. Since the algebra D^∞_G is dense in $D(\alpha)$ and since the maximal ideals of D^∞_G are of the kind $M(s)$, hence (cf. [5]) such are also and the maximal ideals in $D(\alpha)$. Then $\mathcal{I} = \mathcal{I}(s)$ for some $s \in G$. Let $\mathcal{J}(s)$ be the minimal closed primary ideal of the algebra $D(\alpha)$ at the point $s \in G$. (Such an ideal $\mathcal{J}(s)$ exists since $D(\alpha)$ is an algebra of $c-v$ functions of type C, i. e., also a regular Banach algebra (cf. [5]). Let N (resp. Q) be the least (resp. the largest) integer for which $D^N_G \subset D(\alpha) \subset D^Q_G$. Such N (resp. Q) exists as this inclusion is true when N is the largest order of the operators in α (resp. for $Q=0$ since $I \in \alpha$. Let $\mathcal{J}^N_G(s)$, $\mathcal{J}^Q_G(s)$ be the minimal closed primary ideals at the point $s \in G$ of the algebras D^N_G and D^Q_G correspondingly. Further we shall use.

Theorem 11. (G. E. Shilov [5]). *Let $R_1 \subset R_2$ be two regular Banach algebras without radicals and with a same space G of their maximal ideals. Then there exists a continuous algebraic homomorphism s for which the following diagram is commutative*

$$\begin{array}{ccc} R_1 & \xrightarrow{\theta} & R_2 \\ \gamma_1 \downarrow & & \downarrow \gamma_2 \\ R_1/\mathcal{J}_1(s_0) & \xrightarrow{s} & R_2/\mathcal{J}_2(s_0) \end{array}$$

where $\mathcal{J}_1(s_0)$ and $\mathcal{J}_2(s_0)$ are the minimal closed primary ideals of R_1 and R_2, respectively, at the arbitrary fixed point $s_0 \in G$; γ_i, $i=1, 2$, are the canonical homomorphisms; θ is the including map.

Applying this theorem at first to $R_1 = D(\alpha)$, $R_2 = D_G^Q$, and afterwards to $R_1 = D_G^N$, $R_2 = D(\alpha)$ we receive continuous algebraic homomorphisms s_1 and s_2 such that the following diagram is commutative

$$\begin{array}{ccccc} D_G^N & \xrightarrow{\theta_1} & D(\alpha) & \xrightarrow{\theta_2} & D_G^Q \\ \gamma_1 \downarrow & & \gamma \downarrow & & \downarrow \gamma_2 \\ K^N \cong D_G^N/\mathcal{J}_G^N(s_0) & \xrightarrow{s_1} & D(\alpha)/\mathcal{J}(s_0) & \xrightarrow{s_2} & D_G^Q/\mathcal{J}_G^Q(s_0) \cong K^Q. \end{array}$$

The composition of these homomorphisms s_1 and s_2 is the natural projection (inclusion) as follows from the commutativity of the diagram in Shilov's theorem 11. Thus $D(\alpha)/\mathcal{J}(s_0) \cong K^N/J_{s_0}$, where J_{s_0} is an ideal in K^N, generated by a finite number of elements of the form $\sum_{|k| \leq Q}^{N} a_k X^k$. Let $\mathcal{J}^*(s_0)$ be the closed primary ideal at the point s_0 in D_G^N which is the prototype of $\mathcal{J}(s_0)$ in the natural including homomorphism $D_G^N \to D(\alpha)$; Let $I_{s_0}^*$ be the image of $\mathcal{J}^*(s_0)$ in the canonical homomorphism $D_G^N \to D_G^N / \mathcal{J}_G^N(s_0)$.

