Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

DIFFERENCE METHOD FOR THE LINEAR DYNAMICAL PROBLEM OF COUPLED THERMOELASTICITY

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A family of difference schemes for the linear dynamical problem of coupled thermoelasticity is constructed. The error estimate of the approximate solution is obtained. Some numerical results are given and discussed.

1. Introduction. Thermoelasticity deals with processes in which, on one hand, deformation and stresses are produced not only by mechanical forces, but by temperature variation as well, on the other hand, deformation acts as a source or sink of heat. So, in thermoelastic problems the mechanical and thermal aspects are coupled and inseparable.

In practical applications it is usually permissible to disregard the influence of coupling and to evaluate the temperature and deformation fields separately. This assertion, however, is not always true. First of all, for some synthetic materials, such as plastics, the effect of coupling may not necessarily be negligible. The coupling also plays a noticeable part in the phenomena of wave propagation and thermoelastic damping.

Although the foundations of thermoelasticity have been laid in the first half of the nineteenth century by Duhamel [1] and Neumann, only during the last four decades the theory reached a certain completeness and many applications to engineering problems have been successfully made.

The investigations of thermoelastic problems have brought forth numerous theoretical and experimental publications [2—7]. They represent various facets of the theory of thermoelasticity and some analytical approaches are applied for studying the raised problems.

Very often rigorous solutions cannot be obtained and one must resort to approximate methods or to numerical procedures. In recent years considerable attention has been paid to the numerical treatment of thermoelastic problems [8—14].

2. Basic equations of thermoelasticity. For the one-dimensional case the governng equations of thermoelasticity are:

$$(1)\qquad \rho\frac{\partial^2 u}{\partial t^2}-(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}+(3\lambda+2\mu)\alpha\frac{\partial\theta}{\partial x}=F_1(x,t), \qquad \text{(motion)}$$

$$(2)\qquad c\rho\frac{\partial\theta}{\partial t}-\frac{\partial}{\partial x}\left(k\frac{\partial\theta}{\partial x}\right)+(3\lambda+2\mu)\alpha T_0\frac{\partial^2 u}{\partial x\partial t}=F_2(x,t), \qquad \text{(heat transfer)}$$

where x is the space variable, t — time, u — displacement, $\theta=T-T_0$ — the temperature increment, T — absolute body temperature, T_0 — the temperature of the natural state, ρ — density, λ and μ — Lamé elastic constants, α — coefficient of linear thermal expansion, c — specific heat, k — heat conduction coefficient, F_1 — body forces and F_2 — heat sources.

Supplemented by appropriate initial and boundary conditions, the equations (1), (2) represent the entire mathematical model of the thermoelasticity.

SERDICA Bulgaricae mathematicae publicationes. Vol. 14, 1988, p. 161—167.

The presence of the derivative $\partial^2 u/\partial x\partial t$ in equation (2) is a sign of the coupling, existing between the temperature and deformation fields. A second such sign is visible in equation (1) in the form of temperature gradient. The existence of coupling implies that the solution of the system (1), (2) must proceed simultaneously. This fact contributes considerably to the complexity of the coupled thermoelasticity problems.

The purpose of the present paper is first, to give an approximation of the dynamic problem of coupled thermoelasticity based on the replacement of the fundamental differential equations by corresponding difference equations and, secondly, to present some numerical results.

Our attention will be centered on one-dimensional problems. It is convenient to write the governing equations (1), (2) in a dimensionless form:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+a_1\frac{\partial \theta}{\partial x}=F_1(x,t), \tag{3}$$

$$\frac{\partial \theta}{\partial t}-\frac{\partial^2 \theta}{\partial x^2}+a_2\frac{\partial^2 u}{\partial x\partial t}=F_2(x,t), \tag{4}$$

where

$$a_1=(3\lambda+2\mu)\,\alpha\,T_0/(\lambda+2\mu),\quad a_2=(3\lambda+2\mu)\,\alpha/c\rho$$

and the same letter is retained to denote the corresponding dimensionless quantity.

The parameter $\eta=a_1 a_2$ is called a coupling parameter. For some materials $\eta\ll 1$. For example, according to [5] the data for four common metals at 20°C are:

Aluminium	3.56×10^{-2};
Iron (Steel)	1.14×10^{-2};
Copper	1.68×10^{-2};
Lead	7.33×10^{-2}.

