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SOME RESULTS ON α-PRODUCTS OF DISTRIBUTIONS

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In the following, definitions 1-5 and theorems 1, 2 and 4 were originally considered using the real field. However, they hold equally well using the complex field and we will consider them for this case.

The following definitions and theorem were given in [2].

Definition 1. *Let h_r be a distribution for $r=0, 1, \dots$. We say that $\underset{\sim}{h}=[h_0, h_1, \dots, h_r, \dots]$ is a distribution vector.*

If $h_{r+i}=0$ for $i=1, 2, \dots$, we write $\underset{\sim}{h}=[h_0, h_1, \dots, h_r, 0, \dots]=[h_0, h_1, \dots, h_r]$, and if $h_i=0$ for $i=1, 2, \dots$, we write $\underset{\sim}{h}=[h_0]=h_0$.

The set of all distribution vectors is made into a vector space by defining the sum and product by a scalar in the usual way.

Definition 2. *Let $\underset{\sim}{h}=[h_0, h_1, \dots h_r, \dots]$ be a distribution vector and let φ be an arbitnary test function with compact support. We define $(\underset{\sim}{h}, \varphi)$ to be the sequence of complex numbers*

$$(\underset{\sim}{h}, \varphi)=((h_0, \varphi), (h_1, \varphi), \dots, (h_r, \varphi), \dots).$$

Definition 3. *Let $\underset{\sim}{h}=[h_0, h_1, \dots, h_r, \dots]$ be a distribution vector. We define the derivative $\underset{\sim}{h}'$ of $\underset{\sim}{h}$ by*

$$\underset{\sim}{h}'=[h'_0, h'_1, \dots, h'_r, \dots]$$

Theorem 1. *Let $\underset{\sim}{h}=[h_0, h_1, \dots, h_r, \dots]$ be a distribution vector and let φ be an arbitrary test function with compact support. Then*

$$(\underset{\sim}{h}', \varphi)=-(\underset{\sim}{h}, \varphi').$$

Now let ρ be a fixed infinitely differentiable function having the properties

(i) $\rho(x)=0$ for $|x|\geqq 1$,

(ii) $\rho(x)\geqq 0$,

(iii) $\rho(x)=\rho(-x)$,

(iv) $\int_{-1}^{1}\rho(x)dx=1$.

The function δ_n is defined by $\delta_n(x)=n\rho(nx)$ for $n=1, 2, \dots$. It is obvious that the sequence $\{\delta_n\}$ is regular and converges to the Dirac delta — function δ.

The scalar ρ_r is defined for $r=-1, 0, 1, \dots$ by

$$\rho_r=\begin{cases}-1/2, & r=-1,\\ \rho^{(r)}(0)=0, & r=1, 3, \dots,\\ \rho^{(r)}(0), & r=0, 2, \dots\end{cases}$$

The next two definitions were given in [4].

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D e f i n i t i o n 4. *Let f and g be distributions and let* $g_n = g * \delta_n$. *We say that the* α*-product* $f \underset{\circ}{\alpha} g$ *of f and g exists and is equal to the distribution vector* $\underset{\sim}{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *on the open interval* (a, b) *if*

$$(f, g_n\varphi) = (h_0, \varphi) + \sum_{r=1}^{\infty} (h_r, \varphi) . n^{\alpha+r} + \varepsilon_n$$

for all test functions φ *with compact support contained in the interval* (a, b), *where* $-1 < \operatorname{Re} \alpha \leqq 0$ *and* $\lim_{n\to\infty} \varepsilon_n = 0$.

In particular, if $h_r = 0$ for $r = 1, 2, \ldots,$ we simply say that the product $f \circ g$ of f and g exists and we then write

$$f \circ g = h_0$$

on the interval (a, b).

It follows that this definition of the product $f \circ g$ is equivalent to definition 4 of the product $f \circ g$ given in [1].

