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ESTIMATES ON THE INITIAL TRACE FOR THE SOLUTIONS OF THE FILTRATION EQUATION

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Necessary condition for the existence of solution of the Cauchy problem with increasing initial data is obtained for the filtration equation of unsteady type.

1. Introduction. The purpose of this paper is to study nonnegative, continuous weak solutions of the equation

$$u_t = \Delta\varphi(u) \tag{1.1}$$

in the strip $S_T = R^d \times (0, T]$, where the function φ is monotonly increasing and $\varphi(0)=0$!

For such solutions we prove the existence and uniqueness of a nonnegative Bore. measure μ on R^d such that

$$\lim_{t\searrow 0} \int_{R^d} u(x, t)\eta(x)dx = \int_{R^d} \eta(x)\mu(dx) \tag{1.2}$$

for all functions $\eta \in C_0(R^d)$.

Moreover, we establish an estimate on the growth of μ which is a necessary condition for the existence of the solution of Cauchy problem for the equation (1.1).

The existence and uniqueness of the initial trace for the equation (1.1) was studied by W. Widder in [1] and by D. Aronson in [2] for $\varphi(s)=s$, and for nonlinear case, when $\varphi(s)=s^m$, $m>1$ by D. Aronson and L. Caffarelli in [3] and by M. Ughi in [4] for φ close to s^m. In the work of A. Kalashnikov [5] for $d=1$ and recently by Ph. Benilan, M. Crandall and M. Pierre in [6] for $d>1$ is shown that the necessary condition from [3] is also sufficient for the existence of solution of (1.1).

This paper is inspired from the work of D. Aronson and L. Caffarelli [3] and we extend their results for nonpower φ which satisfy some usual conditions.

2. Main result. In this work we deal with the functions φ in the equation (1.1) which satisfies the following conditions:

$$\text{(H1)} \quad \begin{cases} \varphi(s) \in C^2(R\setminus 0), & \varphi'(s)>0 \text{ for } s\neq 0 \\ \varphi(0)=\varphi'(0)=0, & s\varphi''(s)\geq 0 \end{cases}$$

$$\text{(H2)} \quad \int_0^\sigma \frac{\varphi'(s)}{s} ds < \infty \text{ for } \sigma \in R_+$$

(H3) There exist a monotonly decreasing function $F(\lambda)$ and a constant $\alpha \in (0, 1)$ such that

$$\alpha s \int_0^s \frac{\varphi'(\lambda)}{\lambda} d\lambda \geq \int_0^s F(\lambda)\varphi'(\lambda)d\lambda \geq \frac{s(\varphi'(s))^2}{s\varphi''(s) + \frac{2}{d}\varphi'(s)}$$

for every $s>0$.

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Define the function ψ such that $\sigma=\int_0^{\psi(\sigma)} s^{-1}\varphi'(s)ds$ and denote by ψ^{-1} the function inverse for ψ.

Definition. *The function u is a weak solution of* (1.1) *in* S_T *if u is nonnegative, continuous and for every* $\tau_1, \tau_2, 0<\tau_1<\tau_2<T$ *is fulfilled*

$$\iint_{R^d\times(\tau_1,\tau_2)} (\varphi(u)\Delta\chi + u\frac{\partial\chi}{\partial t})dxdt=\int_{R^d}(u\chi)\Big|_{\tau_2}^{\tau_1}dx$$

where $\chi\in c^{2,1}(S_T)$ *and* $\operatorname{supp}\chi(\cdot,t)$ *is compact for every* $t\in(\tau_1, \tau_2]$.

Denote by $B_r(x_0)$ the ball $\{x\in R^d;\ |x-x_0|\leq r\}$ and by $|B_r(x_0)|$ its volume.

Our main result is the next:

Theorem 2.1. *Let u be a nonnegative, continuous weak solution of* (1.1) *in* S_T, *for some* $T>0$ *and the function* φ *satisfy conditions* (H1)–(H3). *Then there exists a unique, nonnegative Borel mesure* μ *in* R^d *and* (1.2) *is fulfilled for every* $\eta\in C_0(R^d)$. *Moreover, the next estimate takes place*

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}\mu(dx)\leq C\psi\{(\frac{T}{r^2})^{\frac{\alpha}{1-\alpha}}[\psi^{-1}(u(x_0, T))]^{\frac{1}{1-\alpha}}+\frac{r^2}{T}\} \tag{2.1}$$

for every $x_0\in R^d$, $r\in R_+$ *where the constant C depends only on* φ, *d and does not depend on u.*

Examples.

1. For $\varphi(s)=s^m$, $m>1$ we obtain the main result of [3] and $\alpha=(m-1)/(m-1+(2/d))$.

2. For $\varphi(s)=s^p+s^q$, $1<p<q$ the conditions of [3] are not fulfilled, but we take the theorem 2.1 with

$$\alpha=(q-1)/(q-1+2/d)$$

and the estimate (2.1) is as in [4].

3. When $\varphi(s)=s^p\ln(1+s)$, $p>(d-2)/2$, $d>3$ the conditions in [4] are not fulfilled, but the theorem 2.1 takes place with $\alpha=p^2/(p+1)(p-1+2/d)$.

The proof of the theorem 2.1 is based on the next fundamental properties of the solution of (1.1) which have also some independent sense: estimations on the extention of initial mass and regularizing effect for the solutions of (1.1). We use these properties to prove a Harnak-type inequaluty from which the result follows immediately. The main tool for the provement are different comparison principles for the solutions of (1.1) as in [3].

3. Auxiliary notices. Here we state some of the properties of the boundary value problem for the equation (1.1) which will be used in the proofs in the next sections. Some of the statements are fulfilled for equation (1.1) as well as in the case $\varphi(s)=s^m$ and their proofs are similar to that of [3, 7, 8, 14].

Note that the next definitions and lemmas are fulfilled in cilinder $\Gamma=\Omega\times(0, T]$, where Ω is a domain in R^d with smooth boundary, but for simplicity we shall deal with the case when Ω is a ball. For arbitrary $\xi\in R^d$, $r\in R_+$ and τ_1, τ_2 such that $0<\tau_1<\tau_2\leq T$ denote $\Gamma_r=\Gamma_r(\xi;\ \tau_1, \tau_2)=B_r(\xi)\times(\tau_1, \tau_2]$ and $\partial\Gamma_r=\bar{\Gamma}_r\setminus\Gamma_r$.

Let $f(x, t)$ be continuous function for $x\in R^d$, $t>0$ and $g(x, t)$ is continuous nonnegative function on $\partial\Gamma$, then consider the boundary value problem

$$u_t=\Delta\varphi(u)+f(x, t) \text{ in } \Gamma_r \tag{3.1}$$

$$u(x, t)=g(x, t) \text{ on } \partial\Gamma_r. \tag{3.2}$$

Through this section we assume that the function φ satisfy (H1).

