Serdica
Bulgariacae mathematicae
publicationes

Сердика
Българско математическо
списание

STABILIZERS OF σ-FIELDS OF SETS IN SEPARABLE METRIC SPACES AND COMPLETE GROUPS

V. V. MIŠKIN

Let X be an uncountable separable metric space and let $\mathcal{F}$ be a strongly homogeneous σ-field of sets in X of uncountable co-finality containing the σ-field of Borel sets in X. Then the stabilizer of $\mathcal{F}$, consisting of those autobijections of X leaving $\mathcal{F}$ globally invariant, or in other words, the Borel automorphism group of the Borel space $(X, \mathcal{F})$ is a complete subgroup of the general symmetric group S_X coinciding with its normalizer in S_X. In particular, so does the stabilizers of the classical σ-fields of Borel and Lebesgue-measurable sets, and of the sets with the Baire property in Euclidean spaces, furthermore, in uncountable Polish spaces.

1. Introduction. We recall that a group G is complete if the center of G is trivial and every automorphism of G is inner [9, p. 56]. The classical examples of complete groups are, by Hölder's theorem [3], the finite symmetric groups S_n, for all $n \neq 2$ and 6. Furthermore, by the Schreier-Ulam theorem [11] the symmetric group S_X of an infinite set X is complete as well. Let us denote by A_X the general alternating group of an infinite set X consisting of compositions of an even (finite) number of transpositions of X. Then the following generalization of the Schreier-Ulam theorem holds [12]. Every automorphism of a group G such that $A_X \subset G \subset S_X$ is induced by an inner automorphism of S_X. We recall that a Polish space is a separable completely metrizable topological space and an absolute Borel topological space is a space homeomorphic to a Borel set in a Polish space. It was shown by P. S. Alexandroff [1] and F. Hausdorff [4] that every uncoutable Borel set in a Polish space contains a copy of the Cantor set D^ω and so is of the cardinal of the continuum. On the other hand, it was established by K. Kuratowski [6] that for any two uncountable Borel sets B_1, B_2 in a Polish space the Borel spaces $(B_1, \mathcal{B}_{B_1})$ and $(B_2, \mathcal{B}_{B_2})$ are isomorphic and so are isomorphic to the Borel space $(D^\omega, \mathcal{B}_{D^\omega})$, where $\mathcal{B}_X$ denotes the σ-field of Borel sets in a topological space X. On can apply the above-mentioned generalization of the Schreier-Ulam theorem and the results of Alexandroff-Hausdorff and Kuratowski to obtain the following assertion, discovered in fact by E. R. Lorch and Hing Tong [7]. The stabilizer of the σ-field of Borel sets in an absolute Borel topological space X of cardinality c is a complete subgroup of S_X. In the present paper, stimulated by [7], a more general result is established concerning the stabilizers of strongly homogeneous extensions of the σ-fields of Borel sets in uncountable separable metric spaces. Some applications of this general result are given to the study of strongly homogeneous σ-fields of sets in Polish spaces and, in particular, to the σ-fields of Lebesgue-measurable sets and of the sets with the Baire property in Euclidean spaces.

2. Notation and terminology. We denote as usual by $\mathcal{P}(X)$ the power-set of a set X, by ω the first infinite ordinal, and by c the cardinal number 2^ω. The cardinality of a set X will be denoted by $|X|$. We recall that a family of subsets of a set X closed under the operations of countable union and complementation (relative to X)

is called a σ-field of sets in X [10] or a σ-algebra. If $\mathscr{F}$ is a σ-field of sets in a set X, then the pair $(X, \mathscr{F})$ is caled a Borel space [8]. Countable means finite or countably infinite. The σ-field of Borel sets in a topological space $(X, \mathscr{T})$ is the smallest σ-field of sets in X containing $\mathscr{T}$. This σ-field will be denoted by $\mathscr{B}_X$. Let $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ be two Borel spaces. A bijection $f: X \to Y$ with $f(\mathscr{F}) = \mathscr{G}$, where $f(\mathscr{F}) = \{f(F) : F \in \mathscr{F}\}$, is called a Borel isomorphism of the Borel spaces $(X, \mathscr{F})$ and $(Y, \mathscr{G})$. If $X = Y$ and $\mathscr{F} = \mathscr{G}$, then f is called a Borel automorphism of the Borel space $(X, \mathscr{F})$. If $(X, \mathscr{F})$ is a Borel space and $Y \subset X$, then $\mathscr{F}|_Y = \{Y \cap F : F \in \mathscr{F}\}$ fs a σ-field of sets in Y and the pair $(Y, \mathscr{F}|_Y)$ is called a subspace of the Borel space $(X, \mathscr{F})$. For every Borel space $(X, \mathscr{F})$ the family $J_{\mathscr{F}} = \{F \in \mathscr{F} : \mathscr{P}(F) \subset \mathscr{F}\}$ is obviously a σ-ideal of sets in X (i. e. the family $J_{\mathscr{F}}$ is closed under the operation of countable union and if $A \in J_{\mathscr{F}}$ and $B \subset A$, then $B \in J_{\mathscr{F}}$). We define the co-finality $cf(\mathscr{F})$ of the σ-field of sets $\mathscr{F}$ in X as follows: $cf(\mathscr{F}) = \min\{cf(|F|) : F \in \mathscr{F} \setminus J_{\mathscr{F}}\}$. A σ-field of sets $\mathscr{F}$ in X and the corresponding Borel space $(X, \mathscr{F})$ are said to be strongly homogeneous provided that any two subspaces F_1 and F_2 of the Borel space $(X, \mathscr{F})$ with $F_1, F_2 \in \mathscr{F} \setminus J_{\mathscr{F}}$ and $|F_1| = |F_2|$ are Borel isomorphic. Let $(X, \mathscr{F})$ be a Borel space and let μ be a measure on $\mathscr{F}$. If $\bar{\mu}$ is the completion of μ, then the σ-field $\{F \cup N : F \in \mathscr{F}$ and there is a set $E \in \mathscr{F}$ such that $E \supset N$ and $\mu(E) = 0\}$ of sets in X on which $\bar{\mu}$ is defined will be denoted by $\bar{\mathscr{F}}$. For the definitions of other standard notions of measure theory the reader is referred to [5] and [8]. A set S in a topological space X is said to have the Baire property, if there is an open set U in X such that the symmetric difference $S \triangle U = (S \setminus U) \cup (U \setminus S)$ is of the first category in X. The σ-field of sets with the Baire property in X will be denoted by $\mathscr{BP}_X$. By J^n_ω, $\mathfrak{M}^n_0$, and $\mathfrak{M}^n_1$ we denote, respectively, the σ-ideals of countable sets, the sets of Lebesgue measure zero, and the sets of the first category in the Euclidean space R^n. We recall that a Polish space is a separable, completely metrizable topological space and an absolute Borel space is a Borel space isomorphic to a Borel subset B in a Polish space equipped with its relative Borel structure $\mathscr{B}_B$. We denote by S_X the general symmetric group of all autobijections of an infinite set X. This group acts naturally on $\mathscr{P}(X)$. Let $\mathscr{A}$ be a family of subsets of X. The subgroup $\mathscr{G}(\mathscr{A}) = \{s \in S_X : s(\mathscr{A}) = \mathscr{A}\}$ in S_X is called the stabilizer of $\mathscr{A}$. If G is a group and $H \subset G$, then by $N_G(H)$ and $C_G(H)$ we denote, respectively, the normalizer and the centralizer of H in G. The center $C_G(G)$ of G will be denote by $C(G)$. A group G is said to be perfect, if $C(G) = \{e\}$ and for every of its automorphism φ there is an element $h \in G$ such that $\varphi(g) = h . g . h^{-1}$ for all $g \in G$.

