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# Сердика

## Българско математическо списание

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## A CLASS OF UNIMODAL DISTRIBUTIONS

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**1. Introduction.** A distribution function  $F(x)$  is called unimodal at the point  $x=0$  [the mode of the distribution function], in short: (0) unimodal, if  $F(x)$  is convex in the interval  $(-\infty, 0)$  and is concave in the interval  $(0, \infty)$ . A function  $\varphi(u)$  is the characteristic function of a (0) unimodal distribution function  $F(x)$  if and only if

$$(1.1) \quad \varphi(u) = \frac{1}{u} \int_0^u \mathfrak{A}(y) dy, \quad u \in R,$$

where  $\mathfrak{A}(u)$  is some characteristic function [1, Theorem 4.5. 1.].

Let  $F(x)$  be a distribution function with finite second moment and characteristic function  $\varphi(u)$ , then

$$(1.2) \quad \psi(u) = \frac{\varphi'(u) - \varphi'(0)}{\varphi''(0)u}$$

is also a characteristic function [1, Theorem 12. 2. 5]. From (1.1) and (1.2) it easily follows that (1.2) is the characteristic function of a (0) unimodal distribution function. Characteristic functions of the form (1.2) will be called members of class  $C$ .

This paper is devoted to the study of the class  $C$ . Certain relationships between characteristic functions of (0) unimodal distributions and characteristic functions of class  $C$  are established. Moreover, the paper investigates the limiting behavior of certain sequences of class  $C$ .

**2. Results.** First we establish certain relationships between characteristic functions of (0) unimodal distributions and characteristic functions of class  $C$ .

**Theorem 1.** *Let  $\varphi(u)$  be the characteristic function of a (0) unimodal distribution function  $F(x)$ . Then there exists a characteristic function  $\gamma(u)$  of a distribution function  $G(x)$  on  $[0, \infty)$  such that*

$$\varphi(u)\gamma\left(i \int_0^u \varphi(y) y dy\right)$$

is a characteristic function of class  $C$ .

**Proof.** From (1.1) and theorem 12. 2. 8 of [1] it follows that

$$k(u) = \exp\left\{-\int_0^u \int_0^y \mathfrak{A}(x) dx dy\right\} = \exp\left\{-\int_0^u \varphi(y) y dy\right\}$$

is the characteristic function of an infinitely divisible distribution function with finite second moment. Let

$$v(u) = \int_0^\infty \exp\left\{-x \int_0^u \varphi(y) y dy\right\} dH(x)$$

with  $H(x)$  a distribution function on  $[0, \infty)$  having finite second moment, then  $v(u)$  is the characteristic function of a distribution function with finite second moment. Set  $\gamma(u)$  for the characteristic function of the distribution function

$$G(x) = \int_0^x y dH(y) / \int_0^\infty y dH(y).$$

Since

$$(2.1) \quad \frac{v'(u) - v'(0)}{v''(0)u} = \varphi(u) \gamma \left( i \int_0^u \varphi(y) y dy \right),$$

from (1.2) it follows that  $\varphi(u) \gamma \left( i \int_0^u \varphi(y) y dy \right)$  is a characteristic function of class C.

**Corollary 1.** *Let  $\varphi(u)$  be the characteristic function of a distribution function  $F(x)$  on  $[0, \infty)$  with finite second moment. Then*

$$\frac{\varphi(u) - 1}{i\mu u} \varphi' \left( i \int_0^u \frac{\varphi(y) - 1}{i\mu y} y dy \right) / i\mu$$

is the characteristic function of an (0) unimodal distribution function.

**Proof.** The function  $\varphi'(u)/i\mu$  is the characteristic function of the distribution function  $F_1(x) = \int_0^x y dF(y)/\mu$ , where  $\mu$  is the mean value of  $F(x)$ . Letting  $\vartheta(u) = \varphi'(u)/i\mu$  in (1.1), we get that  $[\varphi(u) - 1]/i\mu u$  is the characteristic function of an (0) unimodal distribution function. Substituting  $\varphi(u)$  by  $[\varphi(u) - 1]/i\mu u$  and  $\gamma(u)$  by  $\varphi'(u)/i\mu$  in (2.1), we get that

$$\frac{\varphi(u) - 1}{i\mu u} \varphi' \left( i \int_0^u \frac{\varphi(y) - 1}{i\mu y} y dy \right)$$

is the characteristic function of an (0) unimodal distribution function.