Definition. *The function u is a weak solution of* (3.1), (3.2) *if*

(i) $u \in C([\tau_1, \tau_2];\ L^1(B_r)) \cap L^\infty(\Gamma_r)$,

$$\text{(3.3)}\quad \text{(ii)}\quad \iint_{\Gamma_r} (\varphi(u)\Delta\chi + u\frac{\partial\chi}{\partial t})\,dxdt = \int_{\tau_1}^{\tau_2}\int_{\partial B_r} \varphi(g)\frac{\partial\chi}{\partial\nu}\,dsdt + \int_{B_r}(g\chi)\Big|_{\tau_1}^{\tau_2}dx - \int_{\Gamma_r} f\chi\,dxdt$$

for every function $\chi \in C^{1,0}(\bar\Gamma_r) \cap C^{2,1}(\Gamma_r)$ and $\chi \equiv 0$ on $\partial B_r \times [\tau_1, \tau_2]$, where $\partial/\partial\nu$ denotes the exterior normal derivative,

(iii) $u(x, t) \geqq 0$ *on* Γ_r.

For solutions of (3.1) there is an a priori comparison principle. In the case $\varphi(s) = s^m$, $m > 1$, $f(x, t) \equiv 0$, $d \geqq 1$, this was established in [8], and the proof is as in [14].

Lemma 3.1. *Let u_i, $i = 1, 2$ are two weak solutions of* (3.1), (3.2) *with data g_i and f_i, $i = 1, 2$, such that $g_1 \leqq g_2$, a. e. on $\partial\Gamma_r$, $f_1 \leqq f_2$, a. e. in $\bar\Gamma_r$. Then $u_1 \leqq u_2$, a. e. in Γ_r.*

Proof. From the inequality (3.3) for the functions u_i, $i = 1, 2$ we get

$$\iint_{\Gamma_r} \{[\varphi(u_1) - \varphi(u_2)]\Delta\chi + [u_1 - u_2]\frac{\partial\chi}{\partial t}\}\,dxdt = \int_{B_r} [u_1(x, \tau_2) - u_2(x, \tau_2)]\chi(x, \tau_2)dx$$

$$+ \int_{\tau_1}^{\tau_2}\int_{\partial B_r} [\varphi(g_1) - \varphi(g_2)]\frac{\partial\chi}{\partial\nu}\,dxdt - \int_{B_r} [g_1(x, \tau_1) - g_2(x, \tau_1)]\chi(x, \tau_1)dx$$

$$- \iint_{\Gamma} [f_1(x, t) - f_2(x, t)]\chi\,dxdt.$$

Then for every test function $\chi(x, t)$ such that $\chi \geqq 0$ on Γ_r and $\partial\chi/\partial\nu \leqq 0$ on $\partial B_r \times (\tau_1, \tau_2]$ is fulfilled

$$\text{(3.4)}\qquad \int_{B_r} b(x, \tau_2)\chi(x, \tau_2)dx \leqq \iint_{\Gamma_r} \{[\varphi(u_1) - \varphi(u_2)]\Delta\chi + b(x, t)\frac{\partial\chi}{\partial t}\}\,dxdt$$

where $b(x, t) = u_1(x, t) - u_2(x, t)$.

Since $u_i \in L^\infty(\Gamma_r)$, then there exists a constant $C_1 > 0$ such that $|u_i(x, t)| \leqq C_1$ for $i = 1, 2$ and $(x, t) \in \Gamma_r \setminus \gamma$ and the measure of the set γ is zero. Denote by D the set

$$\{(x, t)\colon u_1(x, t) - u_2(x, t) \neq 0\}.$$

Let

$$a(x, t) = \begin{cases} \dfrac{\varphi(u_1(x, t)) - \varphi(u_2(x, t))}{u_1(x, t) - u_2(x, t)} & \text{for } (x, t) \in D \\ \varphi'(u_1(x, t)) & \text{for } (x, t) \in \Gamma_r \setminus D. \end{cases}$$

Hence $a \in L^\infty(\Gamma_r)$ and $\|a\|_\infty \leqq M$. We can find a sequence $\{a_k\}_{k=1}^\infty$ such that

(i) $a(x, t) + k^{-1} \leqq a_k(x, t) \leqq M + k^{-1}$,

(ii) $a_k(x, t) \in C^\infty(\Gamma_r)$,

(iii) $(a_k(x, t) - a(x, t))(a_k(x, t))^{-1/2} \to 0$ in $L^2(\Gamma_r)$.

Fix an arbitrary function $\varkappa(x)$, such that $\varkappa(x) \geqq 0$, $\varkappa(x) \in C_0^\infty(B_r)$, and let w_k be the solution of the boundary value problem

(3.5)
$$\begin{cases} w_{kt}+a_k(x,\ t)\Delta w_k=0 & \text{in } \Gamma_r \\ w_k(x,\ \tau_2)=\varkappa(x) & \text{for } x\in B_r \\ w_k(x,\ t)=0 & \text{for } (x,\ t)\in \partial B_r\times[\tau_1,\ \tau_2). \end{cases}$$

From [9] it follows the existence and the uniqueness of the solution $w_k(x,\ t)$ of the problem (3. 5) and $w_k\in C^\infty(\Gamma_r)$.

From the maximum principle it follows that

(3.6)
$$0\leq w_k(x,\ t)\leq \max_{x\in B_r} \varkappa(x).$$

If $x_0\in\partial B^r$ and ν is exterior normal to ∂B_r at x_0 then $\partial w_k(x_0,\ t)/\partial\nu=-\lim_{\varepsilon\to 0+}(w_k(x_0-\varepsilon\nu,\ t)-w_k(x_0,\ t))/\varepsilon$ hence

(3.7)
$$\partial w_k(x_0,\ t)/\partial\nu\leq 0.$$

From (3.6), (3.7) it follows that $w_k(x,\ t)$ are test functions and (3.4) is fulfilled

(3.8)
$$\int_{B_r} b(x,\ \tau_2)w_k(x,\ \tau_2)\,dx\leq \iint_{\Gamma_r} b(x,\ t)\,[a(x,\ t)\,\Delta w_k+\frac{\partial w_k}{\partial t}]\,dxdt$$

$$=\iint_{\Gamma_r} b(x,\ t)[a(x,\ t)-a_k(x,\ t)]\Delta w_k dxdt\leq\{\iint_{\Gamma_r} b^2(x,t)a_k(x,\ t)(\Delta w_k)^2\,dxdt\,.\iint_{\Gamma_r}\frac{(a_k-a)^2}{a_k}dxdt\}^{1/2}$$

Let us estimate the integral

$$\iint_{\Gamma_r} a_k(x,\ t)(\Delta w_k)^2dxdt=-\iint_{\Gamma_r}\Delta w_k\frac{\partial w_k}{\partial t}\,dxdt=\iint_{\Gamma_r} w_k\,\frac{\partial}{\partial t}\,(\Delta w_k)dxdt$$

$$+\int_{B_r} w_k(x,\ \tau)\Delta w_k(x,\ t)dx\,|_{\tau_2}^{\tau_1}=\iint_{\Gamma_r} w_k\,\Delta\frac{\partial w_k}{\partial t}dxdt-\int_{B_r}|\nabla_x w_k(x,\tau_1)|^2dx+\int_{B_r}|\nabla_x w_k(x,\tau_2))|^2dx$$

$$\leq\iint_{\Gamma_r}\Delta w_k\frac{\partial w_k}{\partial t}\,dxdt-\int_{\tau_1}^{\tau_2}\int_{\partial B_r}\frac{\partial w_k}{\partial\nu}\,\frac{\partial w_k}{\partial t}\,dS\,dt+\int_{B_r}|\nabla_x\varkappa(x)|^2dx.$$