3. The stabilizers of strongly homogeneous σ-fields of sets in separable metric spaces. Main theorem. Let (X, d) be an uncountable separable metric space and let $\mathscr{F}$ be a strongly homogeneous σ-field of sets in X of uncountable co-finality containing the σ-field of Borel sets $\mathscr{B}_X$. Then the stabilizer of $\mathscr{F}$ is a complete subgroup of S_X coinciding with its normalizer in S_X.

We shall divide the proof of the theorem into four steps.

Step 1. Since every singleton in X is closed, it is contained in $\mathscr{B}_X$ and hence in $\mathscr{F}$. It follows that $\mathscr{G}(\mathscr{F})$ contains all transpositions of X. Therefore we have $C(\mathscr{G}(\mathscr{F})) = \{id\}$ and $A_X \subset \mathscr{G}(\mathscr{F})$. By the generalization of the Schreier-Ulam theorem mentioned above for each $\varphi \in \mathrm{Aut}(\mathscr{G}(\mathscr{F}))$ there exists an element $h \in S_X$ such that $\varphi(g) = h \circ g \circ h^{-1}$ for all $g \in \mathscr{G}(\mathscr{F})$. Thus we have that $h \in N_{S_X}(\mathscr{G}(\mathscr{F}))$ and it suffices to prove that $N_{S_N}(\mathscr{G}(\mathscr{F})) = \mathscr{G}(\mathscr{F})$. It can easily be verified that $\mathscr{G}(\mathscr{F}) \subset \mathscr{G}(J_{\mathscr{F}})$.

Step 2. We first show that $h(J_{\mathscr{F}}) = J_{\mathscr{F}}$ for all $h \in N_{S_X}(\mathscr{G}(\mathscr{F}))$ or, in other words, that $N_{S_X}(\mathscr{G}(\mathscr{F})) \subset \mathscr{G}(J_{\mathscr{F}})$. Let $h \in N_{S_X}(\mathscr{G}(\mathscr{F}))$, then $h \cdot \mathscr{G}(\mathscr{F}) \cdot h^{-1} = \mathscr{G}(\mathscr{F})$ and clearly

$h\cdot\mathcal{G}(\mathcal{F})\cdot h^{-1}=\mathcal{G}(h(\mathcal{F}))$. Let us verify the inclusion $h(J_{\mathcal{F}})\subset\mathcal{F}$ by transfinite induction on the cardinality of the sets $A\in J_{\mathcal{F}}$. Let $\alpha=\min\{|A|:A\in J_{\mathcal{F}}$ and $h(A)\notin\mathcal{F}\}$. Obviously $\alpha>\omega$. Let us consider a set $A\in J_{\mathcal{F}}$ such that $|A|=\alpha$ and $h(A)\notin\mathcal{F}$. One can find two subsets A_1 and A_2 of A such that $A_1\cup A_2=A$, $A_1\cap A_2=\emptyset$, and $|A_1|=|A_2|=\alpha$. Let $f:A_1\to A_2$ be a bijection and let $g'\in S_X$ be defined as follows: $g'|A_1=f$, $g'|A_2=f^{-1}$, and $g'|_{X\setminus A}=id_{X\setminus A}$. Then for each $B\in\mathcal{F}$ we have $g'(B)=(B\setminus A)\cup g'(B\cap A_1)\cup g'(B\cap A_2)$. Since $B\setminus A\in\mathcal{F}$, $g'(B\cap A_1)\subset A$, and $g'(B\cap A_2)\subset A$, we have that $g'(B\cap A_1)$, $g'(B\cap A_2)\in\mathcal{F}$ and hence $g'(B)\in\mathcal{F}$. Since $g'^{-1}=g'$, we have also that $g'^{-1}(B)\in\mathcal{F}$ for all $B\in\mathcal{F}$. Thus $g'(\mathcal{F})\subset\mathcal{F}$ and $g'^{-1}(\mathcal{F})\subset\mathcal{F}$ (or $\mathcal{F}\subset g'(\mathcal{F})$) and so $g'\in\mathcal{G}(\mathcal{F})$. If we set $g=h\circ g'\circ h^{-1}$, then $g\in h\cdot\mathcal{G}(\mathcal{F})\cdot h^{-1}=\mathcal{G}(h(\mathcal{F}))=\mathcal{G}(\mathcal{F})$. If $cf(\alpha)=\omega$, then $A=\bigcup_{i\in\omega}B_i$, where $|B_i|<\alpha$ and obviously $B_i\in J_{\mathcal{F}}$, $i\in\omega$. By the definition of α we have $h(B_i)\in\mathcal{F}$, $i\in\omega$, and hence $h(A)=\bigcup_{i\in\omega}h(B_i)\in\mathcal{F}$. A contradiction. So we may assume that $cf(\alpha)>\omega$. If we set $K=h(A)$, then it is evident that $K=\{x\in X:\ g(x)\neq x\}$ and $|K|=|A|=\alpha$. Let Q_+ denote the set of positive rationals and let $K_r=\{x\in X:\ d(x,g(x))>r\}$, $r\in Q_+$. Then $K=\bigcup_{r\in Q_+}K_r$. Since $cf(\alpha)>\omega$, there is an $r\in Q_+$ such that $|K_r|=\alpha$ and there is a point $z\in K_r$ such that for every of its neighbourhood U_z we have $|U_z\cap K_r|=\alpha$. Let us consider the open ball $B=B(z;r/4)$. It is easily seen that $|B\cap K_r|=\alpha$ and $B\cap g(B\cap K_r)=\emptyset$. By the construction g is an involution, so $g=g^{-1}$ and hence $B\cap g^{-1}(B\cap K_r)=\emptyset$ or $(B\cap K_r)\cap g(B)=\emptyset$. Let $C=B\setminus g(B)$. Since $B\cap K_r\subset B\cap(X\setminus g(B))=C$ and $|B\cap K_r|=\alpha$, we have that $|C|\geq\alpha$. On the other hand, $C=B\setminus g(B)=B\setminus g((B\cap K)\cup(B\setminus K))\subset B\setminus g(B\setminus K)=B\setminus(B\setminus K)=B\cap K\subset K$ and hence $|C|\leq\alpha$ and $C\in h(J_{\mathcal{F}})\subset h(\mathcal{F})$. Therefore $|C|=\alpha$ and, since $B\in\mathcal{F}$ and $g\in\mathcal{G}(\mathcal{F})$, we have $C=B\cap(X\setminus g(B))\in\mathcal{F}$. If $|K\setminus C|<\alpha$, then obviously $|h^{-1}(K\setminus C)|<\alpha$ and $h^{-1}(K\setminus C)\in J_{\mathcal{F}}$, and again by the definition of α we have $h(h^{-1}(K\setminus C))=K\setminus C\in\mathcal{F}$. Thus $h(A)=C\cup(K\setminus C)\in\mathcal{F}$ and we obtain a contradiction. If now $|K\setminus C|=\alpha$, then we may consider a bijection $t:\ C\to K\setminus C$ and define the map $s\in S_X$ by setting $s|_C=t$, $s|_{K\setminus C}=t^{-1}$, and $s|_{X\setminus K}=id_{X\setminus K}$. It is easily seen that $s\in\mathcal{G}(h(\mathcal{F}))=\mathcal{G}(\mathcal{F})$. Since $C\in\mathcal{F}$, we have $s(C)=K\setminus C\in\mathcal{F}$ and hence $h(A)=C\cup(K\setminus C)\in\mathcal{F}$. Contradiction again. Thus $\{A\in J_{\mathcal{F}}:h(A)\notin\mathcal{F}\}=\emptyset$ and we obtain $h(J_{\mathcal{F}})\subset\mathcal{F}$. Since $h^{-1}\in N_{S_X}(\mathcal{G}(\mathcal{F}))$, by repeating the argument, with h instead of h^{-1}, we obtain that $h^{-1}(J_{\mathcal{F}})\subset\mathcal{F}$. It follows that $h(J_{\mathcal{F}})\subset J_{\mathcal{F}}$ and $h^{-1}(J_{\mathcal{F}})\subset J_{\mathcal{F}}$ and we obtain the desired equality $h(J_{\mathcal{F}})=J_{\mathcal{F}}$ and hence the inclusion $N_{S_X}(\mathcal{G}(\mathcal{F}))\subset\mathcal{G}(J_{\mathcal{F}})$.