D e f i n i t i o n 5. *Let f and g be distributions and suppose that the* α*-product* $f \underset{\circ}{\alpha} g$ *exists and is equal to the distribution vector* $\underset{\sim}{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *on the open interval* (a, b). *We say that* h_0 *is the finite part of* $f \underset{\circ}{\alpha} g$ *and we then write*

$$\text{p. f. } f \underset{\circ}{\alpha} g = h_0$$

on the interval (a, b).

The following two theorems hold, see [2] and [4].

T h e o r e m 2. *Let f and g be distributions and suppose that the* α*-product* $f \underset{\circ}{\alpha} g$ *and* $f' \underset{\circ}{\alpha} g$ *(or* $f \underset{\circ}{\alpha} g'$*) exist on the open interval* (a, b). *Then the* α*-product* $f \underset{\circ}{\alpha} g$ *(or* $f' \underset{\circ}{\alpha} g$*) exists and*

$$(f \underset{\circ}{\alpha} g)' = f' \underset{\circ}{\alpha} g + f \underset{\circ}{\alpha} g'$$

on the interval (a, b).

T h e o r e m 3. *The 0-products* $x_+^p \underset{\circ}{0} \delta^{(q)}$ *and* $\delta^{(q)} \underset{\circ}{0} x_+^p$ *exist as distribution vectors and*

$$x_+^p \underset{\circ}{0} \delta^{(q)} = \underset{\sim}{h}(p, q) = [h_0(p, q), h_1(p, q), \ldots, h_{q-p}(p, q)]$$

or $p = 0, 1, \ldots, q$ *and* $q = 0, 1, \ldots,$ *where*

$$h_i(p, q) = (-1)^{p-1} \binom{q-i}{p} p! \, \rho_{i-1} \delta^{(q-p-i)}, \quad 0 \leqq i \leqq q-p$$

and

$$\delta^{(q)} \underset{\circ}{0} x_+^p = \underset{\sim}{k}(p, q) = [k_0(p, q), k_1(p, q), \ldots, k_{q-p}(p, q)]$$

for $p = 0, 1, \ldots, q$ *and* $q = 0, 1, \ldots,$ *where*

$$k_i(p,q) = (-1)^{p-1} \binom{q}{p+i} p! \, \rho_{i-1} \delta^{(q-p-i)}, \quad 0 \leqq i \leqq q-p.$$

The next theorem was proved in [4] for real numbers λ, μ, but the proof is easily modified to deal with complex numbers λ, μ. The theorem is also rewritten using a more convenient notation.

T h e o r e m 4. *Let* λ, μ *be complex numbers such that* $\lambda, \mu, \operatorname{Re}(\lambda+\mu) \neq -1, -2, \ldots,$ $\operatorname{Re}(\lambda+\mu) < -1$ *and let* $s = [\operatorname{Re}(-\lambda-\mu)]$, $\alpha = -\lambda - \mu - s - 1$. *Then the* α*-product* $x_+^\lambda \underset{\circ}{\alpha} x_-^\mu = \underset{\sim}{h}(\lambda, \mu) = [0, h_1(\lambda, \mu), \ldots, h_s(\lambda, \mu)]$, *where*

$$h_i(\lambda, \mu)=\Big(\frac{(-1)^i\,.\,\Gamma(\mu+1)\,.\,\Gamma(\lambda+s-i+1)}{(s-i)!\,\Gamma(s-\alpha-i+1)}\cdot\int_0^1 u^{s-\alpha-1}\rho^{(s)}(u)du\Big)\delta^{(s-i)}$$

for $i=1, 2, \ldots, s$.

We now prove the following theorem.