Hence

$$2\iint_{\Gamma_r} a_k(x,\ t)\,(\Delta w_k(x,\ t))^2dxdt\leq\int_{B_r}|\nabla_x\varkappa(x)|^2dx.$$

Since u_i are bounded on Γ_r then there exists a constant M such that $|b(x,\ t)|^2\leq M$ on Γ_r and

(3.9)
$$\iint_{\Gamma_r} b^2(x,\ t)a_k(x,\ t)(\Delta w_k)^2dxdt\leq M\int_{B_r}|\nabla_x\varkappa(x))|^2dx.$$

From (3.8), (3.9) and initial condition for $w_k(x,\ t)$ it follows

$$\int_{B_r} b(x,\ \tau_2)\varkappa(x)dx\leq[M\int_{B_r}|\nabla_x\varkappa(x)|^2dx]^{1/2}[\iint_{\Gamma_r}\frac{(a_k-a)^2}{a_k}\,dxdt]^{1/2}.$$

Letting $k\to\infty$ we obtain $\int_{B_r} b(x,\ \tau_2)\varkappa(x)dx\leq 0$. From the choice of $\varkappa(x)$ we conclude that $u_1(x,\ \tau_2)-u_2(x,\ \tau_2)\leq 0$, a. e. on B_r. But if u is a solution of (3.1), $u(x,\ t)=g(x,\ t)$

on $\partial\Gamma_r$ then u is a solution of the mixed problem with the same data on the smaller domain $B_r\times(\tau_1, \tau_3]$ for $\tau_1<\tau_3<\tau_2$. Lemma 3.1 is proved.

Remark 3.1. It follows from lemma 3.1 that the boundary value problem (3.1), (3.2) have no more than one weak solution.

The next lemma is proved in a similar way as in [8, 10, 13].

Lemma 3.2. *Let $g\in C(\partial\Gamma_r)$, $g\geqq 0$, $\Gamma_r=\Gamma_r(0;\ 0, T)$, then the boundary value problem*

$$(3.10)\qquad \begin{cases} v_t=\Delta\varphi(v) & in\ \Gamma_r \\ v(x,\ t)=g & on\ \partial\Gamma_r \end{cases}$$

has unique weak solution.

Proof. Consider the functions $w(x,\ t)=\varphi(v(x,\ t))$, $w_0(x,\ t)=\varphi(g(x,\ t))$ and denote $A=\sup_{\partial\Gamma}(w_0)+1$. Then the problem (3.10) changes into

$$(3.11)\qquad \begin{cases} w_t=\varphi'(\varphi^{-1}(w(x,\ t)))\Delta w & \text{in } \Gamma_r \\ w(x,\ t)=w_0(x,\ t) & \text{on } \partial\Gamma_r. \end{cases}$$

First we construct approximate solutions of (3.11). Let $\{w_{0,p}(x,\ t)\}_{p=1}^{\infty}$ is a sequence of function with next properties:

(a) $w_{0,p}\in C^{\infty}(R^{d+1})$

(b) $\frac{1}{p}\leqq w_{0,p}(x,\ t)\leqq A$

(c) $w_{0,1}\geqq w_{0,2}\geqq\ldots$ and $\lim\limits_{p\to\infty} w_{0,p}(x,\ t)=w_0(x,\ t)$

(d) $\sup\limits_{\Gamma_r}\{|\nabla w_{0,p}(x,\ t)|^2\}\leqq k$ where k doesn't depend on p.

The construction is as in [15]. We fix the sequence of functions $\{a_p(s)\}_{p=1}^{\infty}$ such that $a_p(s)=\varphi'(\varphi^{-1}(s))$ if $s\in[1/2p,\ A+1]$, $a_p(s)\in[\varphi'(\varphi^{-1}(1/2p+1)),\ \varphi'(A+2)]$ if $s\notin[1/2p,\ A+1]$, $a_p\in C^{\infty}(R)$, $a_p'(s)\geqq 0$, $\lim_{s\to\infty} a_p'(s)=0$. From [9] it follows that there exists a unique solution $w^p(x,\ t)$ of the problem

$$(3.12)\qquad \begin{cases} w_t^p-a_p(w^p(x,\ t))\Delta w^p=0 & \text{in } \Gamma_r \\ w^p(x,\ t)=w_{0,p}(x,\ t) & \text{on } \partial\Gamma_r \end{cases}$$

and $w^p\in H^{2+\beta,\ 1+\beta/2}(\Gamma_r)$, $0<\beta<1$, $\partial w^p/\partial x_j\partial t\in L^2(\Gamma_r)$, $j=1,\ldots,\ d$. Applying the comparison principle for the nondegenerate parabolic problems we get

$$(3.13)\qquad \begin{cases} 1/p\leqq w^p(x,\ t)\leqq A \\ w^p(x,\ t)\geqq w^{p+1}(x,\ t) \text{ for every } p \text{ and every } (x,\ t)\in\Gamma_r. \end{cases}$$

From (3.12) and (3.13) it follows that w^p is a solution of the equation (3.11) and function $u^p=\varphi^{-1}(w^p)$ is a solution of the problem

$$(3.14)\qquad \begin{cases} u_t^p-\Delta\varphi(u^p)=0 & \text{in } \Gamma_r \\ u^p(x,\ t)=\varphi^{-1}(w_{0,p}(x,\ t)) & \text{on } \partial\Gamma_r. \end{cases}$$

Moreover, from the monotony of φ^{-1} and (3.13) it follows

$$(3.15)\qquad \varphi^{-1}\left(\frac{1}{p+1}\right)\leqq u^{p+1}(x,\ t)\leqq u^p(x,\ t)\leqq\varphi^{-1}(A) \text{ on } \bar{\Gamma}_r.$$

Now we estimate the gradient of $\varphi(u^p)$. Multiplying equation in (3.14) by $\varphi(u^p)$ and integrating by parts we have

$$0=\iint_{\Gamma_r}[u_t^p-\Delta\varphi(u^p)]\,\varphi(u^p)\,dxdt=-\int_{S_r\times[0,T]}\varphi(u^p)\frac{\partial\varphi(u^p)}{\partial\nu}\,ds+\iint_{\Gamma_r}[\frac{\partial}{\partial t}F(u^p)+|\nabla_x\varphi(u^p)|^2]\,dxdt$$

$$=\iint_{\Gamma_r}|\nabla_x\varphi(u^p)|^2dxdt+\int_{B_r}F(u^p)|_0^T dx-\int_{S_r\times[0,T]}\varphi(u^p)\frac{\partial\varphi(u^p)}{\partial\nu}\,ds$$

where $F(\sigma)=\int_0^\sigma\varphi(\lambda)d\lambda$. Hence

$$\iint_{\Gamma_r}|\nabla_x\varphi(u^p)|^2\,dxdt\leqq|B_r|\,F(\varphi^{-1}(A))+T\,|S_r|\,K.A. \tag{3.16}$$

