Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

BOUNDS FOR THE PROBABILISTIC CHARACTERISTICS OF LATENT FAILURE TIMES WITHIN COMPETING RISKS FRAMEWORK

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A survival model of two dependent competing risks is discussed. Bounds are given for crude survival probabilities when marginal distributions of the latent times T_1, T_2 and the distribution of the value T-min (T_1, T_2) are fixed. The bounds obtained have been used for interval estimation of the probabilities pr $(T_1>T_2)$ and pr $(T_2>T_1)$ for complete and censored samples resulting from studying the survival of animals after combined injuries.

1. Introduction. The most common approach to constructing a competing risks model (R. L. Prentice et al., 1978) is based on introducing unobserved values — the latent failure times $T_1, \dots, T_n$ — corresponding to each type of failure $i=1, \dots, n$ under n possible causes of failure. In other words, it is assumed that each cause i induces the latent time T_i, $i=1, \dots, n$. The latent times $T_1, \dots, T_n$ induced by all the causes operative form a system of nonnegative mutually dependent random variables with the arbitrary joint distribution $W(t_1, \dots, t_n)=\text{pr}\ (T_1 \leq t_1, \dots, T_n \leq t_n)$. The observed life length T of the subject is assumed to be equal to min $(T_1, \dots, T_n)$. The distribution function $F(t)=\text{pr}\ (T \leq t)$ of the value $T=\min(T_1, \dots, T_n)$ and the corresponding survial function $\bar{F}(t)=1-F(t)$ are defined for $t \in [0, \infty)$ and $F(0)=0$.

In works on the theory of competing risks it is generally assumed that failure is, eventually, due to just one of the n causes, and a pair of the observed values (T, J), where $J=(j\colon T_j<T_k,\ k=1, \dots, n)$ are recorded for each subject. Joint distribution of the values (T, J) is given by the so-called crude survival functions

$$\bar{Q}_i(t)=\text{pr}\ \{(T_i > t),\ \bigcap_{j \neq i} (T_j > T_i)\}. \tag{1}$$

Obviously $\bar{F}(t)=\sum_{i=1}^{n} \bar{Q}_i(t)$. Corresponding to marginal distribution functions for latent failure times $F_i(t)=\text{pr}(T_i \leq t)$, $t \geq 0$, are the functions $\bar{F}_i(t)=\text{pr}(T_i>t)$ which are sometimes referred to as net survial functions. N. Langberg et al. [8] analyzed a more complex model structure and sample observations associated with it which admits of a situation when the failure of different types may coincide in time.

It is generally recongized that the principal applied problem of the theory of competing risks consists in obtaining net, i. e. marginal, survival functions from observation of the pair of variables (T, J). Formulated in such general terms, the problem appears to be nonidentifiable (A. Peterson [11]; A. Tsiatis [17]; R. Prentice et al. [13]). Attempts are usually made to overcome this difficulty either with the aid of the hypothesis of the independence of the random variables $T_1, \dots, T_n$ or by means of specifying a parametric form of their joint distribution $W(t_1, \dots, t_n)$. N. Langberg [9] succeeded in slackening somewhat the requirement on the independence of latent times in the course of identification of marginal distributions, their results, however, fall far short of solving the problem exhaustively.

On the other hand, A. Peterson [12] constructed the following boundaries for the marginal survival functions $\bar{F}_i(t)$ at fixed $\bar{Q}_i(t)$:

(2) $$\bar{F}(t)\leq\bar{F}_i(t)\leq\bar{Q}_i(t)+\{1-\bar{Q}_i(0)\},$$

and showed that these bounds are sharp in the sense of the existence of cases which demonstrate the possibility of their attainment. By replacing in inequality (2) the functions $\bar{F}(t)$ and $\bar{Q}_i(t)$ with their empirical estimators, A. Peterson [12] obtained consistent nonparametric bounds for $\bar{F}_i(t)$. In their study E. Slud and L. Rubinstein [16] considered certain nonparametric assumptions permitting construction, at $n=2$, of bounds of marginal survival which can be much narrower than those of Peterson [12].