Step 3. We say that a set $F\in\mathcal{F}\setminus J_{\mathcal{F}}$ is split if it can be divided into two disjoint subsets $F_1,F_2\in\mathcal{F}\setminus J_{\mathcal{F}}$ of cardinality $|F|$. We shall show that every $F\in\mathcal{F}\setminus J_{\mathcal{F}}$ is split. Let $\beta=\min\{|F|:F\in\mathcal{F}\setminus J_{\mathcal{F}}$ and F is not split$\}$. Let us consider now a set $F\in\mathcal{F}\setminus J_{\mathcal{F}}$ such that $|F|=\beta$ and F is not split. Then there is a point $x\in F$ such that for every of its neighbourhood U_x we have $|U_x\cap F|=\beta$ and $U_x\cap F\notin J_{\mathcal{F}}$. Indeed, if on the contrary for each $x\in F$ there is either a neighbourhood U_x with $|U_x\cap F|<\beta$ or a neighbourhood U'_x with $U'_x\cap F\in J_{\mathcal{F}}$, then we can take some basic neighbourhoods $B_x\subset U_x$ and $B'_x\subset U'_x$ from a countable base $\mathcal{B}$ of open sets in X. Obviously $|B_x\cap F|<\beta$ and $B'_x\cap F\in J_{\mathcal{F}}$. Let $B=\{x\in F:$ there is a neighbourhood U_x of x such that $|U_x\cap F|<\beta\}$ and let $B'=\{x\in F:$ there is a neighbourhood U_x of x such that $U_x\cap F\in J_{\mathcal{F}}\}$. Clearly $B=\bigcup_{x\in B}(B_x\cap F)$ and $B'=\bigcup_{x\in B}(B'_x\cap F)$ and $F=B\cup B'$. Since for each $x\in B$ $B_x\cap F\in\mathcal{F}$, we have $B\in\mathcal{F}$ and hence $B\in\mathcal{F}\setminus J_{\mathcal{F}}$, for $B'\in J_{\mathcal{F}}$ and $F\notin J_{\mathcal{F}}$. Since

$cf(\mathscr{F})>\omega$, we have that $cf(\beta)>\omega$ and hence $|B|<\beta$, for B is a countable union of sets $B_x \cap F$. Thus, by the definition of β, the set B is split. Let $B=B_1 \cup B_2$, $B_1 \cap B_2=\emptyset$, B_1, $B_2 \in \mathscr{F} \setminus J_{\mathscr{F}}$, and $|B_1|=|B_2|=|B|$. Let us consider a partition $B'=B_1' \cup B_2'$, $B_1' \cap B_2' =\emptyset$, and $|B_1'|=|B_2'|=\beta$. If we put $F_1=B_1 \cup (B_1' \setminus (B_1 \cup B_2))$ and $F_2=B_2 \cup (B_2' \setminus (B_1 \cup B_2))$, then $|F_1|=|F_2|=\beta$, $F_1, F_2 \in \mathscr{F} \setminus J_{\mathscr{F}}$, $F_1 \cap F_2=\emptyset$, and $F_1 \cup F_2=F$. Therefore F is split and we obtain a contradiction. Thus we can choose a point $x \in F$ such that for every of its neighbourhood U_x we have $|U_x \cap F|=\beta$ and $U_x \cap F \notin J_{\mathscr{F}}$. Let $V_n=B(x; 1/n+1)$, $n \in \omega$, and let $F_n=F \setminus V_n$. It is evident that $F=\bigcup_{n \in \omega} F_n \cup \{x\}$. Since $cf(\beta)>\omega$, there is a number $n_0 \in \omega$ such that $|F_{n_0}|=\beta$. Since $J_{\mathscr{F}}$ is a σ-ideal of sets in X containing all singletons, there is a number $m_0 \in \omega$ such that $F_{m_0} \notin J_{\mathscr{F}}$. Thus, by setting $m=\max\{n_0, m_0\}$, we obtain that $F=(V_n \cap F) \cup F_n$, $V_n \cap F \notin J_{\mathscr{F}}$, $F_n \notin J_{\mathscr{F}}$, and $|V_n \cap F|=|F_n|=\beta$. It follows that F is split. Contradiction. Thus the set $\{F \in \mathscr{F} \setminus J_{\mathscr{F}} : F \text{ is not split}\}$ is empty and hence each $F \in \mathscr{F} \setminus J_{\mathscr{F}}$ is split.