3. Method of finite differences. Finite difference method is an effective method for investigating problems of mathematical physics.

The construction of a difference scheme which approximates a given differential problem is usually performed in two steps: 1) discretisation of the domain of independent variables; 2) replacement of the differential equations by corresponding difference equations and also formulation of difference analogous to the boundary and initial conditions. The set of difference equations approximating the differential equations and boundary and initial conditions is called a difference scheme. In fact it is a system of algebraic equations.

To the system (3), (4) we associate homogenous boundary conditions

$$u(0,t)=u(1,t)=0,\qquad \theta(0,t)=\theta(1,t)=0,\quad t>0 \tag{5}$$

and the following initial conditions

$$u(x,0)=f(x),\ \frac{\partial u}{\partial t}(x,0)=\varphi(x),\ \theta(x,0)=\psi(x),\ x\in(0,1), \tag{6}$$

where $f(x)$, $\varphi(x)$, $\psi(x)$ are given real functions.

Consider in the domain $\bar{D}=\{0\leq x\leq 1,\ 0\leq t\leq t^*\}$ the mesh $\bar{\omega}$ of points with coordinates $(x_i=ih,\ t_j=j\tau)$, i, j being integers, τ and h being time step and the step in x-direction. More precisely $\bar{\omega}=\bar{\omega}_h\times\bar{\omega}_\tau$, where $\bar{\omega}_h$ is the mesh in the x-direction and $\bar{\omega}_\tau$ is the time mesh defined by

$$\bar{\omega}_h=\{x_i=ih,\ i=0,1,\ldots,N_1,\ h=1/N_1\},$$
$$\bar{\omega}_\tau=\{t_j=j\tau,\ j=0,1,\ldots,N_2,\ \tau=t^*/N_2\}.$$

Let's recall the nonsubscript finite difference notations and formulas introduced in [15]:

$$y(x, t_j)=y^j=y, \quad y(x, t_{j+1})=y(x, t_j+\tau)=\widehat{y},$$

$$y(x, t_{j-1})=y(x, t_j-\tau)=\check{y}, \quad y_{\bar{t}}=(y-\check{y})/\tau,$$

$$v_t=(\widehat{y}-y)/\tau, \quad y_{\bar{t}t}=(\widehat{y}-2y+\check{y})/\tau^2, \quad y_{\overset{\circ}{t}}=(y_t+y_{\bar{t}})/2,$$

$$y(x_i, t)=y_i, \quad y(x_{i\pm1}, t)=y(x_i\pm h, t)=y_{i\pm1},$$

$$y_{\bar{x}}=(y_j-y_{i-1})/h, \quad y_x=(y_{i+1}-y_i)/h,$$

$$y_{\overset{\circ}{x}}=(y_{i+1}-y_{i-1})/2h=(y_x+y_{\bar{x}})/2,$$

$$y_{\bar{x}x}=(y_{i+1}-2y_i+y_{i-1})/h^2=(y_x-y_{\bar{x}})/h.$$

For every fixed value of $t\in\omega_\tau$ we introduce the finite dimensional Hilbert space $\overset{\circ}{H}_h=L_h^2(\bar{\omega}_a)$ of the mesh functions defined on $\bar{\omega}_h$ and vanishing at the boundary points x_0, x_{N_1} with the following inner product and norm:

$$(v, p)=\sum_{x\in\omega_h} v(x)\,p(x)\,h, \quad \|v\|^2=(v, v).$$

We introduce also the Hilbert space $\overset{\circ}{H}_{hA}$ with the energy inner product $(v, p)_A=(Av, p)$ and the corresponding energy norm $\|v\|_A^2=(v, v)_A$ which is equivalent to the first given norm. The operator A is an arbitrary self-adjoint and positive defined discrete operator: $A=A^*>0$.

Making use of the theory (see e. g. [16]) the following two-parametric family of difference schemes approximating the problem (3)—(6) is constructed:

$$\begin{aligned} &y_{\bar{t}t}+Ay^{(\sigma,\sigma)}+a_1Mw=\Phi_1(x, t);\\ (7)\qquad &w_t+Aw^{(\beta)}+a_2\,My_{\overset{\circ}{t}}=\Phi_2(x, t), \quad (x, t)\in\omega\\ &y_0=y_{N_1}=0, \quad w_0=w_{N_1}=0,\\ &y_i^0=f_i, \quad y_{t,i}=\varphi_i, \quad w_i^0=\psi_i.\end{aligned}$$

For convenience the abbreviations $Av=-v_{\bar{x}x}$, $Nv=v_{\overset{\circ}{x}}$, $v^{(\beta)}=\beta\widehat{v}+(1-\beta)\,v$, $v^{(\sigma,\sigma)}=\sigma(\widehat{v}+\check{v})+(1-2\sigma)\,v$ are used.