As $\mathcal{J}(s_0) \supset \mathcal{J}(s_0)$, then $I_{s_0}^* \supset J_{s_0}$. Let fix an arbitrary local chart (U, φ) of G with $s_0 \in U$. The ideal $I_{s_0}^*$ uniquely determines the C-linear differential-invariant finite-dimensional space $\beta_{s_0}^{**}$ of linear constant-coefficient differential operators such that the ideal $\mathcal{J}^{**}(\varphi(s_0))$ in D^N, $\mathcal{J}^{**}(\varphi(s_0)) = \{g\colon g \in D^N,\ g \mid \varphi(U) = (f \mid U) \circ \varphi^{-1}$ for some $f \in \mathcal{J}_{s_0}^*\}$, is determined by the space $\beta_{s_0}^{**}$ as in Proposition 1, (1). Let $\alpha_{s_0}^{**}$ be the C-linear differential-invariant finite-dimensional space of linear constant-coefficient differential operators A^* on R^n, such that for every $A^* \in \alpha_{s_0}^{**}$ there exists $A \in \alpha$ with $A \circ \varphi^{-1}$ at the point $\varphi(s_0)$ is equal to A^*. According to [12] $\alpha_{s_0}^{**}$ is uniquely determined by the ideal J_{s_0}, as the corresponding prototype of the space $\mathcal{A}_{s_0}$ of all linear functionals on K^N, annulling on J_{s_0}. Since $J_{s_0} \subset I_{s_0}^*$, hence $\beta_{s_0}^{**}$ is a subspace of $\alpha_{s_0}^{**}$. The space $\beta_{s_0}^{**}$ (as resp. the space $\alpha_{s_0}^{**}$) uniquely determines a C-linear differential-invariant finite-dimensional space $\beta_{s_0}^*$ (resp. $\alpha_{s_0}^*$) of linear differential operators on G with coefficients in C_G^∞. Evidently $\beta_{s_0}^*$ is a subspace of $\alpha_{s_0}^*$. Moreover, $\beta_{s_0}^*$ is such that $\mathcal{J}^*(s_0)$ is determined by $\beta_{s_0}^*$, i. e. $\mathcal{J}^*(s_0) = \{f\colon f \in D_G^N,\ Bf(s_0) = 0$ for $\forall B \in \beta_{s_0}^*\}$. Since the space α is pointwise equal to the spaces α_s^* (α is constructed by α_s^* — cf. [12]), hence there exists a C-linear differential-invariant subspace β_{s_0} of the space α which is "equal" at the point s_0 to $\beta_{s_0}^*$. As $D(\alpha)$ is a completion of $D_G^\infty \subset D_G^N$ by the norm q (4) and as the ideal $\mathcal{J}^*(s_0)$ is determined by the space β_{s_0} such must be also the ideal $\mathcal{J}$ and we receive (5). Thus

$$\mathcal{J}(s_0) = \{f\colon f \in D(\alpha),\ Bf(s_0) = 0 \text{ for } \forall B \in \beta_{s_0}\} \text{ and also}$$

$$\mathcal{J}(s_0) = \{f\colon f \in D(\alpha),\ Af(s_0) = 0 \text{ for } \forall A \in \alpha\}.$$

The inverse part of Proposition 5 is almost evident: each such space β uniquely determines an ideal $\mathscr{I}^*(s_0)$ of D_G^N after Proposition 1 and as for the corresponding to $\mathscr{I}^*(s_0)$ and to $\mathscr{J}(s_0)$ ideals in K^N, is true $I^*_{s_0} \supset J_{s_0}$ then the set $\mathscr{I}(s_0)=\{f: f\in D(\alpha), Bf(s_0)=0$ for $\forall B\in\beta\}$ is a closed primary ideal of the algebra $D(\alpha)$ at the point s_0.

