Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

GEOMETRIC QUANTIZATION OF THE SPHERICAL PENDULUM

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The Kostant-Souriau geometric quantization of the spherical pendulum is discussed and some of its connections with the Marsden-Weinstein reduction are pointed out.

1. Preliminaries. It is a classical theorem going back to Jacobi and Liouville that if one has k-first integrals in involution, i. e. their Poisson brackets all vanish, then it is possible to reduce Hamilton's equations to a set of Hamiltonian equations in which 2k variables have been eliminated. Similarly in the celestial mechanics rotational invariance allows one to eliminate four variables from Lagrange's equations — Jacobi's celebrate "elimination of the node".

Both of the above examples are special cases of the Marsden-Weinstein' reduction technique which can be briefly sketched as follows.

Let (M, ω, Φ, J) be an Hamiltonian G-space, with Ad^* — equivariant momentum map. Denote by $G_\mu=\{g\in G \mid Ad^*_{g^{-1}}\mu=\mu\}$ the isotropy subgroup of the co-adjoint action at $\mu\in g^*$. Assume that μ is a regular value for J so that $J^{-1}(\mu)$ is a $(\dim M - \dim G)$-dimensional submanifold of M. By Ad^*-equivariance G_μ acts on $J^{-1}(\mu)$. Assume that this action is proper and free so that $M_\mu=J^{-1}(\mu)/G_\mu$, the G_μ — orbit space of $J^{-1}(\mu)$, is a smooth $(\dim M-\dim G-\dim G_\mu)$-dimensional manifold with the canonical projection $\pi_\mu: J^{-1}(\mu)\to M_\mu$ a surjective submersion.

The theorem of Marsden and Weinstein states then that M_μ has a unique symplectic structure ω_μ satisflying $\pi^*_\mu\omega_\mu=i^*_\mu\omega$, where $i_\mu: J^{-1}(\mu)\hookrightarrow M$, is the canonical inclusion. (M_μ, ω_μ) is called the reduced phase space.

Under all the hypotheses above, let $H: M\to \mathbf{R}$ be a G-invariant Hamiltonian. Then the flow F_t of X_H leaves $J^{-1}(\mu)$ invariant (since J is a conserved quantity) and commutes with the G_μ-action on $J^{-1}(\mu)$ (since $\Phi^*_g X_H=X_H$) so it induces canonically a a flow $H_t: M_\mu\to M_\mu$ defined by $F_t\circ i_\mu=H_t\circ\pi_\mu$. Then the theorem of Marsden and Weinstein, [1], asserts that H_t is a Hamiltonian flow on M_μ with the Hamiltonian H_μ induced by H, i. e., $H_\mu\circ\pi_\mu=H\circ i_\mu$, and the Hamiltonian vector field X_H on M_μ is π_μ-related to $X_H|J^{-1}(\mu)$, i. e., $\pi^*_\mu X_H=X_{H_\mu}\circ\pi_\mu\cdot H_\mu$ is called the reduced Hamiltonian and X_{H_μ} the reduced Hamiltonian vector field.

The above theorem tells us that the dynamic on the extended phase space of a mechanical system is projected in a canonical way on its reduced phase space. It is an open and very tempting problem to try to obtain the quantic equivalent of this theorem, or more precisely to give an answer at the following problem:

"Are the reduction and the quantization interchangeable processes?"

The main goal of this paper is to study this problem for the particular case of the spherical pendulum.

2. Spherical pendulum. The classical problem of the spherical pendulum consists in the study of the motion of a matherial point on the 2-sphere, under the action of the gravitational field. The configuration space of the corresponding mechanical system is a 2-sphere in $\mathbf{R}^3$, say the unit sphere S^2, is we choose our unit of length so that the pendulum has length 1. The phase space is the cotangent bundle, T^*S^2, with the symplectic structure, induced by the canonical symplectic structure of $\mathbf{R}^4$.

If p denotes a point on the 2-sphere and α an element of $T^*_pS^2$, then the Hamiltonian for the spherical pendulum is:

$$H(\alpha, p)=\frac{1}{2}\|\alpha\|^2+p_3,$$

where the first term is the kinetic energy and the second term is the potential energy.

If $SO(2)$ $(\approx S^1)$ is the classical group, of rotations in plane which acts on S^2 rotating the points around z-axis, then $(T^*S^2, \omega, SO(2))$ is an $SO(2)$-Hamiltonian system and its corresponding moment map, which is an independent constant of the motion, is given by:

$$J(\alpha, p)=\alpha_1 p_2-\alpha_2 p_1.$$

Finally taking $\rho=(J, H)$, $\rho\colon T^*S^2\to\mathbf{R}^2$, it is not hard to see that for each regular value (x, y) of ρ, its inverse image, $\rho^{-1}(x, y)$, is a 2-torus in T^*S^2, thus, the inverse image of the regular set of ρ is a bundle of 2-dimensional tori. Indeed, the spherical pendulum is a complete integrable mechanical system and then the property results via the well known Arnold-Liouville' theorem, [1].

