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A CLASS OF UNIMODAL DISTRIBUTIONS

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1. Introduction. A distribution functon $F(x)$ is called unimodal at the point $x=0$ [the mode of the distribution function], in short: (0) unimodal, if $F(x)$ is convex in the interval $(-\infty, 0)$ and is concave in the interval $(0, \infty)$. A function $\varphi(u)$ is the characteristic function of a (0) unimodal distribution function $F(x)$ if and only if

$$(1.1) \qquad \varphi(u)=\frac{1}{u}\int_0^u \vartheta(y)\,dy, \quad u\varepsilon R,$$

where $\vartheta(u)$ is some characteristic function [1, Theorem 4. 5. 1.].

Let $F(x)$ be a distribution function with finite second moment and characteristic function $\varphi(u)$, then

$$(1.2) \qquad \psi(u)=\frac{\varphi'(u)-\varphi'(0)}{\varphi''(0)\,u}$$

is also a characteristic function [1, Theorem 12. 2. 5]. From (1.1) and (1.2) it easily follows that (1.2) is the characteristic function of a (0) unimodal distribution function. Characteristic functions of the form (1.2) will be called members of class C.

This paper is devoted to the study of the class C. Certain relationships between characteristic functions of (0) unimodal distributions and characteristic functions of class C are established. Moreover, the paper investigates the limiting behavior of certain sequences of class C.

2. Results. First we establish certain relationships between characteristic functions of (0) unimodal distributions and characteristic functions of class C.

Theorem 1. *Let $\varphi(u)$ be the characteristic function of a (0) unimodal distribution function $F(x)$. Then there exists a characteristic function $\gamma(u)$ of a distribution function $G(x)$ on $[0, \infty)$ such that*

$$\varphi(u)\gamma\left(i\int_0^u \varphi(y)\,y\,dy\right)$$

is a characteristic function of class C.

Proof. From (1.1) and theorem 12. 2. 8 of [1] it follows that

$$k(u)=\exp\{-\int_0^u\int_0^y \vartheta(x)\,dx\,dy\}=\exp\{-\int_0^u \varphi(y)\,y\,dy\}$$

is the characteristic function of an infinitely divisible distribution function with finite second moment. Let

$$v(u)=\int_0^\infty \exp\{-x\int_0^u \varphi(y)\,y\,dy\}\,dH(x)$$

with $H(x)$ a distribution function on $[0, \infty)$ having finite second moment, then $\nu(u)$ is the characteristic function of a distribution function with finite second moment. Set $\gamma(u)$ for the characteristic function of the distribution function

$$G(x)=\int_0^x ydH(y)/\int_0^\infty ydH(y).$$

Since

$$\frac{\nu'(u)-\nu'(0)}{\nu''(0)u}=\varphi(u)\gamma(i\int_0^u\varphi(y)ydy), \tag{2.1}$$

from (1.2) it follows that $\varphi(u)\gamma(i\int_0^u\varphi(y)ydy)$ is a characteristic function of class C.

Corollary 1. *Let $\varphi(u)$ be the characteristic function of a distribution function $F(x)$ on $[0, \infty)$ with finite second moment. Then*

$$\frac{\varphi(u)-1}{i\mu u}\varphi'(i\int_0^u\frac{\varphi(y)-1}{i\mu y}ydy)/i\mu$$

is the characteristic function of an (0) unimodal distribution function.

Proof. The function $\varphi'(u)/i\mu$ is the characteristic function of the distribution function $F_1(x)=\int_0^x ydF(y)/\mu$, where μ is the mean value of $F(x)$. Letting $\vartheta(u)=\varphi'(u)/i\mu$ in (1.1), we get that $[\varphi(u)-1]/i\mu u$ is the characteristic function of an (0) unimodal distribution function. Substituting $\varphi(u)$ by $[\varphi(u)-1]/i\mu u$ and $\gamma(u)$ by $\varphi'(u)/i\mu$ in (2.1), we get that

$$\frac{\varphi(u)-1}{i\mu u}\varphi'(i\int_0^u\frac{\varphi(u)-1}{i\mu y}ydy)$$

is the characteristic function of an (0) unimodal distribution function.

