Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

AN EXTREMAL PROBLEM IN THE SET OF BLASHKE PRODUCTS WITH FIXED MULTIPLICITIES OF THE ZEROS

RUMEN ULUCHEV

We study the problem of existence of Blashke product of minimal L_p-norm with fixed miltiplicities of its zeros. This problem may be formulated in the terms of the theory of optimal recovery in the following way: recovering functions from the Hardy space H^∞ on the basis of N pieces Hermitian type of information to find the optimal information operator. A comparison theorem is proved.

1. Introduction. This note is concerned with the problem of optimal recovery of functions from the Hardy space H^∞ on the basis of N pieces Hermitian type of information

$$T_N(f):=\{f^{(\lambda)}_{(x_i)},\quad \lambda=0,\dots,\nu_i-1,\quad i=1,\dots,n\},$$

where $\{\nu_i\}_1^n$ are positive integers,

$$\bar{\nu}:=(\nu_1,\dots,\nu_n),\quad |\bar{\nu}|:=\nu_1+\dots+\nu_n=N,$$

and $\bar{x}=(x_1,\dots,x_n)\in\Omega_n$, $\Omega_n:=\{(t_1,\dots,t_n):\ -1<t_1<\dots<t_n<1\}$.

It is well known (see [3]) that the error $R_N(\bar{\nu},\bar{x};t)$ of the recovery in fixed point $t\in[-1,1]$ is given by the Blashke product

$$B(\bar{x};t):=\prod_{i=1}^{n}\Big(\frac{t-x_i}{1-tx_i}\Big)^{\nu_i}.$$

More precisely, $R_N(\bar{\nu},\bar{x};t)=|B(\bar{x};t)|$, $t\in[-1,1]$.

We show first the existence of extremal nodes $\bar{x}^*\in\Omega_n$ for which the infimum

$$R_N(\bar{\nu}):=\inf_{\bar{x}\in\Omega_n}\|R_N(\bar{\nu},\bar{x};\cdot)\|_{L_p[-1,1]}=\inf_{\bar{x}\in\Omega_n}\|B(\bar{x};\cdot)\|_{L_p[-1,1]}\tag{1.1}$$

is attained and next, that

$$\min\{R_N(\bar{\nu}):|\bar{\nu}|=N,\quad 1\le n\le N\}\tag{1.2}$$

is attained for $n=N$, $\nu_1=\dots=\nu_n=1$. The number p is fixed, $1\le p<\infty$.

In other words, we seek Blashke product with minimal $L_p[-1,1]$-norm varying the zeros and the multiplicities. The problem of uniqueness of the extremal nodes $\bar{x}^*$ in (1.1) remains open.

Analogous problems for polynomials with fixed multiplicities of zeros have been considered by L. Chakalov [5], T. Popoviciu [4], A. Ghizzetti and A. Ossicini [2], ($p=2$ only). B. Boyanov [1] extend the mentioned results for the case $1\le p<\infty$.

To show the existence of $\bar{x}^*$ in (1.1) we shall follow an idea of L. Chakalov.

SERDICA Bulgaricae mathematicae publicationes. Vol. 14, 1988, p. 98—101.

2. Main results. The next lemma is the central moment in our study.

Lemma 2.1. *Let $\{\nu_i\}_1^n$ be given positive integers, $\nu_k \geqq 2$ for some k, $1 \leqq k \leqq n$ and $\bar{x} \in \Omega_n$. Let*

$$B(\bar{x};\, t) = \prod_{i=1}^{n} \left(\frac{t-x_i}{1-tx_i}\right)^{\nu_i},$$

$$B(\bar{x}(\varepsilon);\, t) = \left[\frac{t-x_k+c_1\varepsilon}{1-t(x_k-c_1\varepsilon)}\right]^{\alpha_0} \cdot \left[\frac{t-x_k-c_2\varepsilon}{1-t(x_k+c_2\varepsilon)}\right]^{\beta_0} \cdot \prod_{\substack{i=1\\ i\neq k}}^{n} \left(\frac{t-x_i}{1-tx_i}\right)^{\nu_i},$$

where the integers α_0 and β_0 are such that

$$0<\alpha_0<\nu_k, \quad 0<\beta_0<\nu_k, \quad \alpha_0+\beta_0=\nu_k,$$

and c_1, c_2 are real numbers.