3. Prequantization. Let $\{\varphi, \theta\}$, $0<\varphi<\pi$, $0<\theta<2\pi$, be polar coordinates on S^2 and $\{p_\varphi, p_\theta\}$ the corresponding moments, on T^*S^2 so that the functions H and J are given in terms of these coordinates by:

$$H=\frac{1}{2}p_\varphi^2+\frac{1}{2\sin^2\varphi}p_\theta^2+\cos\varphi$$

and $J=p_\theta$.

A straightforward computation show us that the Marsden-Weinstein reduced phase space $M_{p\theta}=J^{-1}(p_\theta)/SO(2)$ can be identified with $T^*\mathbf{R}$ which is parametrized by $\{\varphi, p_\varphi\}$ and its symplectic structure is:

$$\omega_{p_\theta}=dp_\varphi\wedge d\varphi.$$

Since ω_{p_θ} is an exact form, $\omega_{p_\theta}=d(p_\varphi d\varphi)$, M_{p_θ} is a quantizable manifold. Indeed, the line bundle $(L^{\omega_{p_\theta}}, \pi_{p_\theta}, M_{p_\theta})$ is simple the trivial bundle, $L^{\omega_{p_\theta}}=M_{p_\theta}\times\mathbf{C}$, the Hermitian structure on $L^{\omega_{p_\theta}}$ is defined by

$$((x, z_1), (x, z_2))=z_1\bar{z}_2,$$

and the space of smooth sections $\Gamma(L^{\omega_{p_\theta}})$ of the line bundle $L^{\omega_{p_\theta}}$ can be identified with $C^\infty(M_{p_\theta}, \mathbf{C})$.

Let $C_c^\infty(M_{p_\theta}, \mathbf{C})$ be the space of smooth complex functions on M_{p_θ} with compact support. It is a pre-Hilbert space under the inner product given by:

$$\langle f, g\rangle=\int_{M_{p_\theta}} f\cdot\bar{g},$$

and let $\mathscr{H}_{p_\theta}$ be its completion. Then for each $f\in C^\infty(M_{p_\theta}, \mathbf{R})$, the prequantum operator $\delta_f^{p_\theta}$ corresponding to f,

$$\delta_f^{p_\theta}: \mathscr{H}_{p_\theta} \to \mathscr{H}_{p_\theta}$$

is given by:

$$\delta_f^{p_\theta} = -ih\left[\frac{\partial f}{\partial p_\varphi}\frac{\partial}{\partial \varphi} - \frac{\partial f}{\partial \varphi}\frac{\partial}{\partial p_\varphi}\right] - \frac{\partial f}{\partial p_\varphi} p_\varphi + f.$$

Let $(\delta, \mathscr{H})$ be the prequantum operator and the Hilbert space of T^*S^2 given by Kostant geometric prequantization. Then we have:

Theorem 1. *Let* $f: T^*S^2 \to \mathbb{R}$ *[resp.* $g \in \mathscr{H}$*] be* $SO(2)$ *invariant functions on* T^*S^2 *and* f_{p_θ} *[resp.* g_{p_θ}*] the induced functions on* M_{p_θ}, *i. e.* $f_{p_\theta} \circ \pi_{p_\theta} = f \circ i_{p_\theta}$ *[resp.* $g_{p_\theta} \circ \pi_{p_\theta} = g \circ i_{p_\theta}$*], where* $\pi_{p_\theta}: J^{-1}(p_\theta) \to M_{p_\theta}$ *and* $i_{p_\theta}: J^{-1}(p_\theta) \hookrightarrow T^*S^2$ *are respectively the projection and the inclusion. Then we have:*

(i) $\delta_f(g)$ *is a smooth,* $SO(2)$ *— invariant function on* T^*S^2;

(ii) $[\delta_f(g)]_{p_\theta} = \delta_{f_{p_\theta}}^{p_\theta}(g_{p_\theta})$, *or in other words the reduction and the geometric prequantization are interchangeable processes.*

Proof.

(i) It is a straightforward computation and we shall omit it.