Theorem 5. *Let* λ *be a complex number such that* $\operatorname{Re}\lambda\neq 0, \pm 1, \pm 2, \ldots$, *let* q *be a non-negative integer such that* $\operatorname{Re}\lambda<q$ *and let* $s=[\operatorname{Re}(-\lambda)]+q+1$, $\alpha=-\lambda+q-s$. *Then the* α*-products* $x_+^\lambda \underset{\circ}{\alpha} \delta^{(q)}$ *and* $\delta^{(q)} \underset{\circ}{\alpha} x_+^\lambda$ *exist and*

$$x_+^\lambda \underset{\circ}{\alpha} \delta^{(q)}=[0,\ h_1(\lambda, q), \ldots,\ h_s(\lambda, q)],$$

where

$$h_i(\lambda, q)=\Big(\frac{(-1)^{q+i}\,.\,\Gamma(\lambda+s-i+1)}{(s-i)!\,.\,\Gamma(s-\alpha-i+1)}\cdot\int_0^1 u^{s-\alpha-i}\rho^{(s)}(u)du\Big)\delta^{(s-i)} \tag{1}$$

for $i=1, 2, \ldots, s$, *and*

$$\delta^{(q)} \underset{\circ}{\alpha} x_+^\lambda=[0,\ k_1(\lambda, q), \ldots,\ k_s(\lambda, q)],$$

where

$$\begin{cases} k_i(\lambda, q)=0 \text{ for } 1\leqq i\leqq s-q-1, \text{ and} \\ k_i(\lambda, q)=\Big(\binom{q}{s-i}\frac{(-1)^{q+i}\,.\,\Gamma(\lambda+1)}{\Gamma(s-\alpha-i+1)}\cdot\int_0^1 u^{s-\alpha-1}\rho^{(s)}(u)du\Big)\delta^{(s-i)} \\ \text{for } s-q\leqq i\leqq s. \end{cases} \tag{2}$$

Proof. Suppose first of all that $\operatorname{Re}\lambda>-1$. Then for arbitrary test function with compact support

$$(x_+^\lambda,\ \delta_n(x)\varphi(x))=\int_0^{1/n} x^\lambda\delta_n^{(q)}(x)\varphi(x)dx=$$

$$=\int_0^{1/n} x^\lambda n^{q+1}\rho^{(q)}(nx)\varphi(x)\,dx=\int_0^1 n^{q-\lambda}u^\lambda\rho^{(q)}(u)\Big\{\sum_{i=0}^{s-1}\frac{u^i\varphi^{(i)}(0)}{n^i\,.\,i!}\Big\}du$$

$$+\varepsilon_n=\Big\{\sum_{i=0}^{s-1}\frac{(-1)^i n^{q-i-\lambda}}{i!}\cdot\int_0^1 u^{\lambda+i}\rho^{(q)}(u)du\Big\}(\delta^{(i)},\ \varphi)+\varepsilon_n$$

$$=\sum_{i=1}^{s}\frac{(-1)^{s-i}n^{\alpha+1}}{(s-i)!}\cdot\int_0^1 u^{\lambda+s-i}\rho^{(q)}(u)du(\delta^{(s-i)},\ \varphi)+\varepsilon_n$$

$$=\sum_{i=1}^{s}\frac{(-1)^{q+i}n^{\alpha+i}\,.\,\Gamma(\lambda+s-i+1)}{(s-i)!\,\Gamma(s-\alpha-i+1)}\cdot\int_0^1 u^{s-\alpha-i}\rho^{(s)}(u)du(\delta^{(s-i)},\ \varphi)+\varepsilon_n,$$

where $\lim_{n\to\infty}\varepsilon_n=0$. Equation (1) follows for $\operatorname{Re}\lambda>-1$ and $i=1, \ldots, s$.

Now assume that equation (1) holds for $-p-1<\operatorname{Re}\lambda<-p$ and $i=1, \ldots, s$. This is certainly true when $p=0$. Then with $-p-1<\operatorname{Re}\lambda<-p$ we have using theorem 2

$$\lambda x_+^{\lambda-1} \underset{\circ}{\alpha} \delta^{(q)}=(x_+^\lambda \underset{\circ}{\alpha} \delta^{(q)})'-x_+^\lambda \underset{\circ}{\alpha} \delta^{(q+1)}=[0,\ h_1'(\lambda, q)-h_1(\lambda, q+1), \ldots,\ h_s'(\lambda, q)-h_s(\lambda, q+1),$$