Step 4. Finaly we will show that $h(\mathscr{F} \setminus J_{\mathscr{F}}) \subset \mathscr{F}$ for all $h \in N_{S_X}(\mathscr{G}(\mathscr{F}))$. Let us consider the cardinal number $\gamma=\min\{|F| : F \in \mathscr{F} \setminus J_{\mathscr{F}}$ and $h(F \notin \mathscr{F}\}$. We can find a set $A \in \mathscr{F} \setminus J_{\mathscr{F}}$ such that $|A|=\gamma$ and $h(A) \notin \mathscr{F}$. Making use of Step 3, we can divide the set A into four mutually disjoint subsets $A_i \in \mathscr{F} \setminus J_{\mathscr{F}}$ of common cardinality γ $i=1, 2, 3, 4$. Since $\mathscr{F}$ is strongly homogeneous, there is a bijection $f\colon A_1 \to A_2$ such that $f(\mathscr{F}|_{A_1})=\mathscr{F}|_{A_2}$. Define the map $g' \in S_X$ as follows: $g'|_{A_1}=f$, $g'|_{A_2}=f^{-1}$, and $g'|_{X \setminus (A_1 \cup A_2)}=id_{X \setminus (A_1 \cup A_2)}$. It is clear that $g' \in \mathscr{G}(\mathscr{F})$. Let $A'=A_1 \cup A_2$, then $A' \notin J_{\mathscr{F}}$ and $A'=\{x \in X\colon g'(x) \neq x\}$. Let $K=h(A')$ and let $g=h \circ g' \circ h^{-1}$. Then $g \in h \cdot \mathscr{G}(\mathscr{F}) \cdot h^{-1} =\mathscr{G}(h(\mathscr{F}))=\mathscr{G}(\mathscr{F})$ and $K=\{x \in X\colon d(x, g(x))>0\}=h(A_1) \cup h(A_2) \in h(\mathscr{F})$. If we set $E_r =\{x \in X\colon d(x, g(x))>r\}$, $r \in \mathbf{Q}_+$, then $K=\bigcup_{r \in \mathbf{Q}_+} E_r$. Since $A' \notin J_{\mathscr{F}}$ and $h(J_{\mathscr{F}})=J_{\mathscr{F}}$, we have $K=h(A') \notin J_{\mathscr{F}}$. Since $J_{\mathscr{F}}$ is a σ-ideal of sets, there is an $r_0 \in \mathbf{Q}_+$ such that $E_{r_0} \notin J_{\mathscr{F}}$ and, since $cf(\gamma)>\omega$, there is $r_1 \in \mathbf{Q}_+$ such that $|E_{r_1}|=\gamma$. If we take $r=\min\{r_0, r_1\}$, then $|E_r|=\gamma$ and $E_r \notin J_{\mathscr{F}}$. Therefore there is a point $z_1 \in E_r$ such that for every of its neighbourhood U_{z_1} we have $U_{z_1} \cap E_r \notin J_{\mathscr{F}}$ and there is a point $z_2 \in E_r$ such that for every of its neighbourhood U_{z_2} we have $|U_{z_2} \cap E_r|=\gamma$. Let us consider the balls $B_1=B(z_1; r/4)$ and $B_2=B(z_2; r/4)$ and let $C_1=B_1 \setminus g(B_1)$, $C_2=B_2 \setminus g(B_2)$, and $C=C_1 \cup C_2$. Clearly $C_1, C_2 \in \mathscr{F}$ and hence $C \in \mathscr{F}$. By repaeting the argument of Step 2 we obtain that $|C_2|=\gamma$, $B_1 \cap E_r \subset C_1$ and hence $C_1 \notin J_{\mathscr{F}}$. Therefore $|C|=\gamma$ and $C \in \mathscr{F} \setminus J_{\mathscr{F}}$. Similarly, making use of A_3 and A_4 instead of A_1 and A_2, we can find a set $D \subset h(A_3 \cup A_4) \subset X \setminus C$ such that $D \in \mathscr{F} \setminus J_{\mathscr{F}}$ and $|D|=\gamma$. Since $\mathscr{F}$ is strongly homogeneous, there is a Borel isomorphism $f'\colon C \to D$ of the Borel subspaces C and D of the Borel space $(X, \mathscr{F})$. Define the map $f \in S_X$ putting $f|_C=f'$, $f|_D=f'^{-1}$, and $f|_{X \setminus (C \cup D)}=id_{X \setminus (C \cup D)}$. It is easily seen that $f \in \mathscr{G}(\mathscr{F})=\mathscr{G}(h(\mathscr{F}))$ and hence $f(K) \in h(\mathscr{F})$, so $K \setminus f(K) \in h(\mathscr{F})$, but $K \setminus f(K)=K \setminus ((K \setminus C) \subset D)=C$. Thus we have $C \in (\mathscr{F} \cap h(\mathscr{F})) \setminus J_{\mathscr{F}}$, $|C|=\gamma$, and $|X \setminus C| \geq \gamma$, hence $|X \setminus C|=|X|$. It follows that $h^{-1}(C) \in (h^{-1}(\mathscr{F}) \cap \mathscr{F}) \setminus J_{\mathscr{F}}$ and $|h^{-1}(C)|=\gamma$. Let $f_1\colon h^{-1}(C) \to A$ be a bijection such that $f_1(\mathscr{F} \setminus h^{-1}(C))=\mathscr{F}|_A$. If $X \setminus A \in J_{\mathscr{F}}$, then by Step 2 $h(X \setminus A)=X \setminus h(A) \in \mathscr{F}$ and hence $h(A) \in \mathscr{F}$. If $X \setminus A \notin J_{\mathscr{F}}$ and $|X \setminus A|<\gamma$, then by the definition of γ, we have $h(X \setminus A) \in \mathscr{F}$ and again $h(A) \in \mathscr{F}$. Finally, if $X \setminus A \notin J_{\mathscr{F}}$ and $|X \setminus A| \geq \gamma$, then $|X \setminus A|=|X|$ and, since $D \notin J_{\mathscr{F}}$, $h^{-1}(J_{\mathscr{F}})$