The proof of the Proposition 6 will use the construction and the notations in the proof of Proposition 5 for the arbitrary fixed point $s\in G$. Let N be the largest order of the operators in α. Let $\mathscr{I}^*=D_G^N\cap\mathscr{I}$. Evidently $\mathscr{I}^*$ is a closed ideal in D_G^N. After the Proposition 2 the ideal $\mathscr{I}^*$ is determined by the sets $N_{\beta(p)}$, $\forall p$, for some C-linear differential-invariant finite-dimensional space β of linear differential operators on G with coefficients which absolute values are upper semicontinuous on G. For each point $s\in G$ we can construct the spaces $\beta_s'^*$ and β_s^* corresponding to the ideals $\mathscr{I}^*$ and $\mathscr{I}$ at the point s. Evidently $\beta_s'^*=\beta_s^*$. As $\mathscr{I}$ is an ideal of $D(\alpha)$ then β_s^* is a subspace of α_s^*, where α_s^* is a C-linear differential-invariant finite-dimensional space of linear differential operators on G with coefficients in C_G^∞, constructed by the ideals $\mathscr{J}(s)$ and J_s (see the former proofs), or by the space α (see [12]). Nevertheless for each $A_s^*\in\alpha_s^*$ there exists an operator $A_s\in\alpha$ equal to A_s^* at the point s. Then if $\mathscr{P}$ is the space of all $c-v$ functions with upper semicontinuous absolute values on G, the space β is a $\mathscr{P}$-linear subspace of the space α. Since $D(\alpha)$ is a completion of $D_G^\infty\subset D_G^N$ by the norm q (4), and $\mathscr{I}^*=D_G^N\cap\mathscr{I}$, hence the ideal $\mathscr{I}$ is also determined by the same sets $N_{\beta(p)}$, $\forall p$.

The inverse part of Proposition 6 is a consequence of the existence of a closed ideal $\mathscr{I}^*$ of D_G^N, determined by β after the Proposition 2. The sought for the ideal $\mathscr{I}$ of $D(\alpha)$ is the completion of $\mathscr{I}^*$ by the norm q (4).

Proof of Proposition 7. All maximal ideals in $C[x]$ are determined by the points of R^n, i. e., I. each maximal ideal M of $C[x]$ is of the kind $M=M(x^*)$ for some $x^*\in\mathsf{R}^n$ (cf. [2], which may be proved also directly); II. for each point $x^*\in\mathsf{R}^n$ the set $M(x^*)=\{p,\ p\in C[x],\ p(x^*)=0\}$ is the maximal ideal at x^*. (The latter is evident, since there exist polynomials annulling at and only at x^*, for instance $p=\Sigma_{i=1}^n(x_i-x_i)^2$). Let the primary ideal $\mathscr{I}$ belong to the maximal ideal $M=M(x^*)$. Then $\mathscr{I}=\mathscr{I}(x^*)$. The ideal $\mathscr{I}$ is finitely generated after the Hilberts' theorem. Let $\mathscr{I}$ be generated by the polynomials $p_1,\ldots,p_m$. (So $p_j(x^*)=0$, $j=1,\ldots,m$.) Let N be the largest degree of $p_1,\ldots,p_m$.

As $M\supset\mathscr{I}\supset M^{N+1}$ then the natural homomorphisms $C[x]/M\to C[x]/\mathscr{I}\to K^N=C[x]/M^{N+1}$ imply that there exists an ideal I^* in K^N — the image of $\mathscr{I}$ in canonical homomorphism $C[x]\to C[x]/M^{N+1}$, that $C[x]/\mathscr{I}=K^N/I^*$. Let $\mathscr{A}_{x^*}$ be the unique C-linear finite-dimensional space of all linear functionals on K^N, annulling on I^*. This space $\mathscr{A}_{x^*}$ uniquely generates the C-linear finite-dimensional space $\alpha=\alpha(\mathscr{I})$ of all linear constant-coefficient differential operators, annuling on $\mathscr{I}$ at the point x^*. α is the prototype of $\mathscr{A}_{x^*}$, where formally to X^k corresponds D^k. So the order of operators in α is not larger than N. This space α is moreover differential-invariant. Let $A\in\alpha$. It is sufficient to prove that if $A=\Sigma\, a_k D^k$ with some of $k_1>0$ for corresponding $a_k\neq0$, $k=(k_1,\ldots,k_n)$, then the operator $A^{(1,0,\ldots,0)}\in\alpha$. Let f, $g\in C[x]$, and $f\in\mathscr{I}$. We have $A(fg)=\Sigma\frac{1}{r!}g^{(r)}A^{(r)}f$. For $g=(x_1-x_1^*)$ we have $0=A(fg)(x^*)=A^{(1,0,\ldots,0)}f(x^*)=0$, i. e., $A^{(1,0,\ldots,0)}\in\alpha$. Then inductively all $A^{(r)}\in\alpha$, $\forall x$. Moreover, the operator $I\in\alpha$ since $\mathscr{I}=\mathscr{I}(x^*)$.