The limiting behavior of certain sequences of characteristic functions of class C is investigated in the following theorems.

Theorem 2. *Let $\varphi(u)$ be the characteristic function of an (0) unimodal distribution function $F(x)$. Then there exists a sequence $\{\omega_n(u)\colon n=1, 2, \dots\}$ of characteristic functions of class C such that $\lim_{n\to\infty}\omega_n(u)=\varphi(u)$, $u\varepsilon R$.*

Proof. The characteristic function $\varphi(u)$ can be written in the form (1.1). Set

$$\varphi_n(u)=\exp\{-\frac{1}{n}\int_0^u\int_0^y\vartheta(x)dxdy\},\quad n=1, 2, \dots,$$

then from theorem 12. 2. 8 of [1] it follows that $\varphi_n(u)$ is the characteristic function of an infinitely divisible distribution function with finite second moment. Hence

$$\omega_n(u)=\frac{\omega_n'(u)-\varphi_n'(0)}{\varphi_n''(0)u}=[\frac{1}{u}\int_0^u\vartheta(y)dy]\exp\{-\frac{1}{n}\int_0^u\int_0^y\vartheta(x)dxdy\} \tag{2.2}$$

is a sequence of characteristic functions of class C. From (2.2) it follows that

$$\lim_{n\to\infty}\omega_n(u)=\frac{1}{u}\int_0^u\vartheta(y)dy=\varphi(u),\quad u\varepsilon R.$$

Corollary 2. *Let $\varphi(u)$ be the characteristic function of a distribution function having a convex density in the half lines $x>0$ and $x<0$. Then there exists a sequence $\{\omega_n(u)\colon n=1, 2, \dots\}$ of characteristic functions of class C such that $\lim_{n\to\infty}\omega_n(u)=\varphi(u)$, $u\varepsilon R$.*

Proof. From theorem 1 of [2] it follows that the characteristic function $\varphi(u)$ can be written in the form

$$\varphi(u)=\frac{2}{u^2}\int_0^u \beta(uy)\,(1-y)\,dy, \tag{2.3}$$

where $\beta(u)$ is a characteristic function. Set

$$\vartheta(u)=\frac{2}{u^2}\int_0^u \beta(y)\,y\,dy \tag{2.4}$$

in (2.2) of theorem 2. Then from (2.3) and (2.4) it follows that

$$\lim_{n\to\infty}\omega_n(u)=u^{-1}\int_0^u \vartheta(y)\,dy=2u^{-2}\int_0^u \beta(uy)\,(1-y)\,dy=\varphi(u),\quad u\varepsilon R.$$

The representation of every member of class C as the limit of a sequence of characteristic functions of class C is given in the following theorem.

Theorem 3. *Let $\omega(u)$ be a characteristic function of class C. Then there exists a sequence $\{\omega_n(u):\ n=1, 2, \dots\}$ of characteristic functions of class C such that*

$$\lim_{n\to\infty}\omega_n(u)=\omega(u),\quad u\varepsilon R.$$

Proof. Consider the sequence of Poisson-type characteristic functions $\varphi_n(u)=\exp\{n^{-1}[\varphi(u)-1]\}$, $\varphi(u)$ is the characteristic function of a distribution function with finite second moment. Set

$$\omega_n(u)=\frac{\varphi_n'(u)-\varphi_n'(0)}{\varphi_n''(0)u}=\frac{\varphi'(u)\exp\{n^{-1}[\varphi(u)-1]\}-\varphi'(0)}{\varphi''(0)u+[\varphi''(0)]^2u/n}\,. \tag{2.5}$$

Then $\{\omega_n(u):\ n=1, 2, \dots\}$ is a sequence of characteristic functions of class C From (2.5) it follows that $\lim_{n\to\infty}\omega_n(u)=\dfrac{\varphi'(u)-\varphi'(0)}{\varphi''(0)u}=\omega(u)$, $u\varepsilon R$.

REFERENCES

1. E. Lukacs. Characteristic functions. London, 1970.
2. G. N. Sakovic. On the characteristic functions of concave distributions. *Theor. Probability and Math. Statistics*, **6**, 1972, 103-108.

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