$=J_{\mathcal{F}}$, we have $h^{-1}(D)\notin J_{\mathcal{F}}$. Since $h^{-1}(D)\subset X\setminus h^{-1}(C)$, we have $X\setminus h^{-1}(C)\notin J_{\mathcal{F}}$ and obviously $|X\setminus h^{-1}(C)|=\gamma=|X\setminus A|$. Therefore there is a bijection $f_2\colon X\setminus h^{-1}(C)\to X\setminus A$ such that $f_2(\mathcal{F}|_{X\setminus h^{-1}(C)})=\mathcal{F}|_{X\setminus A}$. Now we can define the map $p\in S_X$ by setting $p|_{h^{-1}(C)}=f_1$ and $p|_{X\setminus h^{-1}(C)}=f_2$. Obviously $p\in\mathcal{G}(\mathcal{F})=\mathcal{G}(h(\mathcal{F}))=\mathcal{G}(h^{-1}(\mathcal{F}))$ and hence $A=p(h^{-1}(C))\in h^{-1}(\mathcal{F})$. Thus $h(A)\in\mathcal{F}$ and we obtain a contradiction. So the set $\{F\in\mathcal{F}\setminus J_{\mathcal{F}}: h(F)\notin\mathcal{F}\}$ is empty and we have that $h(\mathcal{F}\setminus J_{\mathcal{F}})\subset\mathcal{F}$ for all $h\in N_{S_X}(\mathcal{G}(\mathcal{F}))$, and hence $h^{-1}(\mathcal{F}\setminus J_{\mathcal{F}})\subset\mathcal{F}$. Since $h(J_{\mathcal{F}})=h^{-1}(J_{\mathcal{F}})=J_{\mathcal{F}}$, we have $h(\mathcal{F})\subset\mathcal{F}$ and $h^{-}(\mathcal{F})\subset\mathcal{F}$ and hence $h(\mathcal{F})=\mathcal{F}$, for all $h\in N_S{}^X(\mathcal{G}(\mathcal{F}))$. Thus $N_{S_X}(\mathcal{G}(\mathcal{F}))=\mathcal{G}(\mathcal{F})$ and this completes the proof.

It is independent of ZFC that every uncountable cardinal $\leq 2^{\omega}$ has uncountable cofinality or, in other words, that $2^{\omega}<\omega_{\omega}$.

Corollary 3.1. ($2^{\omega}<\omega_{\omega}$) *If a second-regular strongly homogenous σ-field $\mathcal{F}$ of subsets of an uncountable set X contains a second-countable topology on X, then $N_{S_X}(\mathcal{G}(\mathcal{F}))=\mathcal{G}(\mathcal{F})$ and $\mathcal{G}(\mathcal{F})$ is complete.*

4. The stabilizers of the σ-fields of Borel sets in uncountable Polish spaces. By the Alexandroff-Hausdorff theorem every uncountable Borel set in a Polish space X contains a homeomorphic image of the Cantor discontinuum D^{ω} and hence has the cardinality c of the continuum. Thus $cf(\mathcal{B}_X)>\omega$. By Kuratowski's theorem the σ-field $\mathcal{B}_X$ of Borel sets in an uncountable Polish space X is strongly homogeneous. Thus we deduce from the main theorem the following

Corollary 4.1. (cf. [7]) *The stabilizer of the σ-field of Borel sets in an uncountable absolute Borel space X is a perfect subgroup coinciding with its normalizer in the general symmetric group S_X.*

Remark. By the Alexandroff-Hausdorff theorem and Kuratowski's theorem in an absolute Borel space X we $J_{\mathcal{B}_X}=[X]^{<\omega_1}$.

5. The stabilizers of the σ-fields of Lebesgue-measurable sets in Euclidean spaces.

Lemma 5.1. *Let X be an uncountable Polish space and let $(X,\mathcal{B}_X,\mu)$ be a space with a regular, non-atomic, σ-finite measure μ. If $\bar{\mu}$ is the completion of μ, then the σ-field $\bar{\mathcal{B}}_X$ of $\bar{\mu}$-measurable sets in X is strongly homogeneous.*