Inversely, each fixed point $x^*\in R^n$ and each C-linear finite-dimensional differential-invariant space α of linear constant-coefficient differential operators with $I\in\alpha$, determine a primary ideal $\mathscr{I}=\mathscr{I}(x^*)$ in $C[x]$: $\mathscr{I}(x^*)=\{p: p\in C[x], Ap(x^*)=0$ for $\forall A\in\alpha\}$. Let N be the largest order of operators in α. α uniquely determines the corresponding space $\mathscr{A}$ of linear functionals on K^N. Let I^* be the ideal in K^N on which all functionals of $\mathscr{A}$ are annulling. Then $\mathscr{I}(x^*)$ is the prototype of I^* in the canonical homo morphism $C[x]\to K^N=C[x]/(M(x^*))^{N+1}$. $\mathscr{I}(x^*)$ is an ideal as the space α is differential-invariant.

P r o o f o f P r o p o s i t i o n 8. Let F be the analytic variety of all zeros of the ideal $\mathscr{I}$. If $F=\emptyset$ then $\mathscr{I}=C[x]$, and the space $\alpha=\{0\}$. Let now $F=\emptyset$ and $x^*\in F$. For the arbitrary fixed $x^*\in F$, let the ideal J_{x^*} be the image of J in the canonical homomophism $C[x]\to C[x]/M_*^{N+1}$, where N is the largest degree of the polynomials in a finite basis of $\mathscr{I}$ (such a finite basis exists after the Hilbert's Theorem.); $M_*=M(x^*)$ is the maximal ideal of $C[x]$ at the point x^*. At first we receive the C-linear finite-dimensional space $\mathscr{A}_{x^*}$ of all linear functionals on K^N annulling on J_{x^*}. $\mathscr{A}_{x^*}$ uniquely determines the C-linear finite-dimensional space α_{x^*} of all linear constant-coefficient differential operators on R^n annulling on $\mathscr{I}$ at the point x^*. Moreover, since $\mathscr{I}$ is an ideal, hence the space α_{x^*} is also differential-invariant (cf. the proof of Proposition 7). Let if $x^*\notin F$ $\alpha_{x^*}=\{0\}$. By the set of the spaces $\{\alpha_x\}_{x\in R^n}$ we can construct a C-linear finite-dimensional differential-invariant space α of linear differential operators on R^n in a way that if $A\in\alpha$ then at each fixed $x^*\in R^n$ there exists an operator $A_{x^*}\in\alpha_{x^*}$ such that A_{x^*} is equal to A at x^*.

Furthermore, α may be so constructed that the coefficients of each operator in α to be with upper semicontinuous absolute values. The proof almost literally follows the corresponding part of the proof of Proposition 2: Let $x^*\in F$ be arbitrary fixed. We have $M(x^*)\supset\mathscr{I}$. Let $\mathscr{I}(x^*)$ be the least primary ideal at x^* with $\mathscr{I}(x^*)\supset\mathscr{I}$. ($\mathscr{I}(x^*)$ is the intersection of all primary ideals at x^* which contain $\mathscr{I}$). Let $\omega_{x^*}: C[x]\to C[x]/\mathscr{I}(x^*)$ be the canonical homomorphism. Let fix arbitrary $p\in C[x]$. Let $F(x^*)=|\omega_{x^*}(p)|_{x^*}$, where $|\cdot|_{x^*}$ is the factor norm in K^N ($\mathscr{I}(x^*)\supset M^{N+1}$), determined by the norm Q in $C[x]$: $Q(\sum_{k=0}^{m} a_k x^k)=\sum_{k=0} k!\,|a_k|$. In the same way as in Proposition 2 we see that the function $F(x^*)$ is upper semicontinuous in the indetermine x^*. Further, repeating the proof of Proposition 2 we can see that α can be chosen so that the absolute values of the coefficients of all operators in α are upper semicontinuous on R^n.