The limiting behavior of certain sequences of characteristic functions of class C is investigated in the following theorems.

**Theorem 2.** *Let  $\varphi(u)$  be the characteristic function of an (0) unimodal distribution function  $F(x)$ . Then there exists a sequence  $\{\omega_n(u) : n = 1, 2, \dots\}$  of characteristic functions of class C such that  $\lim_{n \rightarrow \infty} \omega_n(u) = \varphi(u)$ ,  $u \in R$ .*

**Proof.** The characteristic function  $\varphi(u)$  can be written in the form (1.1). Set

$$\varphi_n(u) = \exp \left\{ -\frac{1}{n} \int_0^u \int_0^y \vartheta(x) dx dy \right\}, \quad n = 1, 2, \dots,$$

then from theorem 12.2.8 of [1] it follows that  $\varphi_n(u)$  is the characteristic function of an infinitely divisible distribution function with finite second moment. Hence

$$(2.2) \quad \omega_n(u) = \frac{\omega'_n(u) - \varphi'_n(0)}{\varphi''_n(0)u} = \left[ \frac{1}{u} \int_0^u \vartheta(y) dy \right] \exp \left\{ -\frac{1}{n} \int_0^u \int_0^y \vartheta(x) dx dy \right\}$$

is a sequence of characteristic functions of class C. From (2.2) it follows that

$$\lim_{n \rightarrow \infty} \omega_n(u) = \frac{1}{u} \int_0^u \vartheta(y) dy = \varphi(u), \quad u \in R.$$

**Corollary 2.** *Let  $\varphi(u)$  be the characteristic function of a distribution function having a convex density in the half lines  $x > 0$  and  $x < 0$ . Then there exists a sequence  $\{\omega_n(u) : n = 1, 2, \dots\}$  of characteristic functions of class C such that  $\lim_{n \rightarrow \infty} \omega_n(u) = \varphi(u)$ ,  $u \in R$ .*

Proof. From theorem 1 of [2] it follows that the characteristic function  $\varphi(u)$  can be written in the form

$$(2.3) \quad \varphi(u) = \frac{2}{u^2} \int_0^u \beta(uy) (1-y) dy,$$

where  $\beta(u)$  is a characteristic function. Set

$$(2.4) \quad \vartheta(u) = \frac{2}{u^2} \int_0^u \beta(y) y dy$$

in (2.2) of theorem 2. Then from (2.3) and (2.4) it follows that

$$\lim_{n \rightarrow \infty} \omega_n(u) = u^{-1} \int_0^u \vartheta(y) dy = 2u^{-2} \int_0^u \beta(uy) (1-y) dy = \varphi(u), \quad u \in \mathcal{R}.$$

The representation of every member of class  $C$  as the limit of a sequence of characteristic functions of class  $C$  is given in the following theorem.

Theorem 3. Let  $\omega(u)$  be a characteristic function of class  $C$ . Then there exists a sequence  $\{\omega_n(u): n=1, 2, \dots\}$  of characteristic functions of class  $C$  such that

$$\lim_{n \rightarrow \infty} \omega_n(u) = \omega(u), \quad u \in \mathcal{R}.$$

Proof. Consider the sequence of Poisson-type characteristic functions  $\varphi_n(u) = \exp\{n^{-1}[\varphi(u)-1]\}$ ,  $\varphi(u)$  is the characteristic function of a distribution function with finite second moment. Set

$$(2.5) \quad \omega_n(u) = \frac{\varphi_n'(u) - \varphi_n'(0)}{\varphi_n''(0)u} = \frac{\varphi'(u) \exp\{n^{-1}[\varphi(u)-1]\} - \varphi'(0)}{\varphi''(0)u + [\varphi''(0)]^2 u/n}.$$

Then  $\{\omega_n(u): n=1, 2, \dots\}$  is a sequence of characteristic functions of class  $C$ . From (2.5) it follows that  $\lim_{n \rightarrow \infty} \omega_n(u) = \frac{\varphi'(u) - \varphi'(0)}{\varphi''(0)u} = \omega(u), \quad u \in \mathcal{R}.$

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