Fix arbitrary p and denote $s_0=2^{-1}\varphi^{-1}(1/p)$, $G(s)=\int_{s_0}^s(\varphi'(\lambda))^{1/2}d\lambda$. Multiplying equation (3.14) by $\partial\varphi(u^p)/\partial t$ and integrating on $\Gamma_r(\tau,t)=B_r\times(\tau,t)$ we get

$$0=\iint_{\Gamma(\tau,t)}\{[(\varphi'(u^p))^{1/2}\frac{\partial u^p}{\partial t}]^2-\frac{\partial\varphi(u^p)}{\partial t}\,\Delta\varphi(u^p)\}dxdt$$

$$=\iint_{\Gamma(\tau,t)}\{(\frac{\partial G(u^p)}{\partial t})^2+\frac{1}{2}\,\frac{\partial}{\partial t}|\nabla_x\varphi(u^p)|^2\}\,dxdt-\int_{S_r\times[\tau,t]}\frac{\partial\varphi(u^p)}{\partial t}\,\frac{\partial\varphi(u^p)}{\partial\nu}ds.$$

Hence for $0\leqq\tau<t\leqq T$ it is fulfilled

$$\iint_{\Gamma(\tau,t)}(\frac{\partial G(u^p)}{\partial t})^2\,dxdt+\frac{1}{2}\int_{B_r}|\nabla_x\varphi(u^p(x,t))|^2\,dx$$

$$\leqq\frac{1}{2}\int_{B_r}|\nabla_x\varphi(u^p(x,\tau)|^2dx+\int_{S_r\times[\tau,t]}\frac{\partial w_{0p}}{\partial t}\,\frac{\partial\varphi(u^p)}{\partial\nu}\,ds.$$

If we use this equality for $\tau=0$ and every $t\in(0,T]$ then we obtain

$$\max\{\iint_{\Gamma_r}(\frac{\partial G(u^p)}{\partial t})^2\,dxdt,\ \sup_{t\in[0,T]B_r}\int_{B_r}|\nabla_x\varphi(u^p)|^2dx\}\leqq C_0 \tag{3.17}$$

where C_0 depends only on K, A, r and φ. From the monotony of $\varphi'(s)$ it follows that

$$(\frac{\partial\varphi(u^p)}{\partial t})^2=(\varphi'(u^p)\frac{\partial u^p}{\partial t})^2\leqq\varphi'(\varphi^{-1}(A))\,[\varphi'(u^p)^{1/2}\frac{\partial u^p}{\partial t}]^2=\varphi'(\varphi^{-1}(A))\,(\frac{\partial G(u^p)}{\partial t})^2.$$

Then from (3.16), (3.17) we obtain $\iint_{\Gamma_r}|\operatorname{grad}\varphi(u^p)|^2dxdt\leqq C$ where $C=|B_r|\,F(\varphi^{-1}(A))$ $+T\,|S_r|\,A\,.K+c_0\varphi'(\varphi^{-1}(A))$.

From (3.15) it follows that for every $(x,t)\in\bar\Gamma_r$ we may define the function $u(x,t)=\lim_{p\to\infty}u^p(x,t)$ and $0\leqq u(x,t)\leqq\varphi^{-1}(A)$.

We show that $u(x,t)$ is a weak solution of (3.10). Indeed, from the Lebegue theorem it follows that $u^p\to u$ in $\Gamma^2(\Gamma_r)$, $\varphi(u^p)\to\varphi(u)$ in $L^2(\Gamma_r)$, $u^p(x,t_0)\to u(x,t_0)$ in $L^2(B_r)$ for fixed $t_0\in(0,T]$. Let $\chi\in C^{1,0}(\bar\Gamma_r)\cap C^{2,1}(\Gamma_r)$. Multiplying equation (3.14) by $\chi(x,t)$ $\chi=0$ on $\partial B_r x(0,T]$ and integrating by parts we obtain

$$\iint_{\Gamma_r}\{u^p(x,t)\frac{\partial\chi}{\partial t}+\varphi(u^p)\Delta\chi\}dxdt=\int_B u^p(x,T)\chi(x,T)dx$$

$$-\int_{B_r} \varphi^{-1}(w_{0,p}(x,\ 0))\,\chi(x,\ 0)\,dx + \int_{S_r\times(0,\ T]} \varphi(w_{0,p}(x,\ t))\frac{\partial\chi}{\partial\nu}ds.$$

Letting $p\to\infty$ we get

$$\iint_{\Gamma_r}\{u(x,\ t)\frac{\partial\chi}{\partial t}+\varphi(u)\Delta\chi\}\,dxdt=\int_{B_r} u(x,\ T)\chi(x,\ T)dx$$

$$-\int_{B_r} g(x,\ 0)\chi(x,\ 0)dx+\int_{S_r\times[0,\ T]} g(x,\ t)\frac{\partial\chi}{\partial\nu}ds.$$

Hence (ii) from (3.3) is fulfilled. Now to verify (i) we consider the function $w^p(\cdot, t)$: $[0,\ T]\to L^2(B_r)$ such that $w^p\in L^2(0,\ T;\ L^2(B_r))$ and $\partial w^p/\partial t\in L^2(0,\ T;\ L^2(B_r))$. Then w^p is absolutely continuous and $w^p(t)=w^p(0)+\int_0^t\partial w^p(\cdot,\ s)/\partial s ds$.

Hence for every τ, t, $0\leqq\tau<t\leqq T$ we have $w^p(t)-w^p(\tau)=\int_\tau^T(\partial w^p/\partial t)(\cdot,\ s)ds$ and from [16]

$$\|w^p(t)-w^p(\tau)\|_{L^2(B_r)}\leqq\int_\tau^t\|\frac{\partial w^p(\cdot,s)}{\partial s}\|_{L^2(B_r)}ds=\int_\tau^t(\int_{B_r}(\frac{\partial w^p}{\partial s}(x,\ s))^2dx)^{1/2}ds$$

$$\leqq(t-\tau)^{1/2}[\int_\tau^t\int_{B_r}(\frac{\partial w^p}{\partial s})^2dxds]^{1/2}\leqq(t-\tau)^{1/2}[\iint_{\Gamma_r}(\frac{\partial w^p}{\partial t})^2dxdt]^{1/2}\leqq C(t-\tau)^{1/2}.$$

Hence by the theorem of Artsela it follows that there exists subsequence p_n and a function $w(x,\ t)\in C(0,\ T;\ L^2(B_r))$ such that $w^{p_n}(x,\ t)\to w(x,\ t)$ in $C(0,\ T;\ L^2(B_r))$, then $w^{p_n}(x,\ t)\to w(x,\ t)$ in $L^2(\Gamma_r)$ and from above we obtain $w(x,\ t)=\varphi(u(x,\ t))$ in Γ_r and $\varphi^{-1}(w^{p_n})\to\varphi^{-1}(w)$ in $C(0,\ T;\ L^2(B_r))$ and $\varphi^{-1}(w)\in C(0,\ T;\ L^2(B_r))$.

Lemma 3.2 is proved.