So: $F(x)=|\omega_x(p)|_x=\inf\{\|q\|,\ (q-p)\in\mathscr{I}(x)\}$. Let $x=x^*$ be fixed. Every ideal I_{x^*} of $K^N=K^N_{x^*}$ one-to-one determines a primary ideal $\mathscr{I}'(x^*)$ of $C[x]$, which primary ideal $\mathscr{I}'(x^*)$ is the prototype of I_{x^*} in the canonical map $C[x]\to C[x]/M_{x^*}^{N+1}$. Proposition 7 yields that the ideals $\mathscr{I}'(x^*)$ and $\mathscr{I}(x^*)$ are determined by the corresponding spaces α'_{x^*} and $\alpha_{x^*}=\alpha'_{x^*}$, moreover for each $p\in\mathscr{I}'(x^*)$ or $p\in\mathscr{I}(x^*)$ we have $Ap(x^*)=0$, $\forall A\in\Re'_{x^*}$ (resp. $\forall A\in\alpha_{x^*}$). The polynomials of $\mathscr{I}$ have the same property at x^*, is i. e. the coefficients of Taylor's formula at x^* for polynomials of $\mathscr{I}'(x^*)$ and of $\mathscr{I}$ satisfy the same requirements. The norm in $C[x]/M_{x^*}^{N+1}$ is determined by the coefficients of Taylor's formula at x^*. But these coefficients are determined by the local means of the polynomials. Hence we can affirm that $F(x)=|\omega_x(p)|_x=\inf\{\|q\|,\ (p-q)\in\mathscr{I}(x)\}=\inf\{\|q\|,\ (p-q)\in\mathscr{I}(x),\ (p-q)\,|\,U_{x,q},\in\mathscr{I}\,|\,U_{x,q}$ where $U_{x,q}$ is an appropriate neighbourhood of $x\}$. Let scrutinize

$$F(x)=\begin{cases}|\omega_x(p)|_x, & x\in F,\\ 0 \text{ if } x\in R^n-F\end{cases}$$

where F is the analytic set of all mutual zeros of the polynomials of $\mathscr{I}$. We shall prove that $F(x)$ is upper semicontinuous. As $F(x)\geq 0$, it is sufficient to prove that on F. Let $A\in R$ be arbitrary fixed. Let $x^*\in F$ be also arbitrary fixed and (x_μ) be a sequence with $x_\mu\in F$, $F(x_\mu)\geq A$, and $(x_\mu)\to x^*$. It is sufficient for the upper semicontinuity of $F(x)$ at the point x^* to prove that $F(x^*)\geq A$. Let $\varepsilon>0$ be arbitrary fixed. Then there exists a polynomial q^* with $(q^*-p)\in\mathscr{I}(x^*)$, $(q^*-p)\,|\,U_{q^*,x^*}\in\mathscr{I}\,|\,U_{q^*,x^*}$, such that $F(x^*)>\|q^*\|-\varepsilon$. We have $x_\mu\in U_{q^*,x^*}$ for every sufficiently large μ, $(\mu>\mu_0)$. For such $\mu>\mu_0$: $F(x_\mu)=\inf\{\|q\|,\ (p-q)\in\mathcal{I}(x_\mu),\ (p-q)\,|\,U_{q,x_\mu}\in\mathcal{I}\,|\,U_{q,x_\mu}\}\leqq\|q^*\|$. **Thus** $F(x^*)>\|q^*\|-\varepsilon\geqq F(x_\mu)-\varepsilon\geqq A-\varepsilon$. As $\varepsilon>0$ is arbitrary fixed, then $F(x^*)\geqq A$. This finishes the proof that the function $F(x)$ is upper semicontinuous.

In the canonical homomorphism ω_{x^*}: $C[x]\to C[x]/M_{x^*}^{N+1}$, we get for

$$g=\sum_k \frac{(x-x^*)^k}{k!}D^k g(x^*)$$

that

$$\omega_{x^*}(g)=\sum_{|k|\leqq N}\frac{X^k}{k!}D^k g(x^*).$$

Each ideal I_{x^*} in K^N has a basis

$$\mathfrak{B}[I_{x^*}]=\{\sum_{Q<|k|\leqq N}B_k^j X^k,\ j=0,1,\dots,t\},$$

where the elements

$$\sum_{Q<|k|\leqq N}B_k^j X^k,\quad j=0,1,\dots,t,$$

with $B_k^0=0$, $Q\geqq0$ if $x^*\in F$, are linearly independent; B_k^j are constants in X but eventually depend on x^*. Therefore we may determine