Proof. Let $A',B'\in\bar{\mathcal{B}}_X\setminus J_{\bar{\mathcal{B}}_X}$. Then $\bar{\mu}(A')>0$ and $\bar{\mu}(B')>0$, for $\bar{\mu}$ is complete. Since μ is regular, $\bar{\mu}$ is also regular and hence there exist two G_δ-sets A and B in X such that $A'\subset A$, $B'\subset B$, and $\bar{\mu}(A\setminus A')=\bar{\mu}(B\setminus B')=0$. Since $A,B\in\mathcal{B}_X$ and $\mu(A)=\bar{\mu}(A')$, $\mu(B)=\bar{\mu}(B')$, we have $\mu(A)>0$ and $\mu(B)>0$. The subspaces A and B of X and Polish spaces. Since μ is non-atomic, A and B are uncountable and hence $|A|=|B|=c$. If $\mu(A)<\infty$, then we put for each $E\in\mathcal{B}_A$ $\lambda(E)=\mu(E)/\mu(A)$. If $\mu(A)=\infty$, then we can find a sequence $\{P_n\}_{n\in\omega}$ of mutually disjoint Borel subsets of X such that $\bigcup_{n\in\omega}P_n=X$ and $0<\mu(P_n\cap A)\leq\mu(P_n)<\infty$, $n\in\omega$, for μ is σ-finite, and we put for each $E\in\mathcal{B}_A$ $\lambda(E)=\sum_{n=1}^{\infty}\mu(E\cap P_n)/2^n\mu(A\cap P_n)$. Thus, in any case, we obtain a non-atomic probability measure λ on $\mathcal{B}_A$. Similarly we define a non-atomic probability measure ν on $\mathcal{B}_B$. By the isomorphism theorem [8, 26 6] the spaces with measures $(A,\mathcal{B}_A,\lambda)$ and $(B,\mathcal{B}_B,\nu)$ are isomorphic (i. e. there is a Borel isomorphism $\varphi\colon A_1\to B_1$ of the Borel spaces $(A_1,\mathcal{B}_{A_1})$ and $(B_1,\mathcal{B}_{B_1})$ such that $\lambda\varphi^{-1}=\nu$, where $A_1\in\mathcal{B}_A$, $B_1\in\mathcal{B}_B$, and $\lambda(A\setminus A_1)=\nu(B\setminus B_1)=0$). In particular, for $F\in\mathcal{B}_{A_1}$ we have that $\lambda(F)=0$ iff $\nu(\varphi(F))=0$. Let now $B''=B'\cap\varphi(A'\cap A_1)$ and $A''=\varphi^{-1}(B'')\subset A'$. Then $\bar\mu(B'')=\bar\mu(B')$ and $\bar\mu(A'')=\bar\mu(A')$. We may consider that $|A'\setminus A''|=|B'\setminus B''|$, for we can find a set $C\subset A'\cap A_1$, $|C|=c$, such that $C\in\mathscr{B}_{A_1}$, $\lambda(C)=0$, and $\varphi(C)\subset B'$. If we put $B''=B'\cap\varphi((A'\cap A_1)\setminus C)$ and $A''=\varphi^{-1}(B'')$, then $C\subset A'\setminus A''$ and $\varphi(C)\subset B'\setminus B''$, and we have $|A'\setminus A''|=|B'\setminus B''|=c$. It can easily be verified that $\varphi|_{A''}: A''\to B''$ is a bijection such that $\varphi(\mathscr{B}_{A''})=\mathscr{B}_{B''}$. Let $\psi: A'\setminus A''\to B'\setminus B''$ be a bijection. We define $f: A'\to B'$ via: $f|_{A''}=\varphi|_{A''}$ and $f|_{A'\setminus A''}=\psi$. Let now $F\in\bar{\mathscr{B}}_X|_{A'}$ i. e. $F=(M\cup N)\cap A'$, where $M\in\mathscr{B}_X$, $N\subset E\in\mathscr{B}_X$ and $\mu(E)=0$. Then $F=(A''\cap(M\cup N))\cup(A'\setminus A'')\cap(M\cup N)=(A''\cap M)\cup(A''\cap N)\cup((A'\setminus A'')\cap(M\cup N))$. Let $D_1=A''\cap M$, $D_2=A''\cap N$, and $D_3=(A'\setminus A'')\cap(M\cup N)$. Since $D_1\in\mathscr{B}_{A''}$ and $f(D_1)=\varphi(D_1)$, we have $f(D_1)\in\mathscr{B}_{B''}$ and hence $f(D_1)=K\cap B''=K\cap B'\cap\varphi(A'\cap A_1)=K\cap\varphi((L\cup N)\cap A_1)\cap B'$, where $K\in\mathscr{B}_{B_1}$, $L\in\mathscr{B}_X$, $N\subset S\in\mathscr{B}_{A_1}$, and $\mu(S)=0$. Since $(L\cup N)\cap A_1=(L\cap A_1)\cup(N\cap A_1)$ and $L\cap A_1\in\mathscr{B}_{A_1}$, we have $\varphi(L\cap A_1)\in\mathscr{B}_{B_1}$. Since $N\cap A_1\subset S$, we have $\varphi(N\cap A_1)\subset\varphi(S)\in\mathscr{B}_{B_1}$ and hence $\mu(\varphi(S))=0$, for $\mu(\varphi(S))=0$ is equivalent to $\nu(\varphi(S))=0$, which in turn is equivalent to $\lambda(S)=0$ or $\mu(S)=0$. Thus $f(D_1)=K\cap(\varphi(L\cap A_1)\cup\varphi(N\cap A_1))\cap B'=((K\cap\varphi(L\cap A_1))\cup(K\cap\varphi(N\cap A_1)))\cap B'\in\bar{\mathscr{B}}_X|_{B'}$. Since $D_2\subset E\in\mathscr{B}_{A_1}$ and $\mu(E)=0$, we have $f(D_2)=\varphi(D_2)\subset\varphi(E)$ and $\mu(\varphi(E))=0$, for this is equivalent to $\nu(\varphi(E))=0$, which in turn is equivalent to $\lambda(E)=0$ or $\mu(E)=0$. Since $D_3\subset A'\setminus A''$, $f(D_3)\subset f(A'\setminus A'')=\varphi(A'\setminus A'')=B'\setminus B''$ and since $\bar\mu(B'\setminus B'')=0$, there is a set $T\in\mathscr{B}_X$ such that $\mu(T)=0$ and $B'\setminus B''\subset T$. Thus $f(D_2\cup D_3)\subset\varphi(E)\cup T$ and $\mu(\varphi(E)\cup T)=0$ and hence $f(F)\in\bar{\mathscr{B}}_X|_{B'}$. So $f(\bar{\mathscr{B}}_X|_{A'})\subset\bar{\mathscr{B}}_X|_{B'}$. Similarly one can show that $f^{-1}(\bar{\mathscr{B}}_X|_{B'})\subset\bar{\mathscr{B}}_X|_{A'}$ and hence $f(\bar{\mathscr{B}}_X|_{A'})=\bar{\mathscr{B}}_X|_{B'}$. This completes the proof.

Theorem 5.2. *The stabilizer of the σ-field of $\bar\mu$-measurable sets with respect to the completion $\bar\mu$ of a regular, non-atomic, σ-finite Borel measure μ in an uncountable Polish space X is a complete subgroup of S_X coinciding with its normalizer in S_X.*

Proof. By lemma 5.1. the σ-field $\bar{\mathscr{B}}_X$ of $\bar\mu$-measurable sets in X is strongly homogeneous. If $F\in\bar{\mathscr{B}}_X\setminus J_{\bar{\mathscr{B}}_X}$, then $F=B\cup N$, where $B\in\mathscr{B}_X$ and $N\subset E$ for some $E\in\mathscr{B}_X$ with $\mu(E)=0$ and $|B|>\omega$. Therefore $|F|=c$ and we have that $cf(\bar{\mathscr{B}}_X)>\omega$. Now we can apply the main theorem.

Remark. Since $|\mathscr{B}_X|=c$, there exists a set $Y\subset X$ that meets every uncountable Borel set $B\in\mathscr{B}_X$ (together with its complement $X\setminus Y$) [2]. For every $\bar\mu$-measurable subset $F\subset Y$ (or $F\subset X\setminus Y$) $\bar\mu(F)=0$ and hence Y is not $\bar\mu$-measurable. Therefore, if $\bar\mu(A)>0$, then one of the sets $A\cap Y$ and $A\cap(X\setminus Y)$ is not $\bar\mu$-measurable so $A\notin J_{\bar{\mathscr{B}}_X}$. Thus we have $J_{\bar{\mathscr{B}}_X}=\{E\in\bar{\mathscr{B}}_X:\bar\mu(E)=0\}$.