Now we propose that for φ is fulfilled not only (H1) but also (H2). Then the weak solutions of (3.1), (3.2) has the finite speed of propagation property [7]. We shall use this property at the domain $\{(x,\ t),\ r_1<|x|<r_2,\ \tau_1\leqq t\leqq\tau_2\}$. The proof of next lemma is the same as in [7, 11].

Lemma 3.3. *Let $u(x,\ t)$ be a continuous weak solution of the problem* (3.1), (3.2) *with* $f=0$ $\operatorname{supp} g(x,\ 0)\subset B_{r_0}$ and $g(x,\ t)\equiv 0$ on $S_r\times[0,\ T]$. *Then* $\operatorname{supp} u(x,\ t)\subset B_{r(t)}$ *where*

$$r(t)=r_0[1+2(\psi^{-1}(l)\,t/r_0^2)^{1/2}]\ \ \textit{and}\ \ l=\sup_{\partial\Gamma_r} g(x,\ t).$$

Proof. Denote $w(s)=\varphi(\psi(s))$, then $w'(s)=\varphi'(\psi(s))\psi'(s)=\varphi'(\psi(s))\,\psi(s)/\varphi'(\psi(s))=\psi(s)$, $w''(s)=\psi'(s)$. Now fix arbitrary point $x^1\in B_r\setminus B_{r_0}$. We shall find such a point $t^1\in(0,\ T]$ for which $u(x^1,\ t)\equiv 0$ for $t\in(0,\ t^1]$.

Fix $s\in(-\infty,\ 0)$ such that $sx^1\notin B_r$ and for $\tau_0\in(0,\ T]$ let $a(\tau_0)=(\psi^{-1}(l)\tau_0)^{1/2}$. To find sufficient conditions for $u(x^1,\ \tau_0)=0$ we set $c_0=\psi^{-1}(l)/a(\tau_0)$, $A_0=a(\tau_0)+r_0+|sx^1|$ and define the function

$$\tilde{u}(x,\ t)=\psi([\xi]_+)=\begin{cases}\psi(\xi) & \text{for } \xi>0\\ 0 & \text{for } \xi\leqq 0\end{cases}$$

where $\xi(x,\ t)=C_0^2t-C_0|x-sx^1|+C_0A_0$. It is clear that the function $\tilde{u}(x,\ t)$ has the properties $\tilde{u}\in C^2(R_+^{d+1}\setminus 0)\cap C^0(R_+^{d+1})$ and $\tilde{u}_t-\Delta\varphi(\tilde{u})=C_0\psi(\xi)\ d-1/|x-sx'|$. Hence if we set $f(x, t)=\psi([\xi])_+ C_0(d-1)/|x-sx^1|$ then $\tilde{u}_t-\Delta\varphi(\tilde{u})=\tilde{f}$ for $\xi\neq 0$ and $|x-sx^1|>0$ Then $\tilde{u}(x, t)$ is a weak solution of the problem

$$\begin{cases} \tilde{u}_t-\Delta\varphi(\tilde{u})=\tilde{f} & \text{in } \Gamma_r \\ \tilde{u}(x, 0)=\psi(C_0[A_0-|x-sx^1|]_+) \end{cases}$$

Let us apply lemma 3.1 for $u(x, t)$ and $\tilde{u}(x, t)$. Since $\tilde{f}\geq 0$, then we must compare the initial data

$$\tilde{u}(x, 0)=\psi(C_0[A_0-|x-sx^1|]_+)\geq\psi(C_0[A_0-|x|-|sx^1|]_+)$$

$$=\psi(C_0[a(\tau_0)+r_0-|x|]_+)\geq\begin{cases} \psi(C_0a(\tau_0))=l & \text{for } |x|\leq r_0 \\ 0 & \text{for } |x|>r_0. \end{cases}$$

Hence $\tilde{u}(x, 0)\geq g(x, 0)$ for $x\in B_r$ and from Lemma 3.1 we obtain $u(x, t)\leq\tilde{u}(x, t)$, then $u(x, t)=0$ for $\xi(x, t)\leq 0$. Therefore it is sufficient to have $\xi(x^1, \tau_0)\leq 0$, i. e. $(\psi^{-1}(l)/a(\tau_0))\tau_0+a(\tau_0)+r_0+|sx^1|\leq(1-s)|x^1|$ and hence if $2(\psi^{-1}(l)\tau_0)^{1/2}+r_0\leq|x^1|$ then $u(x^1, \tau_0)=0$.

Lemma 3.3 is proved.

Remark 3.2. Let $h(x)\in C_0(R^d)$, $h\geq 0$, $u\in C(S_T)$, $u\geq 0$. Denote by $v_r(x, t)$ the unique weak solution of the problem

$$\begin{cases} v_t=\Delta\varphi(v) & \text{in } \Gamma_r=B_r\times(\tau_1, \tau_2] \\ v(x, t)=h(x)u(x, t) & \text{on } \partial\Gamma_r \end{cases}$$

where $0<\tau_1<\tau_2\leq T$.

Then there exists a constant $R^*>0$, which depends only on $h(x)$, $\sup_{x\in \operatorname{supp} h(x)}\{h(x)u(x, t)\}$, φ, such that $\operatorname{supp} v_r(\cdot, t)\subset B_{R^*}$ for $t\in[\tau_1, \tau_2]$ and $r>R^*$. Indeed, denote by $l_0=\sup\{h(x)u(x, t)$: $x\in \operatorname{supp} h(x)$, $t\in(0, T]\}$,

$$r_0=\inf\{r:\ \operatorname{supp} h(x)\subset B_r\} \text{ and } R^*=r_0[1+2(\psi^{-1}(l)\frac{T}{r_0^2})^{1/2}]$$

and consider the problem (3.1)), (3.2) for arbitrary fixed $r>R^*$. From Lemma 3·1 and Remark 3.1 it follows that there exists unique weak solution $v_r(x, t)$, then we apply Lemma 3.3.

4. Properties of the Cauchy problem. Denote by $S(\tau, T)$ the strip $\{(x, t)\in R^{d+1}$, $0<\tau<t\leq T\}$. We consider the Cauchy problem

$$\begin{cases} u_t=\Delta\varphi(u) & \text{in } S(\tau, T) \quad (4.1)\\ u(x, \tau)=u_\tau(x) & \text{in } R^d \quad (4.2)\end{cases}$$

where the function φ satisfy (H1).

Definition. *The function $u(x, t)$ is a weak solution of the problem* (4.1), (4.2) *if the next conditions are fulfilled*

(i) $u\in C(\tau, T;\ L^1(R^d))\cap L^\infty(S(\tau, T))$,

(4.3) (ii) $$\iint_{S(\tau, T)}[u\frac{\partial\chi}{\partial t}+\varphi(u)\Delta\chi]dxdt=\int_{R^d} u(x, T)\chi(x, T)dx-\int_{R^d} u_\tau(x)\chi(x, \tau)dx$$

where $\chi\in C^{2,1}(S(\tau, T))$ and supp $\chi(\cdot, t)$ *is compact for every* t,

(iii) $u(x, t)\geq 0$.