The scheme (7) can be obtained approximating the integral identity which corresponds to the generalized solution of the considered problem:

$$\int_0^1\left(\frac{\partial^2u}{\partial t^2}v+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}-a_1\frac{\partial v}{\partial x}\theta+\frac{\partial\theta}{\partial x}p+\frac{\partial\theta}{\partial x}\frac{\partial p}{\partial x}+a_2\frac{\partial^2u}{\partial x\partial t}p\right)dx=\int_0^1(F_1\,v+F_2\,p)\,dx,$$

where $v(x)$ and $p(x)$ are arbitrary continuous functions with integrable derivativs in $L_2(0, 1)$.

The time derivative $\partial^2u/\partial t^2$ is approximated on the mesh by the usual centred difference representation, The temperature gradient $\partial\theta/\partial x$ is approximated by first order difference.

Varying the weights $0\leq\sigma$, $\beta\leq1$ various schemes can be obtained. If the solution of the differential problem (3)—(6) is sufficiently smooth the approximation error of the scheme (7) is: $O(\tau+h^2)$ for $\Phi_2=F_2$ or $\Phi_2=F_2+O(\tau+h^2)$.

It can be proved that the scheme (7) is stable in the sense of

Theorem 1. *Assume that the inequalities*

$$\sigma \geq 0.25 - 0.25\, h^2/\tau^2, \quad \beta > 0.5 \tag{8}$$

are satisfied. Then the following a priori estimate holds for the approximate solution (y, w):

$$\| y^j \|_*^2 + \| w^{j+1} \|^2 + \tau \| w^{j+1} \|_A^2 + \tau \sum_{n=1}^{j} \| 0.5 (w^{n+1} + w^n) \|_A^2 \tag{9}$$

$$\leq C_1 \{ \| y^0 \|_*^2 + \| w^1 \|^2 + \tau \| w^1 \|_A^2 + \sum_{n=1}^{i} \tau (\| \Phi_1^n \|^2 + \| \Phi_2^n \| \|_{A^{-1}}^2)\},$$

where $\| \cdot \|_*$ is defined with

$$\| y^n \|_* = \| y_t^n \|^2 + \tau^2 (\sigma - 0.25) \| y_t^n \|_A^2 + 0.25 \| y^{n+1} + y^n \|_A^2 \tag{10}$$

and C_1 is a positive constant independent on the data functions, the mesh steps and the solution (y, w).

Proof. The method of "energy" inequalities will be used. Making use of the formulas $y^{(\sigma,\sigma)} = 0.5(\hat{y} + \check{y}) + \tau^2(\sigma - 0.5) y_{\bar{t}t}$, $w^{(\beta)} = w + \beta\tau w_t$ the difference equations (7_1) and (7_2) can be written in the form

$$(E + \tau^2(\sigma - 0.5)A) y_{\bar{t}t} + 0.5\, A(\hat{y} + \check{y}) + a_1 M w = \Phi_1.$$

$$(E + \beta\tau A) w_t + Aw + a_2 M y_t = \Phi_2.$$

Now we multiply the first equation by $2\tau y_{\overset{\circ}{t}} = \tau(y_t + y_{\bar{t}}) = \hat{y} - \check{y}$, the second equation by $2\tau\hat{w}$ and then we add the resulting equations:

$$((E + \tau^2(\sigma - 0.5)A)(y_t - y_{\bar{t}}),\ y_t + y_{\bar{t}}) + 0{,}5\, (A(\hat{y} + \check{y}),\ \hat{y} - \check{y})$$

$$+ 2a_1\tau(Mw,\ y_{\overset{\circ}{t}}) + 2\tau((E + \beta\tau A) w_t,\ \hat{w}) + 2\tau(Aw,\ \hat{w}) \tag{11}$$

$$+ 2a_2\tau(My_t,\ \hat{w}) = 2\tau(\Phi_1,\ y_{\overset{\circ}{t}}) + 2\tau(\Phi_2,\ \hat{w}).$$