$$-h_{s+1}(\lambda, q+1)],$$

where

$$h_i'(\lambda, q)-h_i(\lambda, q+1)=\lambda\frac{(-1)^{q+i}\,.\,\Gamma(\lambda+s-i+1)}{(s-i+1)!\,\Gamma(s-\alpha-i+2)}\cdot\int_0^1 u^{s-\alpha-i+1}\rho^{(s+1)}(u)du.\delta^{(s-i+1)}=\lambda h_i(\lambda-1, q)$$

for $i=1, \ldots, s$ and

$$-h_{s+1}(\lambda,\ q+1)=\{\frac{(-1)^{q+s+1}\ .\ \Gamma(\lambda+1)}{\Gamma(1-\alpha)}\cdot\int_0^1 u^{-\alpha}\,\rho^{(s+1)}(u)du\}\delta=\lambda,\ h_{s+1}(\lambda-1,\ q).$$

Equation (1) follows now by induction, since we got it for $-p-2<\mathrm{Re}\,\lambda<-p-1$ and $i=1,\ldots,\ s+1$.

We will suppose that $\mathrm{Re}\,\lambda>-1$ and put $(x_+^\lambda)_n=x_+^\lambda*\delta_n(x)=:c_n(x)$, so that $c_n^{(i)}(0)=(-1)^i n^{i-\lambda}\,.\int_0^1 u^\lambda\rho^{(i)}(u)du$.

Then

$$(\delta^{(q)},\ (x_+^\lambda)_n\varphi)=(-1)^q\,.\sum_{i=0}^{q}\binom{q}{i}.\,c_n^{(i)}(0)\,.\,\varphi^{(q-i)}(0)=\sum_{i=0}^{q}\binom{q}{i}n^{i-\lambda}\,.\int_0^1 u^\lambda\rho^{(i)}(u)du(\delta^{(q-i)},\ \varphi)$$

$$=\sum_{i=0}^{q}\binom{q}{i}(-1)^{s-i}\,n^{i-\lambda}\frac{\Gamma(\lambda+1)}{\Gamma(s+\lambda-i+1)}\cdot\int_0^1 u^{\lambda+s-i}\,\rho^{(s)}(u)du\,(\delta^{(q-i)},\ \varphi)$$

$$=\sum_{i=s-q}^{s}\binom{q}{s-i}(-1)^{q+1}\,n^{\alpha+i}\frac{\Gamma(\lambda+1)}{\Gamma(s-\alpha-i+1)}\int_0^1 u^{s-\alpha-i}\,\rho^{(s)}(u)du\,(\delta^{(s-i)},\ \varphi)$$

and equation (2) follows for $\mathrm{Re}\,\lambda>-1$ and $i=1,\ 2,\ldots,\ s$.

Now assume that equation (2) holds for $-p-1<\mathrm{Re}\,\lambda<-p$ and $i=1,\ldots,\ s$. This is certainly true when $p=0$. Then with $-p-1<\mathrm{Re}\,\lambda<-p$ we have using theorem 2

$$\lambda\delta^{(q)}\underset{\circ}{\alpha}\,x_+^{\lambda-1}=(\delta^{(q)}\underset{\circ}{\alpha}\,x_+^\lambda)'-\delta^{(q+1)}\underset{\circ}{\alpha}\,x_+^\lambda=[0,\ldots,\ 0,\ k'_{s-q}(\lambda,\ q)-k_{s-q}(\lambda,\ q+1),\ldots,\ k'_s(\lambda,\ q)$$

$$-k_s(\lambda,\ q+1),\ -k_{s+1}(\lambda,\ q+1)]=\lambda\,[0,\ldots 0,\ k_{s-q+1}(\lambda-1,\ q),\ldots,\ k_{s+1}(\lambda-1,\ q)]$$

and equations (2) follow for $-p-2<\mathrm{Re}\,\lambda<-p-1$ and $i=1,\ldots,\ s+1$. The result follows by induction. This completes the proof of the theorem.