In the present paper we are considering the inverse problem, that of constructing bounds for $Q_i(t)$ and some other characteristics of the model at fixed marginal distribution $F_i(t)$, $i=1,\dots,n$ and the distribution $F(t)$ of the value $T=\min(T_1,\dots,T_n)$. One of the most relevant biological fields for the application of the problem is analysis of the survival of organisms after combined injuries. Such applications are discussed in Section 4.

2. Preliminary results and notation. The bounds for joint distribution at fixed marginal distributions were studied by W. Hoeffding [6] and M. Fréchet [3]. In what follows we shall repeatedly use bounds of the kind

(3) $$\max\{0,\ \sum_{i=1}^{N}F_i(x_i)-N+1\}\leq W(x_1,\dots,x_n)\leq\min\{F_1(x_1),\dots,F_N(x_N)\},$$

which came to be known in the literature as the Frechet bounds (J. Galambos [4, Chapter 5]).

For any one-dimensional random variables X' and X'' with the distribution functions F' and F'' respectively, the distance in total variation is defined as

(4) $$\sigma(F',F'')=\sup\{|\operatorname{pr}(X'\in A)-\operatorname{pr}(X''\in A)|:\ A\subset\mathscr{B}\},$$

where $\mathscr{B}$ is the Borel σ-algebra in R^1. In particular, if X' and X'' are the discrete random variables with the same support $\{x_1,x_2,\dots\}$, then (4) takes the form

$$\sigma(F',F'')=\frac{1}{2}\sum_{j=1}^{\infty}|\operatorname{pr}(X'=x_j)-\operatorname{pr}(X''=x_j)|.$$

For any $A\subset\mathscr{B}$ it may be written

$$\operatorname{pr}(X\in A)=\operatorname{pr}(X\in A,\ X=Y)+\operatorname{pr}(X\in A,X\neq Y)\leq\operatorname{pr}(Y\in A)+\operatorname{pr}(X\neq Y),$$

whence, allowing for symmetry considerations, we have

(5) $$\operatorname{pr}(X'\neq X'')\geq\sigma(F',F'').$$

It follows from the Dobrushin theorem [1] that there exist dependent random variables X' and X'' such that

$$\operatorname{pr}(X'\neq X'')=\sigma(F',F''),$$

and in this sense (5) is a sharp estimator.

As regards the distance $\sigma(F',F'')$, it is known that

(6) $$\sigma(F',F'')\geq\rho(F',F''),$$

where $\rho(F',F'')$ is the uniform metric, i. e.,

$$\rho(F,F'')=\sup\{|F'(x)-F''(x)|:\ x\in R^1\}.$$

3. Bounds for crude survival probabilities. Introduce the class $\mathcal{T}_n$ of all the joint distributions $W(t_1, \ldots, t_n)$ with the fixed distributions $F_i(t)$, $i=1, \ldots, n$, and $F(t)$. The lower and upper bounds for the functions $\bar{Q}_i(t)$, $i=1, \ldots, n$ are given by the following theorem.

Theorem. *For any $t \geq 0$, $i=1, \ldots, n$, $n \geq 2$ and $W(t_1, \ldots, t_n) \in \mathcal{T}_n$ there exist inequalities*

$$\bar{Q}_i(t) \geq \max \{ \sum_{j \neq i} \sigma(F_j, F) - n + 2 - F_i(t), 0\},$$

$$\bar{Q}_i(t) \leq 1 - \max \sigma(F_i, F), \max_{j \neq i} \pi_{ij}, \max_{k=1, \ldots, n} F_k(t)\},$$

where $\pi_{ij} = \text{pr}\,(T_i = T_j)$.

Proof. Let us evaluate from below the function $\bar{Q}_i(t)$ using relations (3) and (5):

$$\bar{Q}_i(t) = \text{pr}\,(\bigcap_{j \neq i} (T_j \neq T), T_i > t) = \text{pr}\,\{ \bigcap_{j \neq i} (T_j \neq T)\} - \text{pr}\,\{ \bigcap_{j \neq i} (T_j \neq T), T_i \leq t\}$$

$$\geq \text{pr}\,\{ \bigcap_{j \neq i} (T_j \neq T)\} - \min\,[\text{pr}\,\{ \bigcap_{j \neq i} (T_j \neq T)\}, \text{pr}\,(T_i \leq t)] = \max\,[\text{pr}\,\{ \bigcap_{j \neq i} (T_j \neq T) - F_i(t), 0]$$

$$\geq \max \{ \sum_{j \neq i} \text{pr}\,(T_j \neq T) - n + 2 - F_i(t), 0\} \geq \max \{ \sum_{j \neq i} \sigma(F_j, F) - n + 2 - F_i(t), 0\}.$$