D e f i n i t i o n. *If the set* $\bigcup_{t\in(\tau,T)} \operatorname{supp} u(\cdot, t)$ *is bounded in* R^d *we say that the weak solution of the problem* (4.1), (4.2) *has compact support.*

T h e o r e m 4.1. *Let* $u(x, t)$ *be continuous in* S_T *and is a weak solution of* (4.1) *and let* $g(x)$ *is continuous function with compact support. Suppose that for some* $\tau\in(0, T)$ *is fulfilled* $g(x)\leq u(x, \tau)$, $x\in R^d$ *and if* w *is a weak solution with compact suppport of the problem* (4.1), (4.2) *with* $w(x, \tau)=g(x)$, *then* $w(x, \tau)\leq u(x, t)$ *in* $\overline{S(\tau, T)}$.

P r o o f. Denote $l_0=\sup_x\{g(x)\}$, $r_0=\inf\{r ; \operatorname{supp} g(x)\subset B_r\}$ and $r_1=r_0[1+2(\psi^{-1}(l_0)T/r_0^2)^{1/2}]$. For arbitrary $r>r_1$ let $v_r(x, t)$ is a weak solution of the boundary value problem

$$(4.4)\qquad \begin{cases} v_t=\Delta\varphi(v) & \text{in } \Gamma(0 ; \tau, T ; r) \\ v(x, \tau)=g(x) & \text{on } B_r \\ v(x, t)=0 & \text{in } \partial B_r\times(\tau, T]. \end{cases}$$

It follows from Lemma 3.2 and Remark 3.1, that such a solution exists and is unique, and from Lemma 3.3, that for every $t\in(\tau, T]$ is fulfilled

$$(4.5)\qquad \operatorname{supp} v_r(\cdot, t)\subset B_{r_1}.$$

Then for sufficiently large r we have

$$w(x, t)=\begin{cases} v_r(x, t) & \text{for } (x, t)\in\Gamma(0 ; \tau, T ; r) \\ 0 & \text{for } (x, t)\in S(\tau, T)\setminus\Gamma(0 ; \tau, T : r). \end{cases}$$

Now as in [3] is proved that function u is a weak solution of the boundary value problem

$$(4.6)\qquad \begin{cases} v_t=\Delta\varphi(v) & \text{in } \Gamma(0, \tau, T ; r) \\ v(x, t)=u(x, t) & \text{on } \partial\Gamma(0 : \tau, T ; r). \end{cases}$$

Then we apply Lemma 3.1 and obtain the Theorem 4.1.

The next theorem shows that the estimate on the finite speed of propagation depend as a matter of fact on the initial mass of the solution. Denote $J_r(u)=\frac{1}{|B_r|}\int_{B_r} u\,dx$.

T h e o r e m 4.2. *Let* u *is continuous weak solution of* (4.1) S_T *and* (H1), (H2) *are fulfilled. Then if* $\operatorname{supp} u(\cdot, 0)\subset B_{r_0}$ *and* $\delta>1$ *is arbitrary, for every* t *we have* $\operatorname{supp} u(\cdot, t)\subset B_{r(t)}$ where

$$(4.7)\qquad r(t)=r_0(2+\delta)\,[1+((t/r_0^2)\psi^{-1}((1/\delta^d J_{r_0}(u_0)))^{1/2}].$$

P r o o f. Fix arbitrary point $(\chi^0, t^0)\in S_T\setminus\Gamma(0 ; 0, T ; r_0(2+\delta))$ and $\delta>1$. Let $r=|x^0|-2r_0$. For every point $x^1\in B_r$ we define the plane

$$\Pi(x^0, x^1)=\{x\in R^d,\ \langle x, x^1-x^0\rangle=\langle 2^{-1}(x^0+x^1),\ (x^1-x_0)\rangle\}.$$

Then $\operatorname{dist}\{\Pi(x^0, x^1), (0)\}\geq 2^{-1}(|x^0|-|x^1|)\geq r_0$.

Hence x^1 and $\operatorname{supp} u(\cdot, 0)$ are in the same half space with respect to $\Pi(x^0, x^1)$. Moreover, x^0 is the reflection of x^1 in $\Pi(x^0, x^1)$. Define $v(x, t)=u(\bar{x}, t)$ for $t\in(0, T]$ where $\bar{x}$ is reflection of x with respect to $\Pi(x^0, x^1)$, then $v(y, t)=u(y, t)$ for $y\in\Pi(x^0, x^1)$ and $v(x, 0)=u(\bar{x}, 0)\leq u(x, 0)$ for x in the same half space with $\operatorname{supp} u(x, 0)$ with respect to $\Pi(x^0, x^1)$. Then applying Lemma 3.1 we obtain that $u(x^0, t^0)\leq u(x^1, t^0)$. Since $x^1\in B_r$ is an arbitrary point, then

$$u(x^0, t^0)\leq \inf_{x^1\in B_r} u(x^1, t^0).$$

Now for the solutions with compact support we have that the total mass is conserved and hence

$$\int_{B_{r_0}} u(x,\ 0)dx=\int_{R_d} u(x,\ t^0)dx\geq \int_{B_r} u(x,\ t^0)dx\geq \inf_{x'\in B_r} u(x^1,t^0)\,|B_r|.$$

Then for every $(x_0,\ t^0)\in S_T/\Gamma(0;\ 0,\ T;\ r_0(2+\delta))$ is fulfilled

$$u(x^0,\ t^0)\leqq(r^0/r)^d\,|B_{r_0}|^{-1}\int_{B_{r_0}} u(x,\ 0)dx\leqq(1/\delta^d)J_{r_0}(u_0). \tag{4.8}$$

It follows from Lemma 3.3 that there exists $R_1<\infty$ such that

$\bigcup_{t\in[0,\ T]}\operatorname{supp} u(\cdot,\ t)\subset B_{R_1}$. Set $R_0=\max\{R_1,\ r_0(2+\delta)+2[\psi^{-1}(1/\delta^d)J_{r_0}(u_0)T)]^{1/2}\}$.

Then we apply Lemma 3.3 for the function u in the domain $\Gamma(0;\ 0,\ T,\ R_0)\setminus\Gamma(0;\ 0,\ T;\ r_0(2+\delta))$ with $l=(1/\delta^d)J_{r_0}(u_0)$ and using (4.8) the theorem 4.2 is proved.

Now for the continuous weak solutions of the problem (4.1), (4.2) we established some pointwise estimates of such solutions, using conditions (H1)), (H2), (H3). To obtain a regularizing effect for solutions of (4.1), (4.2) we follow [12].

Theorem 4.3. *Let u is a continuous weak solution of* (4.1), (4.2) *and* (H1), (H2), (H3) *are fulfilled. If $\beta(s)\in C^1([t,\ T])$ is such a function that $\beta(t)=1$, $\beta(T)=0$, then for every $x_0\in R^d$, $r>0$ is fulfilled*

$$t^\alpha v(x,\ t)\leq T^\alpha v(x_0,\ T)+4^{-1}\,|x-x_0|^2\int_t^T s^\alpha\beta'^2(s)ds \tag{4.9}$$

here $v=\psi^{-1}(u)$.