Corollary 5.3. *The stabilizer of the σ-field of Lebesgue-measurable sets in Euclidean space E is a complete subgroup in the symmetric group S_E coinciding with its normalizer in S_E.*

6. The stabilizers of the σ-fields of sets with the Baire property in Polish spaces.

Lemma 6.1. *The σ-field $\mathscr{B}P_X$ of the sets with the Baire property in an uncountable Polish space X is strongly homogeneous.*

Proof. Let $A', B'\in\mathscr{B}P_X\setminus J_{\mathscr{B}P_X}$. Let $[X]^{<\omega_1}$ be the σ-ideal of countable sets in X, let $\mathfrak{M}_1$ be the σ-ideal of the sets of the first category (or meager sets) in X, and let Is_X be the set of all isolated points in X. It is easily seen that $[X]^{<\omega_1}\subset\{C\cup S:C\in\mathfrak{M}_1$ and $S\subset Is_X\}\subset J_{\mathscr{B}P_X}$ and hence A' and B' are uncountable sets of the second category in X. Let us consider two G_δ-sets A and B in X such that $A\subset A'$, $B\subset B'$ and $A'\setminus A$, $B'\setminus B\in\mathfrak{M}_1$. Since A and B are uncountable Borel sets in X, they contain a copy of the Cantor discontinuum D^ω and hence $|A|=|B|=c$. Clearly A and B are separable completely metrizable spaces and hence they are homeomorphic to G_δ-subsets of the Hilbert cube $[0,1]^\omega$. Let I be the subspace of irrationals in $[0,1]$. We shall show that the Borel spaces $([0,1]^\omega,\mathscr{B}P_{[0,1]^\omega})$ and $(I, \mathscr{B}P_I)$ are Borel isomorphic. Let us consider the space $M=\{c_1/3+\cdots+c_n/3^n+\cdots: c_n=0 \text{ or } 2\}\subset D^\omega$ homeomorphic to I and let $t: M\to[0,1]$ be defined via $t(c_1/3+\cdots+c_n/3^n+\cdots)=c_1/2^2+\cdots+c_n/2^{n+1}+\cdots$. Then t is a continuous bijection such that t^{-1} has a countable set of discontinuity points. Let $f: I\to[0,1]$ be a continuous bijection such that $f|_{I\setminus C}$ is a homeomorphism, where $C\subset I$ and $|C|=\omega$. If we consider the map $f^\omega: I^\omega\to[0,1]^\omega$ $(x_0, x_1,\ldots)\to(f(x_0), f(x_1),\ldots)$ then $f^\omega|_{I^\omega\setminus C^\omega}$ is a homeomorphism and C^ω is obviosly a set of the first category in I^ω, for I is dense in itself. The set $f^\omega(C^\omega)$ is also a set of the first category in $[0,1]^\omega$. Since I^ω is homeomorphic to I, we obtain a bijection $\varphi: I\to[0,1]$ such that $\varphi|_{I\setminus F}$ is a homeomorphism, where F is of the first category in I and $\varphi(F)$ is of the first category in $[0,1]^\omega$. Therefore φ is a Borel isomorphism of the Borel spaces $(I, \mathscr{B}P_I)$ and $([0,1]^\omega, \mathscr{B}P_{[0,1]^\omega})$. If now A and B are two G_δ-sets in $[0,1]^\omega$ and $|A|=|B|=c$, then $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$ are separable, completely metrizable, uncountable, O-dimensional spaces and hence $\varphi^{-1}(A)=N_1\cup I_1$ and $\varphi^{-1}(B)=N_2\cup I_2$, where $|N_1|=|N_2|\leq\omega$ and I_1, I_2 are homeomorphic to I [6, § 36, IV]. Thus the Borel spaces $(\varphi^{-1}(A), \mathscr{B}P_{\varphi^{-1}(A)})$ and $(\varphi^{-1}(B), \mathscr{B}P_{\varphi^{-1}(B)})$ are Borel isomorphic and so does the input A and B. It can easily be verified that the Borel spaces $(A', \mathscr{B}P_{A'})$ and $(B', \mathscr{B}P_{B'})$ are also Borel isomorphic and this completes the proof of the lemma.

Remark. The existence of a Bernstein's set Y in X that meets together with its complement every uncountable Borel set in X implies that $J_{\mathscr{B}P_X}=\{C\cup S: C\in\mathfrak{M}_1$ and $S\subset Is_X\}$. Indeed, every set $B\subset Y\setminus Is_X$ with the Baire property in X is of the first category in X and the same holds for $X\setminus Y$. Therefore, if $A\in\mathscr{B}P_X\setminus\{C\cup S: C\in\mathfrak{M}_1$ and $S\subset Is_X\}$, then one of the sets $A\cap Y$, $A\cap(X\setminus Y)$ does not have the Baire property and hence $A\notin J_{\mathscr{B}P_X}$. Thus we obtain that $J_{\mathscr{B}P_X}=\{C\cup S: C\in\mathfrak{M}_1$ and $S\subset Is_X\}$.

Theorem 6.2. *The stabilizer of the σ-field of sets with the Baire property in an uncountable Polish space X is a subgroup of S_X coinciding with its own normalizer in S_X.*

Proof. By lemma 6.1 the σ-field $\mathscr{B}P_X$ of sets with the Baire property in X is strongly homogeneous. If $B\in\mathscr{B}P_X\setminus J_{\mathscr{B}P_X}$, then B contains an uncountable G_δ-set in X and hence $|B|=c$. Thus one have $cf(\mathscr{B}P_X)>\omega$ and we can apply the main theorem.

Corollary 6.3. *The stabilizer of the σ-field of sets with the Baire property in an Euclidean space E is a complete subgroup in the symmetric group S_E coinciding with its normalizer in S_E.*

In conclusion we give a characterization of the elements of the stabilizer $\mathscr{G}(\mathscr{B}P_X)$ for a Polish (in fact second-countable) space X. We may assume that the set of non-isolated points in X is of the second category in X, for otherwise $\mathscr{B}P_X=\mathscr{P}(X)$ and $\mathscr{G}(\mathscr{B}P_X)=S_X$.