Then for some special choise of the parameters c_1 and c_2, and sufficiently small positive ε

$$\| B(\bar{x}(\varepsilon);\cdot)\|_p < \| B(\bar{x};\cdot)\|_p. \tag{2.1}$$

Proof. Set $\alpha=p\alpha_0$, $\beta=p\beta_0$,

$$c_1=1/\alpha, \quad c_2=1/\beta \tag{2.2}$$

$$E=\frac{1}{2}\min\{|1+x_1|,\ |1-x_n|,\ |x_2-x_1|, \dots,\ |x_n-x_{n-1}|\},$$

$$g(t)=\left|\prod_{\substack{i=1\\ i\neq k}}^{n}\left(\frac{t-x_i}{1-tx_i}\right)^{\nu_i}\right|^p, \quad \varphi_1(t;\,\varepsilon)=\frac{|t-x_k+c_1\varepsilon|}{1-t(x_k-c_1\varepsilon)}, \quad \varphi_2(t;\,\varepsilon)=\frac{|t-x_k-c_2\varepsilon|}{1-t(x_k+c_2\varepsilon)}$$

and for $0\leqq\varepsilon<E$

$$f(\varepsilon)=[\|B(\bar{x}(\varepsilon);\cdot)\|_p]^p=\int_{-1}^{1} g(t)\,\varphi_1^{\alpha}(t;\,\varepsilon)\,\varphi_2^{\beta}(t;\,\varepsilon)\,dt.$$

It is clear that $\alpha\geqq 1$, $\beta\geqq 1$, $g(t)\geqq 0$ for $t\in[-1,1]$ and $g(t)=0$ if and only if $t=x_i$ $i=1,\dots,x-1,k+1,\dots,n$.
Our task is to show that $f'(0)=0$ and $f''(0)<0$. Then (2.1) would follow immediately. We have

$$f'(\varepsilon)=\int_{-1}^{1} g(t)\,\varphi_1^{\alpha-1}(t;\,\varepsilon)\,\varphi_2^{\beta-1}(t;\,\varepsilon)\,\varphi(t;\,\varepsilon)\,dt,$$

where

$$\varphi(t;\,\varepsilon)=\alpha\varphi_2(t;\,\varepsilon)\frac{d\varphi_1(t;\,\varepsilon)}{d\varepsilon}+\beta\varphi_1(t;\,\varepsilon)\frac{d\varphi_2(t;\,\varepsilon)}{d\varepsilon}$$

$$=\alpha\frac{|t-x_k-c_2\varepsilon|}{1-t(x_k+c_2\varepsilon)}\cdot\frac{[1-t(x_k-c_1\varepsilon)]c_1\operatorname{sign}(t-x_k+c_1\varepsilon)-tc_1|t-x_k+c_1\varepsilon|}{[1-t(x_k-c_1\varepsilon)]^2}$$

$$+\beta\frac{|t-x_k+c_1\varepsilon|}{1-t(x_k-c_1\varepsilon)}\cdot\frac{[1-t(x_k+c_2\varepsilon)](-c_2)\operatorname{sign}(t-x_k-c_2\varepsilon)+tc_2|t-x_k-c_2\varepsilon|}{[1-t(x_k+c_2\varepsilon)]^2}.$$

Then $\varphi(t;\,0)=(\alpha c_1-\beta c_2)(t-x_k)(1-t^2)/(1-tx_k)^3=0$ because of (2.2) and consequently $f'(0)=0$.