The upper bound follows from the chain of inequalities:

$$\bar{Q}_i(t) = \text{pr}\,\{\bigcap_{j \neq i} (T_j \neq T), \bigcap_{k=1}^{n} (T_k > t)\} \geq \min\,[\text{pr}\,\{ \bigcap_{j \neq i} (T_j \neq T), \text{pr}\,\{ \bigcap_{k=1}^{n} (T_k > t)\}$$

$$= 1 - \max\,[\text{pr}\,\{ \bigcup_{j \neq i} (T_j = T)\}, \text{pr}\,\{ \bigcup_{k=1}^{n} (T_k \leq t)\}] \leq 1 - \max \{ \sigma(F_i, F), \max_{j \neq i} \pi_{ij}, \max_{k=1, \ldots, n} F_k(t)\}.$$

Thus the theorem is proved. One generalization of this theorem is given in Appendix.

Hereinafter we shall consider the case of $n=2$. It should be pointed out that all. the arguments are readily applicable to any n.

Thus, according to the theorem we have for all $t \geq 0$ at $W(t_1, t_2) \in \mathcal{T}_2$ inequalities

$$\max \{\sigma(F_2, F) - F_1(t), 0\} \leq \bar{Q}_1(t) \leq 1 - \max\,[F_1(t), F_2(t), \sigma(F_1, F), \pi_{12}] \tag{7}$$

$$\max \{ \sigma(F_1, F) - F_2(t), 0\} \leq \bar{Q}_2(t) \leq 1 - \max\,[F_1(t), F_2(t), \sigma(F_2, F), \pi_{12}],$$

where $\pi_{12} = \text{pr}\,(T_1 = T_2)$.

It is not difficult to construct examples showing that for certain F_1, F_2 and F the bounds in (7) are sharp.

Note that $\Pi_1 = \text{pr}\,(T_2 > T_1)$ and $\Pi_2 = \text{pr}\,(T_1 > T_2)$ are probabilities of failures of the first and second types, respectively. Assuming that in (7) $t=0$, we have

$$\sigma(F_2, F) \leq \Pi_1 \leq 1 - \sigma(F_1, F) - \pi_{12} \tag{8}$$

$$\sigma(F_1, F) \leq \Pi_2 \leq 1 - \sigma(F_2, F) - \pi_{12}.$$

Following directly from (7) and (6) are the bounds

$$\rho(F_2, F) \leq \Pi_1 \leq 1 - \rho(F_1, F) - \pi_{12} \tag{9}$$

$$\rho(F_1, F) \leq \Pi_2 \leq 1 - \rho(F_2, F) - \pi_{12}$$

which will be used in Section 4 for constructing confidence intervals for the probabilities Π_1 and Π_2.

As to covariance between latent failure times in the case $W(t_1, t_2) \in \mathcal{T}_2$ the similar consideration leads to the following bounds for the value $E(T_1 T_2)$:

$$E(T_1 T_2) \leq \int_0^\infty \int_0^y \min\{\bar{F}(x), \bar{F}_2(y), \bar{F}(y), \bar{F}_1(x) - \bar{F}_1(y)\}\, dx dy$$

$$+ \int_0^\infty \int_y^\infty \min\{\bar{F}(y), \bar{F}_1(x), \bar{F}(x) + \bar{F}_2(y) - \bar{F}_2(x)\}\, dx dy$$

$$E(T_1 T_2) \geq \int_0^\infty \int_0^y \max\{\bar{F}(y), \bar{F}_1(x) + \bar{F}_2(y) - 1, \bar{F}(x) + \bar{F}_2(y) - \bar{F}_2(x)\}\, dx dy$$

$$+ \int_0^\infty \int_y^\infty \max\{\bar{F}(x), \bar{F}_1(x) + \bar{F}_2(y) - 1, \bar{F}(y) + \bar{F}_1(x) - \bar{F}_1(y)\}\, dx dy.$$

4. Applications. A model of combined injury and confidence intervals for the probabilities Π_1 and Π_2. One of the most topical problems of survial data statistical analysis in the cases of combined injuries is evaluation of the contribution of each of the effecting factors to the lethal effect of their joint action. The difficulties involved in its solution are, apparently, due to the presence of competing risks, each capable of causing death in combined injury, and to the impossibility of establishing the exact cause of death in any given experiment. The following experimental situation is typical for biological investigation of lethal effects of combined injury (O. Messerschmidt [10]).