The next definition was given in [3].

Definition 6. *Let f_λ be a distribution depending on a complex parameter λ. Then f_λ is said to be an analytic function of λ on a domain D if $(f_\lambda,\ \varphi)$ is analytic on D for all test functions φ with compact support. If D is the whole complex plane, then f_λ is said to be an entire function of λ. If $\underset{\sim}{f_\lambda}=[f_0(\lambda),\ f_1(\lambda),\ldots,\ f_r(\lambda),\ldots]$ is a distribution vector and $f_r(\lambda)$ is an analytic function of λ on a domain D for $r=0,\ 1,\ldots,$ then $\underset{\sim}{f_\lambda}$ is said to be an analytic vector function of λ on D. If $f_r(\lambda)$ is an entire function of λ for $r=0,\ 1,\ldots,$ then $\underset{\sim}{f_\lambda}$ is said to be an entire vector function of λ.*

We now define the distributions f_+^λ by

$f_+^\lambda=\delta^{(-\lambda-1)}$ for $\lambda=-1,\ -2,\ldots,$ and $f_+^\lambda=\dfrac{x_+^\lambda}{\Gamma(\lambda+1)}$ for $\lambda\neq-1,\ -2,\ldots,$ and the distributions f_-^λ by

$$f_-^\lambda(x)=f_+^\lambda(-x).$$

Then f_+^λ and f_-^λ are entire functions of λ, see [5].

Theorem 6. *Let λ, μ be complex numbers such that $\mathrm{Re}\,\mu\neq-1,\ -2,\ldots,$ $\mathrm{Re}\,\mu<-1$ and let $s=[\mathrm{Re}\,(-\mu)]$, $\alpha=-\mu-s-1$. Then the α-product $f_+^\lambda\underset{\circ}{\alpha}f_-^{\mu-\lambda}$ of the entire functions f_+^λ and $f_-^{\mu-\lambda}$ exists as a distribution vector and*

$$f_+^\lambda\underset{\circ}{\alpha}\,f_-^{\mu-\lambda}=\underset{\sim}{g}(\lambda,\ \mu)=[0,\ g_1(\lambda,\ \mu),\ldots,\ g_s(\lambda,\ \mu)],$$

where $\underset{\sim}{g}(\lambda,\ \mu)$ is an entire vector function of λ and

$$g_i(\lambda,\ \mu)=\Big\{\frac{(-1)^i\,.\,\Gamma(\lambda+s-i+1)}{(s-i)\,!\,.\,\Gamma(s-\alpha-i+1)\,.\,\Gamma(\lambda+1)}\cdot\int_0^1 u^{s-\alpha-1}\,\rho^{(s)}(u)du\Big\}\delta^{(s-i)} \tag{3}$$

for $i=1,\ldots,s$.

Proof. With λ, $\mu-\lambda\neq-1,-2,\ldots$, we have by theorem 4 that the α-product $f_+^\lambda \underset{\circ}{\alpha} f_-^{\mu-\lambda}$ exists and

$$f_+^\lambda \underset{\circ}{\alpha} f_-^{\mu-\lambda}=\frac{x_+^\lambda}{\Gamma(\lambda+1)}\underset{\circ}{\alpha}\frac{x_-^{\mu-\lambda}}{\Gamma(\mu-\lambda+1)}=[0,\ g_1(\lambda,\ \mu),\ldots,\ g_s(\lambda,\ \mu)],$$

where

$$g_i(\lambda,\ \mu)=\frac{h_i(\lambda,\ \mu-\lambda)}{\Gamma(\lambda+1)\,.\,\Gamma(\mu-\lambda+1)}=\Big\{\frac{(-1)^i\,.\,\Gamma(\lambda+s-i-1)}{(s-i)\,!\,\Gamma(\lambda+1)\,.\,\Gamma(s-\alpha-i+1)}\cdot\int_0^1 u^{s-\alpha-i}\rho^{(s)}(u)du\Big\}\delta^{(s-i)}$$

for $i=1,\ldots,s$. Equation (3) follows for λ, $\mu-\lambda\neq-1,-2,\ldots$.