(ii) We can write succesively:

$$X_{f_{p_\theta}}(g_{p_\theta})((\pi_{p_\theta})(x)) = (X_{f_{p_\theta}})_{\pi_{p_\theta}(x)}(g_{p_\theta}) = T_x(X_f)_x(g_{p_\theta}) = (X_f)_x(g_{p_\theta} \circ \pi_{p_\theta})$$

$$= (X_f)_x(g \circ i_{p_\theta}) = (X_f(g))_{i_{p_\theta}}(x);$$

$$[(p_\varphi\, d\varphi)(X_{f_{p_\theta}})(\pi_{p_\theta}(x))] \cdot [g_{p_\theta}(\pi_{p_\theta}(x))]$$

$$= [(p_\varphi\, d\varphi)_{\pi_{p_\theta}(x)}(X_{f_{p_\theta}})_{\pi_{p_\theta}(x)}] \cdot [g \circ i_{p_\theta}(x)]$$

$$= [(p_\varphi\, d\varphi)\pi_{p_\theta}(T_x \pi_{p_\theta}(X_f)_x] \cdot [g \circ i_{p_\theta}(x)]$$

$$= [\pi_{p_\theta}^*(p_\varphi\, d\varphi)_{\pi_{p_\theta}(x)}(X_f)_x] \cdot [g(i_{p_\theta}(x)]$$

$$= [(p_\varphi\, d\varphi)_{i_{p_\theta}(x)}] \cdot [g(i_{p_\theta}(x))] = [p_\varphi d\varphi(X_f) g]_{i_{p_\theta}(x)};$$

$$(f_{p_\theta} \cdot g_{p_\theta})(\pi_{p_\theta}(x)) = f_{p_\theta}(\pi_{p_\theta}(x)) \cdot g_{p_\theta}(\pi_{p_\theta}(x))$$

$$= f(i_{p_\theta}(x)) \cdot g(i_{p_\theta}(x)) = (f \cdot g)_{i_{p_\theta}}(x),$$

by where we obtain the desired result.

q. e. d.

4. Quantization. Let us consider now the particular case of the horizontal motions on the sphere. These motions were first discovered by Huygens, [3], see also Duistermaat [2], the physical explanation being that the centrifugal force, together with the constraining force of the pendulum exactly balances the force of gravity. Huygens has also proved that the surfaces of revolution for which, the horizontal motions have a constant period are the paraboloids.

The equations of the horizontal motions of the spherical pendulum are:

$$\begin{cases} p_\varphi = 0 \\ \sin\varphi + J^2 \dfrac{\cos\varphi}{\sin^3\varphi} = 0, \end{cases}$$

and they represent in fact the equilibrium points of the reduced phase space M_{p_θ}. The Hamiltonian [resp. reduced Hamiltonian] for the Huygens periodical motions [resp. reduced Huygens periodical motions] is given by:

$$H=\frac{p_\theta}{2\sqrt{-\cos\varphi}}+\cos\varphi$$

resp.

$$H_{p_\theta}=\cos\varphi.$$

Let F be the polarization on T^*S^2 generated by $\{\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial p_\varphi}\}$ and F_{p_θ} the vertical polarization on the reduced phase, i. e., $F_{p_\theta}\sim\frac{\partial}{\partial p_\varphi}$. Then it is not hard to see that H and H_{p_θ} are quantizable with respect to F and F_{p_θ}, respectively. Moreover, we have

Theorem 2. *The geometric quantization, operators* $(\delta_F)_H$ *and* $(\delta^{p_\theta}_{F_{p_\theta}})_{H_{p_\theta}}$ *have the diagonal form, and the geometric quantization and the Marsden-Weinstein reduction are interchangeable processes.*

Proof. Let $C^{00}(T^*S^2, F; 1)$ [resp. $C^{00}(M_{p_\theta}, F_{p_\theta}; 1)$] be the space of quantizable functions on T^*S^2 [resp. M_{p_θ}], i. e. $f\in C^{00}(T^*S^2, F; 1]$ [resp. $f\in C^{00}(M_{p_\theta}, F_{p_\theta}; 1)$] if and only if $[X_f, X]$ is tangent to the polarization F [resp. F_{p_θ}] whenever X is tangent to the polarization F [resp. F_{p_θ}], and $\mathscr{H}_F$ [resp. $\mathscr{H}_{F_{p_\theta}}$] the Hilbert representation space. Then a straightforward calculation show us that for each $g\in\mathscr{H}_F$ we have:

$$((\delta_F)_H(g))_{p_\theta}=(\delta^{p_\theta}_{F_{p_\theta}})_{H_{p_\theta}}(g_{p_\theta})=H_{p_\theta}\cdot g_{p_\theta},$$

or in other words, the quantum operators $(\delta_F)_H$ and $(\delta^{p_\theta}_{F_{p_\theta}})_{H_{p_\theta}}$ have the diagonal form. This implies that the reduction and the geometric quantization are interchangeable processes.

q. e. d.

REFERENCES

1. R. Abraham, J. Marsden. Foundations of Mechanics 2-nd ed.. Cummings. 1978.
2. J. J. Duistermaat. On global action-angle coordinates. *Comm. Pure Appl. Math.*, **33**, **1980**, 687-707.
3. Ch. Huygens. L' Horloge à Pendule 1673, Sur la Force Centrifugal Résultant du Mouvement Circulaire. Oeuvres Complétes, t. 18.

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Received 1. 6. 1987