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ON A CLASS OF STRONGLY ACYCLIC MAPS

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1. Introduction. In this note we define a new class of multi-valued maps which we call s t r o n g l y a c y c l i c m a p s (SA-maps). We show that the set of homotopy classes of strongly acyclic maps from a compact, finite dimensional CW-complex into the unit sphere S^n in the euclidean $(n+1)$-dimensional space R^{n+1} is exactly the same as the set of homotopy classes of single-valued continuous maps. In what follows the topological spaces are assumed to be normal and Hausdorff. $\Pi=[0, 1]$ is the unit interval in R.

First, we recall the definitions and some properties of multi-valued maps. Given two spaces X and Y the symbol $\varphi: X\to Y$ will stand for a multi-valued map from X to Y such that each set $\varphi(x)$ is non-empty for all $x\in X$; the single-valued maps will be denoted by $f, g, h, \ldots$. Given $\varphi: X\to Y$ the graph of φ is $\Gamma(\varphi)=\{(x, y)\in X\times Y : y\in\varphi(x)$. The multi-valued map $\varphi: X\to Y$ is called an o p e n (c l o s e d)-g r a p h m a p if the graph $\Gamma(\varphi)$ of φ is an open (closed) subset of $X\times Y$; φ is u p p e r s e m i c o n t i n u o u s (usc) if $\varphi(x)$ is compact for every $x\in X$ and $\{x\in X: \varphi(x)\subset U\}$ is open for every open $U\subset Y$. Assume that $\varphi: X\to Y$ is a multi-valued map such that $Y\setminus\varphi(x)\neq\emptyset$ for every $x\in X$. For such a map φ we define $\varphi^*: X\to Y$ by $\varphi^*(x)=Y\setminus\varphi(x)$. The multi-valued map φ^* is called the m a p c o n j u g a t e w i t h φ.

The following proposition is evident.

1.1. P r o p o s i t i o n. *If $\varphi: X\to Y$ is a closed map such that $Y\setminus\varphi(x)\neq\emptyset$ for each $x\in X$ then the conjugate map $\varphi^*: X\to Y$ is an open graph map and $\Gamma(\varphi^*)=X\times Y\setminus\Gamma(\varphi)$. A single-valued (continuous) map $f: X\to Y$ is a* c o n t i n u o u s s e l e c t i o n *for $\varphi: X\to Y$ if $f(x)\in\varphi(x)$ for all $x\in X$. If $A\subset X$ and $g: A\to Y$ is a single-valued map such that $g(x)\in\varphi(x)$ for all $x\in X$, then g is called a partial selection for φ (A may be empty).*

2. Strongly acyclic multi-valued maps. In this section we introduce a new class of multi-valued maps which we call strongly acyclic maps

2.1. D e f i n i t i o n. *A multi-valued $\varphi: X\to Y$ is called* s t r o n g l y a c y c l i c *(written: SA-map) if the following conditions are satisfied*:

(2.1.1) *φ i usc,*

(2.1.2) *$Y\setminus\varphi(x)\neq\emptyset$ for all $x\in X$*

(2.1.3) *the homotopy groups $\pi_i(Y\setminus\varphi(x))=0$ for each $x\in X$ and $i=0, 1, 2, \ldots$*

Two SA-maps $\varphi, \psi: X\to Y$ are called SA-homotopic if there is a SA-map $\chi: X\times\Pi\to Y$ such that $\chi(x, 0)=\varphi(x)$ and $\chi(x, 1)=\psi(x)$ for each $x\in X$; χ is called a S A-h o m o t o p y b e t w e e n φ a n d ψ.

For $\chi: X\times\Pi\to Y$ by $\chi_t: X\to Y$, $t\in\Pi$, we will denote the SA-map defined as follows: $\chi_t(x)=\chi(x, t)$.

2.2. R e m a r k. Let us observe that if $\varphi: X\to Y$ is a SA-map then the conjugate map $\varphi^*: X\to Y$ is an open graph map and $\pi_i(\varphi^*(x))=0$ for each $x\in X$ and $i=0, 1, 2, \ldots$.

2.3. Definition. *Let $\varphi: X \to Y$ be a SA-map. A continuous single-valued map $f: X \to Y$ is called* associated with φ *if it is a selection of φ^*.*

2.4. Theorem. *Let X be a compact, finite dimensional CW-complex and Y be a topological space. Assume that $\varphi: X \to Y$ is a SA-map. Then*

(2.4.1) *there exists a map $f: X \to Y$ which is accociated with φ,*

(2.4.2) *if $f, g: X \to Y$ are associated with φ, then f and g are homotopic,*

(2.4.3) *if $\varphi, \psi: X \to Y$ are two SA-homotopic SA-maps and f, g are associated with φ and ψ respectively, then f is homotopic with g.*

Proof. Let $\varphi: X \to Y$ be a SA-map and let φ^* be the conjugate map. Then φ^* satisfies all the assumptions of theorem (1.1) in [1], (comp. (2.3)), therefore there exists a selection f for φ^* and (2.1) is proved. Moreover, (2.4.2) is an easy consequence of theorem (1.1) in [1]. Now we prove (2.4.3). Let $\chi: X \times \Pi \to Y$ be an SA-homotopy between φ and ψ and let χ^* be the conjugate map for χ. Then $\chi^*(x, 0) = \varphi^*(x)$ and $\chi^*(x, 1) = \psi^*(x)$.