Denote by φ_0 the function $\varphi_0(t;\,\varepsilon)=\varphi_1^{\alpha-1}(t;\,\varepsilon)\,.\,\varphi_2^{\beta-1}(t;\,\varepsilon)$.
The careful calculations show that

$$f'(\varepsilon)=\int_{-1}^{x_k-c_1\varepsilon}(1-t^2)g(t)\frac{\varepsilon[p(t)+\varepsilon q(t)]\varphi_0(t;\varepsilon)}{[1-t(x_k-c_1\varepsilon)]^2[1-t(x_k+c_2\varepsilon)]^2}dt$$

$$-\int_{x_k-c_1\varepsilon}^{x_k+c_2\varepsilon}(1-t^2)g(t)\frac{\varepsilon[p(t)+\varepsilon q(t)]\varphi_0(t;\varepsilon)}{[1-t(x_k-c_1\varepsilon)]^2[1-t(x_k+c_2\varepsilon)]^2}dt$$

$$+\int_{x_k+c_2\varepsilon}^{1}(1-t^2)g(t)\frac{\varepsilon[p(t)+\varepsilon q(t)]\varphi_0(t;\varepsilon)}{[1-t(x_k-c_1\varepsilon)]^2[1-t(x_k+c_2\varepsilon)]^2}dt,$$

where

(2.3) $p(t)=-(c_1+c_2)(t^2-2tx_k+1)<0$ for $t\in[-1,1]$, $x_k\in(-1,1)$, $q(t)=(c_2^2-c_1^2)t$.

Denote

$$\psi(t;\varepsilon)=\frac{g(t)(1-t^2)\varphi_0(t;\varepsilon)[p(t)+\varepsilon q(t)]}{[1-t(x_k-c_1\varepsilon)]^2[1-t(x_k+c_2\varepsilon)]^2}.$$

Then

$$f''(\varepsilon)=\int_{-1}^{x_k-c_1\varepsilon}(1-t^2)g(t)\frac{d}{d\varepsilon}\{\varepsilon\frac{\varphi_0(t;\varepsilon)[p(t)+\varepsilon q(t)]}{[1-t(x_k-c_1\varepsilon)]^2[1-t(x_k+c_2\varepsilon)]^2}\}dt-c_1\varepsilon\psi(x_k-c_1\varepsilon;\varepsilon)$$

$$-\int_{x_k-c_1\varepsilon}^{x_k+c_2\varepsilon}(1-t^2)g(t)\frac{d}{d\varepsilon}\{\varepsilon\frac{\varphi_0(t;\varepsilon)[p(t)+\varepsilon q(t)]}{[1-t(x_k-c_1\varepsilon)]^2[1-t(x_k+c_2\varepsilon)]^2}\}dt-c_2\varepsilon\psi(x_k+c_2\varepsilon;\varepsilon)-c_1\varepsilon\psi(x_k-c_1\varepsilon;\varepsilon)$$

$$+\int_{x_k+c_2\varepsilon}^{1}(1-t^2)g(t)\frac{d}{d\varepsilon}\{\varepsilon\frac{\varphi_0(t;\varepsilon)[p(t)+\varepsilon q(t)]}{[1-t(x_k-c_1\varepsilon)]^2[1-t(x_k+c_2\varepsilon)]^2}\}dt-c_2\varepsilon\psi(x_k+c_2\varepsilon;\varepsilon).$$

Therefore

$$f''(0)=\int_{-1}^{1}g(t)(1-t^2)\frac{\varphi_0(t;0)p(t)}{(1-tx_k)^4}dt=\int_{-1}^{1}g(t)(1-t^2)p(t)\frac{|t-x_k|^{\alpha+\beta-2}}{(1-tx_k)^{\alpha+\beta+2}}dt$$

and (2.3) yields $f''(0)<0$. The proof is complete.