$$X^{k_j}=\sum_{\substack{Q<|k|\leqq N\\ k\neq k_r;\ |k|\leqq|k_j|}}A_k^j X^k(\operatorname{mod} I_{x^*}),\quad r,j=0,1,\dots,t;\ A_k^0=0,$$

where x^* is fixed; $A_k^j\in C$, and if $|k|>|k_j|$, then $A_k^j=0$. Thus

$$K_{x^*}=K^N/I_{x^*}=\{\sum_{|k|\leqq N}a_k X^k,\ X^{k_j}=\sum_{\substack{Q<|k|\leqq N\\ k\neq k_r;\ |k|\leqq|k_j|}}A_k^j X^k,\ j,r=0,1,\dots,t\}.$$

Let

$$K_{x^*}\ni\sum_{|k|\leqq N}a_k X^k=\sum_{\substack{|k|\leqq N\\ k\neq k_j}}a_k X^k+\sum_{j=0}^{t}$$

$$a_{k_j}\sum_{\substack{0<|k|\leqq N\\ k\neq k_r;\ |k|\leqq|k_j|}}A_k^j X^k=\sum_{|k|\leqq Q}a_k X^k+\sum_{\substack{Q<|k|\leqq N\\ k\neq k_r;\ |k|\leqq|k_j|}}(a_k+\sum_{j=0}^{t}a_{k_j}A_k^j)X^k,\quad j,r=0,1,\dots,t,$$

where if $|k|>|k_j|$, then $A_k^j=0$. The norm in the finite-dimensional algebra K^N/I_{x^*} up to an equivalence is determined by each basis of the linear functionals on K^N, which are zero on I_{x^*}. So the norm in K^N/I_{x^*} is determined uniquely, with precision up to equivalence by:

$$|\sum_{|k|\leqq N}a_k X^k|_{x^*}=\sum_{|k|\leqq Q}b_k(x^*)\,|a_k|+\sum_{\substack{Q<|k|\leqq N\\ k\neq k_r;\ |k|\leqq|k_j|}}d_k(x^*)\,|a_k+\sum_{j=0}^{t}a_{k_j}A_k^j(x^*)|,$$

where $A_k^j=0$ if $|k|>|k_j|$; $b_k(x^*)>0$; $d_k(x^*)>0$. Moreover, as $K^N=C[x]/M_{x^*}^{N+1}$ is scrutinized with its quotient norm Q, then $b_k(x^*)=k!$. This implies for

$$P(x)=\sum_k\frac{(x-x^*)^k}{k!}(D^kP_*)(x^*)$$

that

$$|\omega_{x^*}(P)|_{x^*}=\sum_{|k|\leq Q}|D^kP_*(x^*)|+\sum_{\substack{Q<|k|\leq N\\ k\neq k_r;\ |k|\leq|k_j|}} d_k(x^*)\left|\frac{D^kP_*(x^*)}{k!}+\sum_{j=0}^{t(x^*)}A_k^j(x^*)D^{k}jP_*(x^*)/k_j!\right|.$$