Let f be associated with φ and g be associated with ψ. Then the map $h_1: X \times \{0,1\} \to Y$ defined by $h_1(x, 0) = f(x)$ and $h_1(x, 1) = g(x)$ is a partial selection of χ^*, so by using once again theorem (1.1) in [1] we get a map $h: X \times \Pi \to Y$ such that h is a selection of χ^* and $h(x, t) = h_1(x, t)$, for $x \in X$ and $t = 0,1$. Then h is a homotopy joining f and g. This completes the proof.

For given $x \in S^n$ we denote by $E_x: S^n \setminus \{x\} \to S^n \setminus \{x\}$ the constant map defined by $E_x(y) = -x$, for each $y \in S^n \setminus \{x\}$.

2.5. Lemma. *For every $x \in S^n$ there exists a homotopy*

$$\beta_t^x: S^n \setminus \{x\} \to S^n \setminus \{x\}$$

joining the identity map id_x over $S^n \setminus \{x\}$ with the constant map E_x. Moreover for each $0 \leq t < 1$ the set $S^n \setminus \{x\}$ is mapped homeomorphically onto its image $\beta_t^x(S^n \setminus \{x\})$ by the homeomorphism β_t^x and the map $\beta: (S^n \times S^n \setminus \Delta) \times \Pi \to S^n$ defined by $\beta((x, y), t) = \beta_t^x(y)$ is continuous. $\Delta = \{(x, x)\} \in S^n \times S^n: x \in S^n$ — the diagonal. We omit the easy proof of this lemma.

2.6. Theorem. *Let X be a compact finite dimensional CW-complex and let $\varphi: X \to S^n$ be a SA-map. Then there is a single-valued map $g: X \to S^n$ such that φ and g are SA-homotopic.*

Proof. Let $f: X \to S^n$ be the associated with φ (see (2.4.1)). We define $g: X \to S^n$ by setting $g(x) = -f(x)$, for every $x \in X$.

Define $\chi: X \times \Pi \to S^n$, $\chi(x, t) = \beta(\{f(x)\} \times \varphi(x) \times \{t\}) = \beta_t^{f(x)}(\varphi(x))$. It is SA-homotopy joining φ and g. This completes the proof.

2.7. Theorem. *Let $\varphi: S^n \to S^n$ be an odd SA-map, i. e. $\varphi(-x) = -\varphi(x)$ for each $x \in S^n$. Then there exist an odd single-valued map $g: S^n \to S^n$ such that φ and g are SA-homotopic.*

Proof. We consider the unit n-sphere S^n as a CW-complex with standard decomposition on the $2(n+1)$-cells. Then we have two cells in each dimension from o to n. Let $(S^n)^k$ be the l-skeleton of S^n. Let us observe that $(S^n)^k = S^k$. We construct by induction an odd map associated with φ. Let $x \in (S^n)^0 = S^0$, then we put $f^0(x)$ to be an arbitrary point in $\varphi^*(x)$; moreover, we define $f^0(-x) = -f^0(x)$.

Assume $f^i: S_+^i \to S^n$ has been constructed for all $i < k$. Then by theorem (1.1) in [1], it may be extended to a map $f^k: S_+^k \to S^n$ conjugated with $(\varphi | S^k): S_+^k \to S^n$. If $x \in S_-^k$ we define $f^k(x) = -f(-x)$, where S_+^k and S_-^k denote the upper and lower hemisphere of S^k, respectively. It completes induction; so we get an odd map $f: S^n \to S^n$ which is a selector of φ^*. Then the map $g(x)=-f(x)$ satisfies our hypotesis and the proof is complete.

Let X be a finite dimensional, compact CW-complex. By $[X, S^n]$ ($[X, S^n]_{SA}$) we will denote the set of all homotopy classes of single-valued (SA-maps) maps from X to S^n. Define a map $\theta: [X, S^n] \to [X, S^n]_{SA}$ by $\theta([f])=[f]_{SA}$ for each $[f] \in [X, S^n]$. Since any continuous (single-valued) map is a SA-map the above definition is correct. Now we are able to state the main result of this paper.

2.8. Theorem. *The map* $\theta: [X, S^n] \to [X, S^n]_{SA}$ *is a bijection.*

Proof. First let us observe that in view of theorem (2.4) we obtain that θ is onto. To show that θ is an injection suppose that $\theta([f])=\theta([g])$.

It means that there exist a SA-homotopy $\chi: X \times \Pi \to S^n$ such that $\chi(x, 0)=f(x)$ and $\chi(x, 1)=g(x)$ for each $x \in X$. Consider the conjugate map $\chi^*: X \times \Pi \to S^n$. The map $\chi^*: X \times \Pi \to S^n$. The map $h: X \times \{0,1\} \to S^n$ defined by is a partial selection of χ^*.

$$h(x, t)=\begin{cases} -f(x) & \text{if } t=0 \\ -g(x) & \text{if } t=1 \end{cases}$$

So by using theorem (1.1) in [1], we deduce that $(-f)$ is homotopic with $(-g)$; (by a single-valued homotopy).

It implies that f and g are homotopic (by a single-valued homotopy).

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