Theorem 2.1. *Let $\{\nu_i\}_1^n$ be given positive integers. Then there exist extermal nodes $\bar{x}^*\in\Omega_n$ for the infimum* (1.1)

Proof. Suppose the infimum (1.1) is attained for the Blashke product

$$B(\bar{y};t)=\prod_{i=1}^{m}\left(\frac{t-y_i}{1-ty_i}\right)^{\mu_i},$$

where $1\le m\le n$, $\bar{y}=(y_1,\dots,y_m)$, $-1\le y_1<\dots<y_m\le 1$ and $\mu_i=\nu_{j_i+1}+\dots+\nu_{j_{i+1}}$, $i=1,\dots,m$ with some integers

$$0=:j_1<\dots<j_m<j_{m+1}:=n.$$

Assume, for example, that $y_1=-1$ and set

$$\tilde{\tau}=(\tau_1\dots,\tau_m)=\begin{cases}(0), & m=1\\ ((y_2-1)/2,\ y_2,\dots,y_m), & m>1.\end{cases}$$

Then, using the convention $\Pi_{i=2}^m a_i=1$ for $m=1$, we have

$$\|B(\bar{y};\cdot)\|_p^p-\|B(\tilde{\tau};\cdot)\|_p^p=\int_{-1}^{1}\left|\prod_{i=2}^{m}\left(\frac{t-y_i}{1-ty_i}\right)^{\mu_i}\right|^p.\left[1-\left|\frac{t-\tau_1}{1-t\tau_1}\right|^{p\mu_1}\right]\ dt>0,$$

since

$$0<\left|\frac{t-\xi}{1-t\xi}\right|<1 \quad \text{for all } t\in(-1,\ 1),\ \xi\in(-1,\ 1),\quad t\neq\xi.$$

This proves that $y_1\neq-1$. Similarly we conclude that $y_m\neq 1$. Thus

$$-1<y_1 \text{ and } y_m<1. \tag{2.4}$$

Assume now that $1\leqq m<n$. According to (2.4), we have $\bar{y}\in\Omega_m$ and applying Lemma 2.1, we immediately get a contradiction with the extremality of the nodes $\bar{y}$.

Therefore $m=n$, $\mu_i=\nu_i$, $i=1,\dots,n$ and $\bar{y}=\bar{x}^*\in\Omega_n$. The theorem is proved.

Now we ready to state and prove the comparison theorem.

Theorem 2.2. *Let* $\{\nu_i\}_1^n$ *be arbitrary positive integers and* $\nu_k\geqq 2$ *for some* k, $1\leqq k\leqq n$. *Set* $\bar{\mu}=(\nu_1,\dots,\nu_{k-1},\ \nu_k-1,\ 1,\ \nu_{k+1},\dots,\nu_n)$. *Then* $R_N(\bar{\mu})<R_N(\bar{\nu})$.

Proof. The assertion follows from Theorem 2.1 and Lemma 2.1 with $\alpha_0=\nu_k-1$, $\beta_0=1$.

As an immediate consequence we get.

Corollary 2.1. For every set of positive integers $\bar{\nu}=(\nu_1,\dots,\nu_n)$ with $|\bar{\nu}|=N$ the inequality

$$R_N(\bar{\mu})<R_N(\bar{\nu})$$

holds, where $\bar{\mu}=(1,\ 1,\dots,\ 1)$ (N times "1").

That is, the function evaluations at the simple optimal nodes is better information than any N pieces Hermitian information.

REFERENCES

1. B. Boyanov. Extremal problems in a set of polynomials with fixed multiplicities of zeros. *C. R. Acad. Bulg. Sci.*, **31**, 1978, 4, 377—380.
2. A. Ghizzetti, A. Ossicini. Sull' esistenza e unicita delle formule di quadratura gaussiane, *Rend. Mat.*, **1**, 1975, 1—15.
3. K. Osipenko. Optimal interpolation of analytic functions. *Mat. Zametki.*, **12**, 1972, 4, 465—476.
4. T. Popoviciu. Asurpa unei generalizari a formulei de integrare numerica a lui Gauss. *Acad. R. P. Rumaine Fil. Iasi Stud. Cerc. Sti.*, **6**, 1955, 29—57.
5. L. Chakalov. General quadrature formulae of Gaussian type. *Izv. Mat. Institut,* BAN, 1, *fasc.* 2, 1954, 67—84 (Bulgarian).

Centre for Mathematics and Mechanics
Sofia 1090 P. O. Box 373

Received 5. 11. 1986