Proposition 6.4. *Let X and Y be two second-countable topological spaces such that $\mathscr{B}P_X\neq\mathscr{P}(X)$ and $\mathscr{B}P_Y\neq\mathscr{P}(Y)$. Then a bijection $f: X\to Y$ is a Borel isomorphism of the Borel spaces $(X, \mathscr{B}P_X)$ and $(Y, \mathscr{B}P_Y)$ iff there is a set $A\in J_{\mathscr{B}P_X}$ such that $f(A)\in J_{\mathscr{B}P_Y}$ and $f|_{X\setminus A}$ is a homeomorphism.*

Proof. Let f be a Borel isomorphism of the Borel spaces $(X, \mathscr{B}P_X)$ and $(Y, \mathscr{B}P_Y)$ and let $\{B_i\}_{i\in\omega}$ be a countable base of open sets in Y. Since $f^{-1}(B_i)\in\mathscr{B}P_X$, we have $f^{-1}(B_i)=(U_i\setminus M_i)\cup N_i$, where U_i is open in X and $M_i, N_i\in\mathfrak{M}_1$, $i\in\omega$. Let A'

$=\bigcup_{i\in\omega}(M_i\cup N_i)$. It is easily seen that $f|_{X\setminus A}$ is continuous. Indeed, if V is an open set in Y, then $V=\bigcup_{k\in\omega}B_{i_k}$ and $f|^{-1}_{X\setminus A}(V)=\bigcup_{k\in\omega}f^{-1}(B_{i_k})\setminus A'=\bigcup_{k\in\omega}(U_{i_k}\setminus M_{i_k})\cup N_{i_k}\setminus A'$ $=\bigcup_{k\in\omega}U_{i_k}\setminus A'$, for $M_{i_k}\cup N_{i_k}\subset A'$, $k\in\omega$. Thus $f|^{-1}_{X\setminus A}(V)$ is open in $X\setminus A'$. Similarly, making use of a countable base in X, one can find a set B' of the first category in Y such that $f^{-1}|_{B'}$ is continuous. Since, J is a Borel isomorphism, $f(J_{\mathscr{B}P_X})=J_{\mathscr{B}P_Y}$ and, since $A\in_{\mathscr{B}P_X}$, $B\in J_{\mathscr{B}P_Y}$, we have $f(A)\in J_{\mathscr{B}P_Y}$ and $f^{-1}(B)\in J_{\mathscr{B}P_X}$. Therefore $A=A'\cup f^{-1}(B')$ $\in J_{\mathscr{B}P_X}$ and $f(A)=B'\cup f(A')\in J_{\mathscr{B}P_Y}$, and $f|_{X\setminus A}$: $X\setminus A\to Y\setminus f(A)$ is a homeomorphism.

Let $A\in J_{\mathscr{B}P_X}$ such that $f(A)\in J_{\mathscr{B}P_Y}$ and $f|_{X\setminus A}$ is a homeomorphism. For each $B\in\mathscr{B}P_X$ we have $B\cap(X\setminus A)\in\mathscr{B}P_{X\setminus A}$. Since $f|_{X\setminus A}$ is obviously a Borel isomorphism of the Borel spaces $(X\setminus A, \mathscr{B}P_{X\setminus A})$ and $(Y\setminus f(A), \mathscr{B}P_{Y\setminus f(A)})$, we have $f(B\setminus A)$ $\in\mathscr{B}P_{Y\setminus f(A)}$. Since $Y\setminus f(A)\in\mathscr{B}P_Y$, we have $\mathscr{B}P_{Y\setminus f(A)}\subset\mathscr{B}P_Y$ and hence $f(B\setminus A)\in\mathscr{B}P_Y$. In view of $f(B\cap A)\subset f(A)\in J_{\mathscr{B}P_Y}\subset\mathscr{B}P_Y$ we have $f(B\cap A)\in\mathscr{B}P_Y$. Therefore $f(B)$ $=f(B\setminus A)\cup f(B\cap A)\in\mathscr{B}P_Y$. Thus $f(\mathscr{B}P_X)\subset\mathscr{B}P_Y$ and similarly one can show that $f^{-1}(\mathscr{B}P_Y)\subset\mathscr{B}P_X$. This completes the proof.

Corollary 6.5. *Let X and Y be two second-countable dense in itself topological spaces of the second category. Then a bijection $f\colon X\to Y$ is a Borel isomorphism of the Borel spaces $(X, \mathscr{B}P_X)$ and $(Y, \mathscr{B}P_Y)$ iff there is a meager set A in X such that $f(A)$ is a meager set in Y and $f|_{X\setminus A}$ is a homeomorphism.*

Corollary 6.6. *Let X be an uncountable Polish space. Then a bijection $f\in S_X$ belongs to $\mathscr{G}(\mathscr{B}P_X)$ iff there is a set $A\subset X$ such that $A\cup f(A)\in J_{\mathscr{B}P_X}$ and $f|_{X\setminus(A\cup f(A))}$ is an autohomeomorphism of the subspace $X\setminus(A\cup f(A))\subset X$.*

Corollary 6.7. *Let X be an uncountable dense in itself Polish space and $f\in S_X$. Then $f\in\mathscr{G}(\mathscr{B}P_X)$ iff there is a set $A\subset X$ such that $A\cup f(A)$ is a meager set in X and $f|_{X\setminus(A\cup f(A))}$ is an autohomeomorphism of $X\setminus(A\cup f(A))$.*

I wish to express my gratitude to V. I. Ponomarev and M. M. Čoban for valuable discussions.

REFERENCES

1. P. S. Alexandroff. Sur la puissance des ensembles mesurables (B). *C. R. Acad. Sci. Paris*, 1916, 323-327.
2. F. Bernstein. *Ber. Verh. K. Sachs. Ges. Wiss.*, **60**, 1908.
3. O. Hölder. Bildung zusammengesetzter Gruppen. *Math. Ann.*, **46**, 1895, 321-422.
4. F. Hausdorff. *Math. Ann.*, **77**, 1916.
5. P. R. Halmos. Measure theory. New York, 1950.
6. K. Kuratowski. Topology, v. 1. New York, 1966.
7. E. R. Lorch, Hing Tong. On the automorphisms of the group of Baire equivalences of a complete separable metric space. *Ball. Inst. Math. Acad. Sinica*, **6**, 1978, No. 2, part 1, 333-336.
8. K. R. Parthasarathy. Introduction to probability and measure, New York, 1980.
9. M. Suzuki. Group Theory I.Berlin, 1982.
10. R. Sikorski. Boolean algebras. Berlin, 1964.
11. I. Schreier, S. Ulam. Über die Automorphismen der Permutationsgruppe der natürlichen Zahlenfolge. *Fund. Mand.* **28**, 1937, 258-260.
12. H. Wielandt. Unendliche Permutationsgruppen. Tübingen, 1960.

Department of Algebra and Geometry
Kemerovo State University
Kemerovo 650043, USSR

Received 25. 2. 1985
Revised 5. 11. 1987