Using theorem 5 with $\mu-\lambda=-1,-2,-q-1$ and $\lambda\neq-1,-2,\ldots$, we have

$$f_+^\lambda \underset{\circ}{\alpha} f_-^{-q-1}=\frac{x_+^\lambda}{\Gamma(\lambda+1)}\underset{\circ}{\alpha}(-1)^q\delta^{(q)}=[0,\ f_1(\lambda,\ q),\ldots,\ f_s(\lambda,\ q)],$$

where

$$f_i(\lambda,q)=\frac{(-1)^q h_i(\lambda,\,q)}{\Gamma(\lambda+1)}=\Big\{\frac{(-1)^i\,.\,\Gamma(\lambda+s-i+1)}{(s-i)\,!\,\Gamma(s-\alpha-i+1)\,.\,\Gamma(\lambda+1)}\cdot\int_0^1 u^{s-\alpha-1}\rho^{(s)}(u)du\Big\}\delta^{(s-i)}=g_i(\lambda,\lambda-q-1)$$

for $i=1,\ldots,s$ and equation (3) follows for $\lambda\neq-1,-2,\ldots$ and $\mu-\lambda=-1,-2,\ldots$

Replacing x by $-x$ in the second part of theorem 5, we have

$$\delta^{(q)}\underset{\circ}{\alpha}x_-^\lambda=(-1)^{q+s}[0,\ -k_1(\lambda,\ q),\ldots,\ (-1)^i k_i(\lambda,\ q),\ldots,(-1)^s k_s(\lambda,\ q)].$$

It follows that with $\lambda=-1,-2,\ldots,-q-1$, and $\mu-\lambda\neq-1,-2,\ldots$,

$$f_+^{-q-1}\underset{\circ}{\alpha}f_-^{\mu+q+1}=\delta^{(q)}\underset{\circ}{\alpha}\frac{x_-^{\mu+q+1}}{\Gamma(\mu+q+2)}=[0,\ l_1(\mu,\ q),\ldots,\ l_s(\mu,\ q)]$$

where

$$l_i(\mu,\ q)=\frac{(-1)^{q+s+i}\,.\,k_i(\mu+q+1,\,q)}{\Gamma(\mu+q+2)}=\Big\{\frac{(-1)^s}{\Gamma(s-\alpha-i+1)}\binom{q}{s-i}\int_0^1 u^{s-\alpha-i}\rho^{(s)}(u)\,du\Big\}\,\delta^{(s-i)}$$

$$=\Big\{\frac{(-1)^i\,.\,\Gamma(\lambda+s-i+1)}{(s-i)\,!\,\Gamma(s-\alpha-i+1)\Gamma(\lambda+1)}\cdot\int_0^1 u^{s-\alpha-i}\rho^{(s)}(u)du\Big\}\delta^{(s-i)}=g_i(-q-1,\ \mu)$$

for $i=s-q,\ldots,s$ and

$$l_i(\mu,\ q)=0=g_i(-q-1,\ \mu)$$

for $i=1,\ldots,s-q-1$. Equation (3) follows for $\mu-\lambda\neq-1,-2,\ldots,\lambda=-1,-2,\ldots$

We have therefore proved that $f_+^\lambda \underset{\circ}{\alpha} f_-^{\mu-\lambda}=\underset{\sim}{g}(\lambda,\ \mu)$ for $\operatorname{Re}\mu\neq-1,-2,\ldots,$ and $\operatorname{Re}\mu<-1$. Since $\dfrac{\Gamma(\lambda+s-i+1)}{\lambda(\Gamma+1)}=\prod_{j=1}^{s-i}(\lambda+j)$ for $1\leq i\leq s-1$, respectively 1 for $i=s$, it follows that $\underset{\sim}{g}(\lambda,\ \mu)$ is an entire function of λ. This completes the proof of the theorem.

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