As A is a self-adjoint operator we obtain the relations (see e. g. [16]):

$$((E + \tau^2(\sigma - 0.5)A)(y_t - y_{\bar{t}}),\ y_t + y_{\bar{t}}) = ((E + \tau^2(\sigma - 0.5)A) y_t,\ y_t)$$

$$-((E + \tau^2(\sigma - 0.5)A) y_{\bar{t}},\ y_{\bar{t}}),$$

$$(A(\hat{y} + \check{y}),\ \hat{y} - \check{y}) = 0.5\,[(A(\hat{y} + y),\ \hat{y} + y) + \tau^2(Ay_t,\ y_t)] - 0.5\,[(A(y + \check{y}),\ y + \check{y}) + \tau^2(Ay_{\bar{t}},\ y_{\bar{t}})].$$

Substituting this relations into (11), we arrive at the "energy" identity:

$$((E + \tau^2(\sigma - 0.25)A) y_t,\ y_t) + 0.25(A(\hat{y} + y),\ \hat{y} + y) + 2\tau((E + \beta\tau A) w_t,\ \hat{w})$$

$$+ 2\tau(Aw,\ \hat{w}) + 2a_1\tau(Mw,\ y_{\overset{\circ}{t}}) + 2a_2\tau(My_t,\ \hat{w}) \tag{12}$$

$$= ((E + \tau^2(\sigma - 0.25)A) y_{\bar{t}},\ y_{\bar{t}}) + 0.25(A(y + \check{y}),\ y + \check{y}) + 2\tau(\Phi_1\ y_{\overset{\circ}{t}}) + 2\tau(\Phi_2,\ \hat{w}).$$

The evaluation of the terms in (12) has been performed using the Cauchy-Schwarz inequality and the ε-inequality. We omit details, giving the final results:

$$2\tau((E+\beta\tau A)w_t,\ \widehat{w})=\|\widehat{w}\|^2-\|w\|^2+\beta\tau(\|\widehat{w}\|_A^2-\|w\|_A^2)+\tau^2\|w_t\|^2+\beta\tau^3\|w_t\|_A^2,$$

$$2\tau(Aw,\ \widehat{w})=2\tau\|0.5(\widehat{w}+w)\|_A^2-0.5\tau^3\|w_t\|_A^2,$$

$$2\tau(Mw, y_{\mathring{t}})\leq\tau\varepsilon_1\|0.5(\widehat{w}+w)\|_A^2+(\tau/\varepsilon_1)\|y_{\mathring{t}}\|^2+\varepsilon_2\tau^3\|w_{t\mathring{x}}\|^2+(\tau/4\varepsilon_2)\|y_{\mathring{t}}\|^2,$$

$$2\tau(My_t,\ \widehat{w})\leq\tau\varepsilon_3\|0.5(\widehat{w}+w)\|_A^2+(\tau/\varepsilon_3)\|y_t\|^2+\varepsilon_4\tau^3\|w_{t\mathring{x}}\|^2+(\tau/4\varepsilon_4)\|y_t\|^2,$$

$$2\tau(\Phi_2,\ \widehat{w})\leq\tau\varepsilon_5\|0.5(\widehat{w}+w)\|_A^2+(\tau/\varepsilon_5)\|\Phi_2\|_{A^{-1}}^2+\varepsilon_6\tau^3\|w_t\|_A^2+(\tau/4\varepsilon_6)\|\Phi_2\|_{A^{-1}}^2,$$

$$2\tau(\Phi_1,\ y_{\mathring{t}})\leq\tau\varepsilon_7\|\Phi_1\|^2+(\tau/4\varepsilon_7)\|y_{\mathring{t}}\|^2.$$

Here $\varepsilon_i(i=\overline{1,7})$ are arbitrary real numbers.