The ideal I_{x^*} is the canonical image of the ideal $\mathcal{I}(x^*)\subset C[x]$ in the canonical homomorphism $C[x]\to C[x]/M_{x^*}^{N+1}$. Then the coefficients $k!\,B_k^j(x^*)$ of its basis are D^k-derivatives at the point x^* of the polynomials $p_1,\dots,p_m$. Then the coefficients $A_k^j(x^*)$ are rational functions in these derivatives (which hence are continuous at x^*), and so are rational functions in x. That is why: 1. If we can determine $X^{k_1},\dots,X^{k_t}$ by the others at the point x^*, then we can do so and also in the intersection $F\cap U_0'$ of a hyperball neighbourhood U_0' of x^*; 2. In $F\cap U_0'$ all A_k^j are continuous functions in x; 3. Since $|\omega_{x^*}(p)|_{x^*}=\inf\{\|q\|,\ (q-p)\in\mathcal{I}(x^*)\}$; since the norm $\|\cdot\|$ is the norm Q in $C[x]$; and since $B_k^j(x^*)=0$ if $|k|\leq Q$, then all $b_k(x)=k!$, $|k|\leq Q$; 4. If all derivatives $k!\,B_k^j(x)$, $Q<|k|\leq N$, are not annulling at any point of $U_0'\cap F$ and the rang$(B_k^j(x))_{j,k}$ $\equiv t$ on $U_0'\cap F$, then $d_k(x)$ are equal to $k!$ on $U_0'\cap F$. But this is eventually not true when some of B_k^j is zero at some point of $U_0'\cap F$ or the rang$(B_k^j(x))_{j,k}\neq t$ on $U_0'\cap F$ That is why we shall prove: (i). Let $k\neq k_r$, $Q<|k|\leq N$. There exists a neighbourhood U^0 of x^*, $U^0\subset U_0'$, such that all $d_k(x)$ are upper semicontinuous on $U^0\cap F$; (ii). All $d_k(x)|A_k^j(x)|$ are upper semicontinuous on $U^0\cap F$, as it immediately follows from (i). and from 2. (iii). Let S be a compact in R^n. Hence there exists a finite number of points $t_1,\dots,t_s$ in S, such that $\bigcup_{\mu=1}^{s}U_\mu\supset S$, where U_μ is the neighbourbood of t_μ, corresponding to U^0 of x^*. Then it is determined a C-linear finite-dimensional differential-invariant space α_S of linear differential operators on S, which α_S is generated by the operators $\{D^k,\ \forall k$ with $|k|\leq Q$; $B_l=\sum_{Q<|k|\leq N}B_{lk}D^k$, $l=1,\dots,\mu\}$; B_l are received

by the operators $M_k=(D^k/k!+\sum_{j=0}^{t(t_\nu)}A_k^j(x)D^{k}i/k_j!)$, $Q<|k|\leq N$, $k\neq k_r=k_r(t_\nu)$, $\nu=1,\dots,\mu$, and by a C^∞ decomposition of the unity, submissed to $U_1,\dots,U_s$. That is why the coefficients $|B_{lk}|$ are also upper semicontinuous.

It remain to prove (i) by Induction: Let $|k'|=Q+1$, $k'\neq k_j$. Let put in ω_{x^*} the polynomial $P(x)=(x-x^*)^{k'}$. Then $|\omega_{x^*}(P)|_{x^*}=\sum_{|k|\leq Q}|(D^kP)(x^*)|+d_{k'}(x)$. Since $|\omega_x(P)|_x$ is an upper semicontinuous function in x, hence $d_{k'}(x)$ is also upper semicontinuous in x at least on a neighbourhood U_0^* of x^*, $U_0^*\subset U_0'$. The intersection of all U_0^* for $\forall k'$ with $|k'|=Q+1$, $k'\neq k_j$, are upper semicontinuous in x. Analogously, it follows by inductive arguments that all $d_k(x)$, $Q<|k|\leq N$, $k\neq k_j$, are upper semicontinuous in x on $F\cap U^0$, where U^0 is a "ball" neighbourhood of x^*, $U^0\subset U_0'$.

Remark. Evidently if we suppose in addition that $d_k(x)=0$ for $x\in(R^n-F)$, or $k=k_j$ for some j, then $d_k(x)\geq 0$ is upper semicontinuous on R^n. Furthermore, $d_k(x)$ is a constant equal either to $k!$ or to 0 on any $(N_{\alpha(p^0)}-\bigcup_{p'<p^0}N_{\alpha(p')})$.

That is why the assertion of Remark 2 is now obvious: Let all sets $D\subset R^n$ (or $D\subset G$) be scrutinized with the topologies, generated by the topology of R^n (respectively, by the topology of G). Then the corresponding spaces α in Propositions 2 and 8 can be chosen so that on the sets $(N_{\alpha(p^0)}-\bigcup_{p'<p^0}N_{\alpha(p')})$ the coefficients of the opera-

tors of α are continuous for the case of the closed ideals in D^{ν}_{G} (Proposition 2) and moreover are rational functions in x in the case of the ideals in $C[x]$ (Proposition 8).

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Centre for Mathematics and Mechanics
Sofia 1090 P. O. Box 373

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