Proof. First we obtain as in [12] a regularizing effect. For $\varepsilon>0$ let w is a classical solution of the problem

$$\begin{cases} w_t-\Delta\varphi(w)=0 & \text{in } S_T\\ w(x,\ 0)=u_0(x)+\varepsilon & \text{in } R^d\end{cases}.$$

Then $v_{\varepsilon t}=g(v_\varepsilon)\Delta v_\varepsilon+|\nabla v_\varepsilon|^2$, where $v_\varepsilon=\psi^{-1}(w)$ $g(v_\varepsilon)=\varphi'(\psi(v_\varepsilon))$ and $w_t=w\Delta v_\varepsilon+(\varphi'(w)/w)\,|\nabla w|^2$. The function $p=\Delta v_\varepsilon$ satisfy $p_t\geq g(v_\varepsilon)\Delta p+2(\nabla g(v_\varepsilon)+\nabla v_\varepsilon)\nabla p+g''(v_\varepsilon)\,|\nabla v_\varepsilon|^2p+(g'(v_\varepsilon)+(2/d))p^2\equiv\mathscr{L}(p)$ where we use the inequality $\Delta|\nabla v_\varepsilon|^2\geq(2/d)(\Delta v_\varepsilon)^2+2\nabla v_\varepsilon\nabla(\Delta v_\varepsilon)$.

Let $h(v_\varepsilon)>0$ and for $z=-h(v_\varepsilon)/t$ we have $z_t\leq\mathscr{L}(z)$. Since $p_t\geq\mathscr{L}(p)$ and $z\leq p$ for small t then $z\leq p$ for every $t>0$ which follows from the comparison principles of [9]. For arbitrary function $h(s)$ we calculate $z_t-\mathscr{L}(z)$ with $z=-h(v_\varepsilon)/t$

$$z_t=\mathscr{L}(z)+\left[\frac{1}{h(v_\varepsilon)}-\left(g'(v_\varepsilon)+\frac{2}{d}\right)\right]z^2+[(gh)''+h']\frac{|\nabla v_\varepsilon|^2}{t}.$$

If

$$\begin{cases}(gh)''+h'\leq 0\\ \dfrac{1}{h}\leq g'+\dfrac{2}{g}, \quad h>0 \text{ for } s>0\end{cases} \tag{4.10}$$

then we can apply a comparison principle for $p=\Delta v_\varepsilon$ and $z=-h(v_\varepsilon)/t$.

The solution of inequalities (4.10) where $g(s)=\varphi'(\psi(s))=\psi(s)/\psi'(s)$ is $h(s)=(\psi(s)\varphi'(\psi(s))^{-1}\times\int_{s_0}^s F(\sigma)\psi(\sigma)d\sigma$ with F from (H3). So

$$t\Delta v_\varepsilon \geqq -\Big(\int_{\psi(s_0)}^{\psi(s)} F(\lambda)\varphi'(\lambda)d\lambda\Big)/\psi(s)\varphi'(\psi(s))$$

and using (H3) we obtain

$$t\Delta v_\varepsilon \geqq -\Big(\int_0^{\psi(v_\varepsilon)} F(\lambda)\varphi'(\lambda)d\lambda\Big)/\psi(v_\varepsilon)\varphi'(\psi(v_\varepsilon)). \tag{4.11}$$

Now we get the expression

$$w(s)=s^\alpha v\varepsilon(x_0+\beta(s)(x-x_0),\ s)+4^{-1}|x-x_0|^2\int_0^s \sigma^\alpha\beta'^2(\sigma)d\sigma$$

where $\beta(s)\in C^1$ is arbitrary function. Then

$$\frac{dw}{ds}=\alpha s^{\alpha-1}v_\varepsilon+s^\alpha v_{\varepsilon s}+s^\alpha\beta'(s)\sum_1^d v_{\varepsilon\xi_j}(x_j-x_{0j})+4^{-1}|x-x_0|^2 s^\alpha\beta'^2(s)$$

$$=\alpha s^{\alpha-1}v_\varepsilon+s^\alpha\varphi(\psi(v_\varepsilon))(\Delta v_\varepsilon+s^\alpha|\nabla v_\varepsilon|^2+s^\alpha\beta'(s)\sum_1^d v_{\varepsilon\xi_j}(x_j-x_{0j})+4^{-1}|x-x_0|^2 s^\alpha\beta'^2(s)$$

$$\geqq s^{\alpha-1}\Big[\alpha v_\varepsilon-(\psi(v_\varepsilon))^{-1}\int_0^{\psi(v_\varepsilon)} F(\lambda)\varphi'(\lambda)d\lambda\Big]+s^\alpha\sum_1^d (v_{\varepsilon\xi_j}+2^{-1}|x_j-x_{0j}|_{\beta'(s)})^2.$$

From (H3) we have $\alpha\psi(v_\varepsilon)\int_0^{\psi(v_\varepsilon)}\varphi'(\lambda)/\lambda d\lambda \geqq \int_0^{\psi(v_\varepsilon)} F(\lambda)\varphi'(\lambda)d\lambda$ and hence $\alpha v_\varepsilon\psi(v_\varepsilon) \geqq \int_0^{\psi(v_\varepsilon)} F(\lambda)\varphi'(\lambda)d\lambda$.

Then $dw/ds\geqq 0$ which means that w is nondecreasing function on s and then $w(t)\leqq w(T)$ for $0<t<T$. We choose $\beta(s)$ such that $\beta(t)=1$, $\beta(T)=0$. Then (4.9) is obtained, for v_ε, and letting $\varepsilon\to 0$ the theorem 4.3 is proved.

In the next theorem we study the properties of the continuous weak solutions in $\bar S_T$ with compact support.

Theorem 4.4. *Let* φ *satisfy* (H1)—(H3), *and* u *is continuous weak solution of* (4.1), (4.2) in $\bar S_T$ *with* $\operatorname{supp} u(\cdot,0)\subset B_{r_0}(x_0)$ *and* $\delta>1$. *Then*

$$J_{r_0}(u_0)\leqq C\Psi((T/r_0^2)^{\alpha/(1-\alpha)}[\psi^{-1}(u(x_0,\ T))]^{1/1-\alpha}+r_0^2/T) \tag{4.12}$$

where C *depends only on* φ, d, *and doesn't depend on* u.

Proof. From (4.9) we obtain for $|x-x_0|\leqq\rho$, $u(x,t)\leqq\psi(t^{-\alpha}T^\alpha v(x_0,T)+t^{-\alpha}\rho^2/4\times\int_t^T s^\alpha\beta'^2(s)ds)$.