Substituting these estimations into (12) and after some rearrangements, we arrive at the inequality

$$\begin{aligned}&\|y\|_*^2+\|\widehat{w}\|^2+\tau^2\|w_t\|^2+\beta\tau\|\widehat{w}\|_A^2+\tau^3(\beta-0.5-|a_1|\varepsilon_2-\varepsilon_4-|a_2|\varepsilon_6)\|w_t\|_A^2\\ &+\tau(2-|a_1|\varepsilon_1-|a_2|\varepsilon_3-\varepsilon_5)\|0.5(\widehat{w}+w)\|_A^2\\ &\leq\|\check{y}\|_*^2+\|w\|^2+\beta\tau\|w\|_A^2+\varepsilon'\tau\|y_t\|^2+|a_2|\tau(\varepsilon_5^{-1}+\varepsilon_6^{-1})\|\Phi_2\|_{A^{-1}}^2+\varepsilon_7\tau\|\Phi_1\|^2,\end{aligned}\tag{13}$$

where $\varepsilon'=\varepsilon_1^{-1}+\varepsilon_3^{-1}+0.25(\varepsilon_2^{-1}+\varepsilon_4^{-1}+\varepsilon_7^{-1})$.
Now let's choose ε_1, ε_3 and ε_5 so that $2-|a_1|\varepsilon_1-|a_2|\varepsilon_3-\varepsilon_5>0$ and let's assume that the assumptions of the theorem are fulfilled. To end the proof we sum up (13) for $n=1, 2, \dots, j$ and refer to the Gronwall's lemma. Thus the desired estimation (9) is obtained.

Whenever the conditions of stability (8) are fulfilled and the differential problem (3)—(6) has a solution then the difference problem (7) also has a solution converging to the exact solution. Theorem 1 can be used for estimating the rate of convergence.

The error vector $z=(z_1, z_2)=(y-u, w-\theta)$ is the solution of a problem which is analogous to the original problem (3)—(6). So the estimation (9) holds for (z_1, z_2). Making use of this and by means of a Sobolev's imbedding theorem, it can be shown that

Theorem 2. *Assume that $u(x, t)$, $\theta(x, t)$ belong to the classes $W_2^{6,6}(D)$ and $W_2^{6,3}(D)$, respectively. Then the following estimate holds for the error vector:*

$$\|z_1\|_*+\|z_2\|_{**}\leq C_2(h^2+\tau),\tag{14}$$

where $\|v^j\|_{**}^2=\|v^{j+1}\|^2+\tau\|v^{j+1}\|_A^2+\tau\sum_{n=1}^{j}\|0.5(v^{n+1}+v^n)\|_A^2$ *and $C_2>0$ doesn't depend on the data functions and the solution of the differential problem* (3)—(6).

So the rate of convergence of the approximate solution to the exact solution is of second order with respect to h and of first order with respect to τ.

Note that in [15] the following stability condition for the heat transfer equation was obtained:

$$\beta\geq0.5-0.25h^2/\tau.$$

Thit difference (see (8_2)) and the presence of additional temperature dependent terms in the a priori estimate (9) are due the coupling terms in the heat transfer and motion equations.

4. Numerical results. To illustrate the foregoing results and to examine the algorithmic properties of the constructed family of schemes some numerical evaluations of the temperature and displacement are carried out. As a convenient test an exact similarity solution is used [17].

Table 1 shows the maximum deviation of the approximate solution (y, w) from the corresponding exact solution (u_e, θ_e).

Table 1

Mesh	N_1	h	τ	$\max \varepsilon_u = \max\|(y-u_e)/u_e\|$	$\max \varepsilon_\theta = \max\|(w-\theta_e)/\theta_e\|$
I	11	0.1	0.1	5.0 %	1.5 %
II	21	0.05	0.025	2.7 %	0.5 %
III	41	0.025	0.025	1.7 %	0.4 %

The approximate solution is seen to be in fair agreement with the exact solution. Therefore the considered two-parametric family of difference schemes could be recommended for numerical treatment of practical thermoelastic problems.

The influence of the coupling parameter $\eta = a_1 a_2$ on the accuracy of the computaons is shown in Table 2.

Table 2

N_1	h	τ	a_1	a_2	ε_u (%)	ε_θ (%)
			0.5	0	4.4	0.2
			0.5	0.04	5	2.7
11	0.1	0.1	0.1	0.04	4	0.17
			0.5	0.5	8.5	3.5
			1	1	80	70

It can be observed that the increasing of η above real practical values yields increasing of the error of the approximate solution (it has already been mentioned that $\eta \ll 1$).

So, one should restrict the use of the scheme (7) to cases in which $\eta \ll 1$.

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Received 4. 12. 1986