Integrating over $|x-x_0|\leqq\rho$, then

$$\psi^{-1}(J_\rho(u(x,\ t)))\leqq T^\alpha t^{-\alpha}\psi^{-1}(u(x_0,\ T))+t^{-\alpha}\rho^2/4\int_t^T s'\beta'^2(s)ds.$$

Let $\beta(s)=(T-s)/(T-t)$, then $\int_t^T s^\alpha\beta'^2(s)ds\leqq T^\alpha/(T-t)$. From Theorem 4.2 we have that there exists a function $r(t)\in C([0,\ T])$, such that $\operatorname{supp} u(\cdot,t)\subset B_{r(t)}(x_0)$. Then for $\rho=r(t)$ we obtain

$$\psi^{-1}(J_{r(t)}(u))\leqq t^{-\alpha}\Big[T^\alpha\psi^{-1}(u(x_0,\ T)+4^{-1}r^2(t)\int_t^T s^\alpha\beta'^2(s)ds\Big].$$

Using $\int_{B_{r_0}(x_0)}u_0dx=\int_{B_{r(t)}(x_0)}udx$ we get

$$\psi^{1-}\Big(\frac{|B_{r_0}(x_0)|}{|B_{r_{(t)}}(x_0)|}J_{r_0}(u_0)\Big)\leq\Big(\frac{T}{t}\Big)^{\alpha}\Big[\psi^{-1}(u(x_0,T))+\frac{r^2(t)}{4(T-t)}\Big]. \tag{4.13}$$

Now consider two cases:

(i) $\psi^{-1}(1/\delta^{-d}J_{r_0}(u_0))\leq 2\frac{r_0^2}{T}$,

(ii) $\psi^{-1}(1/\delta^{-d}J_{r_0}(u_0))>2r_0^2/T$.

In the first case the estimate (4.12) is fulfilled. In the second case set $t_\delta=r_0^2/\psi^{-1}((1/\delta^d)J_{r_0}(u_0))$ then $t_\delta\leq T/2$ and $T-t_\delta\geq T/2$ so $r^2(t_\delta)/4(T-t_\delta)\leq r_0^2(2(2+\delta))^2/2T$.
From (4.13) we obtain

$$\psi^{-1}([2(2+\delta)]^{-d}J_{r_0}(u_0))\leq[T^\alpha/[r_0^2/\psi^{-1}(1/\delta^d J_{r_0}(u_0))]^\alpha[\psi^{-1}(u^{-1}(x_0,\ T))+r_0^2(2(2+\delta))^2/2T].$$

From the condition (H3) it follows that for $\beta\in(0,1)$ there exists $A_\beta>0$ such that $\psi^{-1}(\beta l)/\psi^{-1}(l)\geq A_\beta$ for all $l\in R_+$, hence with $\beta_0=\delta^d/(2(2+\delta))^d$ we have

$$\psi^{-1}(1/\delta^{-d}J_{r_0}(u_0))\leq[1/A_{\beta_0}(T/r_0^2)^\alpha\{\psi^{-1}(u(x_0,\ T))+r_0^2(2(2+\delta))^2/2T\}]^{1/1-\alpha}$$

and we obtain (4.12). Theorem 4.4 is proved.

5. **Proof of Theorem 2.1**. First we shall show that if u is a continuous weak solution of (4.1), (4.2) in $\bar S_T$ then the estimate (4.12) take place. From Theorem 4.4 it is clear for solutions with compact initial data. Set the function $h_\varepsilon(x)\in C(R^d)$, and

$$h_\varepsilon(x)=\begin{cases}1, & |x|\leq r_0\\ 0, & |x|>r_0+\varepsilon\end{cases}$$

and let $w^\varepsilon(x,t)$ is a solution of the problem

$$\begin{cases} w_t^\varepsilon=\Delta\varphi(w^\varepsilon) & \text{in } S_T\\ w^\varepsilon(x,0)=h_\varepsilon(x)u(x,0) & \text{in } R^d.\end{cases}$$

Then $\operatorname{supp} w^\varepsilon(x,0)\subset B_{r_0+\varepsilon}$. From Theorem 4.1 it follows that $u(x,t)\geq w^\varepsilon(x,t)$ in $\bar S_T$. From Theorem 4.4 for w^ε we have the estimate (4.12)

$$|B_{r_0+\varepsilon}|^{-1}\int_{B_{r_0+\varepsilon}}w^\varepsilon_{(x,0)}dx\leq C_1\psi[(T/(r_0+\varepsilon)^2)^{\alpha/(1-\alpha)}([\psi^{-1}(u(x_0,T))]^{1/(1-\alpha)}+(r_0+\varepsilon)^2/T].$$

Letting $\varepsilon\to 0$ we obtain the estimate (4.12) for u.
Now for every $\tau\in(0,T]$ we have

$$J_{r_0}(u(x,\tau))\leq C\psi[((T-\tau)/r_0^2)^{\alpha/(1-\alpha)}[\psi^{-1}(u(x_0,\ T-\tau))]^{1/(1-\alpha)}+r_0^2/(T-\tau)]$$

where C is independent of τ.

Then there exists a sequence τ_n such that $\lim_{\tau_n\to 0}\int_{R_d}\chi(x)u(x,\tau_n)dx=\int_{R_d}\chi(x)\mu(dx)$ for every $\chi\in C_0(R^d)$ and μ satisfy the estimate

$$|B_{r_0}|^{-1}\int_{B_{r_0}}\mu(dx)\leq C\psi[(T/r_0^2)^{\alpha/(1-\alpha)}[\psi^{-1}(u(x_0,T))]^{1/(1-\alpha)}+r_0^2/T].$$

The prove of uniqueness of μ is the same as in the case $\varphi(s)=s^m$ in [3]. Indeed, by Theorem 4.4 for $t\in(0,T/2)$ we have

$$\int_{B_1(0)}u(x,t)dx\leq C\psi[(T/2)^{\alpha/(1-\alpha)}[\psi^{-1}(u(0,\ T/2))]^{1/(1-\alpha)}+1/T]. \tag{5.1}$$

Since $u(x, t)\neq 0$, assume that $\int_{B_1(0)} u(x, t)dx\neq 0$, now for each $s\in(0, T/4)$ and arbitrary small $\eta\in R_+$ set $\rho=\int_{B_1(0)} u(x, s)dx-\eta$. From Theorem 4.1 and Theorem 4.2 it follows that there exists $\tau\in R_+$ such that

$$\int_{B_{1+\varepsilon}(0)} u(x, t+s)dx \geqq \int_{B_1(0)} u(x, s)dx-\eta$$

for all t, which satisfy $0<t<\min(T/4, \tau)$.

Suppose that u converges weakly to a mesure μ_1 along the sequence $\{\varepsilon_k^1\}$ and to a mesure μ_2 along a sequence $\{\varepsilon_k^2\}$. Then take the limit of both sides of (5.1) as $\varepsilon_k^1\to 0$, setting t fix and since u is continuous we find

$$\int_{B_{1+\varepsilon}(0)} u(x, t)dx \geqq \mu_1(B_1(0))-\eta.$$

Now set $t=\varepsilon_k^2$ and when $\varepsilon_k^2\to 0$ we get $\mu_2(B_{1+\varepsilon}(0))\geqq\mu_1(B_1(0))-\eta$. Since ε and η are arbitrary we conclude that $\mu_2(B_1(0))\geqq\mu_1(B_1(0))$. On the other side, taking the limit with $t=\varepsilon_k^2$, followed by the limit with $t=\varepsilon_k^1$ we obtain $\mu_1(B_1(0))\geqq\mu_2(B_1(0))$. Therefore $\mu_1(B_1(0))=\mu_2(B_1(0))$. This will be done on every ball in R^d so that $\mu_1\equiv\mu_2$. The theorem 2.1 is proved.

The main results of this paper have been published without proof in [17].

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