3 groups of animals are studied of which two are under observation for the effects of the isolated action of each of two injurious agents A and B, respectively, while the third group consists of animals exposed to the combined action of both agents A and B, the doses used being the same as those chosen for the first two groups. The life lengths of the animals of the first two groups have the distribution functions $H_A(t)$ and $H_B(t)$, respectively. The fact that the joint distribution $W(t_1, t_2)$ of the latent times T_1 and T_2 belongs to the class $\mathcal{T}_2$ implies the acceptance of the following hypotheses:

(i) the distribution function $F_1(t)$ of the latent time T_1 induced by the cause A coincides with $H_A(t)$, whereas the distribution function $F_2(t)$ of the latent time T_2 induced by cause B coincides with $H_B(t)$;

(ii) in the cause of combined injury, i. e. when both causes A and B are operative, the life length $T = \min(T_1, T_2)$ is observed with the distribution function $F(t)$, where T_1 and T_2 are dependent random variables with the joint distribution $W(t_1, t_2)$.

The hypotheses (i) and (ii) may seem restrictive, yet it is difficult to define the actual fields of their application which in reality may well prove to be fairly extensive. In the next section we shall outline some ways of relaxing the hypotheses (i) and (ii), but now we accept them as an indispensable stage in statistical estimation of the probabilities Π_1 and Π_2, which characterize the contribution of each agent to the combined injurious effect. In other words, the model formulated in terms of the propositions (i) and (ii) substantiates in this particular case the observability of the marginal distributions $F_1(t)$ and $F_2(t)$ essential for constructing corresponding statistical estimators. We shall further assume the functions $F_1(t)$, $F_2(t)$, and $F(t)$ to be absolutely continuous.

Let us turn again to inequalities (9). Assume that $F_1^{(n)}(t)$, $F_2^{(k)}(t)$, $F^{(m)}(t)$ are empirical counterparts of the distribution functions $F_1(t)$, $F_2(t)$, $F(t)$ constructed by use of complete samples of the sizes n, k, m, respectively.

Consider the first of inequalities (9), setting $\pi_{12} = 0$.

Using the triangle inequality, we have

(10) $$\Pi_1 \geqq \rho(F_2^{(k)}, F^{(m)}) - \rho(F_2^{(k)}, F) - \rho(F^{(m)}, F).$$

From inequalities (3) it follows that

(11) $$\mathrm{pr}\{\rho(F_2^{(k)}, F_2) \leqq c_2 \cap \rho(F^{(m)}, F) \leqq c\} \geqq \max\{G^{(k)}(c_2) + G^{(m)}(c) - 1, 0\},$$

where $G^{(k)}(x)$ and $G^{(m)}(x)$ are the Kolmogorov distributions for finite samples of the sizes k and m, respectively, C_2 and C being positive constants.

Hence, on the basis of (10) and (11), the lower confidence bound is

(12) $$\mathrm{pr}\{\Pi_1 \geqq \rho(F_2^{(k)}, F^{(m)}) - c_2 - c\} \geqq \max\{G^{(k)}(c_2) + G^{(m)}(c) - 1, 0\}.$$

Similar reasoning makes it possible to obtain from (9) the upper confidence bound as well

(13) $$\mathrm{pr}\{\Pi_1 \leqq 1 - \rho(F_1^{(n)}, F^{(m)}) + c_1 + c\} \geqq \max\{G^{(n)}(c_1) + G^{(m)}(c) - 1, 0\}.$$

The confidence bounds for Π_2 are symmetric in respect to (12) and (13). Using (3), one may joint the estimators obtained and form two-sided confidence limits but at the expense of a reduction in values of confidence probabilities. Taking into account the arbitrary choice of constants C, C_1, C_2, it is reasonable to rewrite the bounds (12) and (13) in the following form:

(14) $$\mathrm{pr}\{\Pi_1 \geqq \rho(F_2^{(k)}, F^{(m)}) - c_2 - c\} \geqq \max\{\sup_h [G^{(k)}(c_2 - h) + G^{(m)}(c + h) - 1], 0\}$$

(15) $$\mathrm{pr}\{\Pi_1 \leqq 1 - \rho(F_1^{(n)}, F^{(m)}) + c_1 + c\} \geqq \max\{\sup_h [G^{(n)}(c_1 - h) + G^{(m)}(c + h) - 1], 0\}.$$

Confidence bounds (14), (15) were constructed for complete samples. In the case of censored observations asymptotic confidence bands may be obtained. Let us consider a scheme of random independent censorship on the right, assuming continuity of the distribution functions of both life length and censoring time. Let $S_1^{(n)}(t)$ and $S^{(m)}(t)$ be the Karlan—Meier estimators for survival functions $F_1(t)$ and $F_2(t)$, respectively. Then let

$$\tau_\Phi = \inf\{t \geqq 0 : \Phi(t) = 1\} \leqq \infty$$

for every distribution function $\Phi(t)$.

Following the ideas of W. Hall and J. Wellner [5], within the interval of time $t \in [0, \tau_1]$ such that $\tau_1 \leqq \min(\tau_{F_1}, \tau_F)$ introduce the statistics,

where

$$r_{\tau_1}(\Phi', \Phi'') = \sup_{t \in [0, \tau_1)} |\Phi'(t) - \Phi''(t)|,$$

$$v_1^{(n)} = \sup_{t \in [0, \tau_1)} \mathscr{D}_1^{(n)}(t), \quad v^{(m)} = \sup_{t \in [0, \tau_1)} \mathscr{D}^{(m)}(t).$$

Introducing the similar notations $S_2^{(k)}(t)$ and $V_2^{(k)}(t)$, for the sample form the distribution $F_2(t)$, for the values $t \in [0, \tau_2)$, $\tau_2 \leqq \min(\tau_{F_2}, \tau_F)$, we can now give the final representation of the confidence bounds for Π_1 and Π_2 as $\min(n, k, m) \to \infty$

$$\mathrm{pr}\{\Pi_1 \geqq r_\tau(S_2^{(k)}, S^{(m)}) - \lambda_2 v_2^{(k)} - \lambda v^{(m)}\} \geqq \max\{G_{a_2}(\lambda_2) + G_a(\lambda) - 1, 0\}$$

$$\mathrm{pr}\{\Pi_1 \leqq 1 - r_\tau(S_1^{(n)}, S^{(m)}) + \lambda_1 v_1^{(n)} + \lambda v^{(m)}\} \geqq \max\{G_{a_1}(\lambda_1) + G_a(\lambda) - 1, 0\}$$

$$\mathrm{pr}\{\Pi_2 \geqq r_\tau(S_1^{(n)}, S^{(m)}) - \lambda_1 v_1^{(n)} - \lambda v^{(m)}\} \geqq \max\{G_{a_1}(\lambda_1) + G_a(\lambda) - 1, 0\}$$

$$\text{pr}\{\Pi_2 \leq 1 - r_\tau(S_2^{(k)}, S^{(m)}) + \lambda_2 v_2^{(k)} + \lambda v^{(m)}\} \geq \max\{G_{a_2}(\lambda_2) + G_a(\lambda) - 1, 0\},$$

where all the estimators are considered at $0 \leq t < \tau$, $\tau = \min(\tau_1, \tau_2)$.

Example. Shown in Fig. 1 are the patterns of empirical counterparts of survival functions $\bar{F}_1(t)$, $\bar{F}_2(t)$ and $\bar{F}(t)$. These patterns correspond to synergism in the action of two damaging agents: $\bar{F}(t) < \bar{F}_1(t)\bar{F}_2(t)$. Using data from Fig. 1, the following confidence bounds for the values Π_1 and Π_2 were obtained:

$$\text{pr}(\Pi_1 \geq 0.25) \geq 0.9, \quad \text{pr}(\Pi_1 \leq 0.74) \geq 0.9;$$

$$\text{pr}(\Pi_2 \geq 0.27) \geq 0.9, \quad \text{pr}(\Pi_2 \leq 0.75) \geq 0.9.$$

These bounds seem to be too broad for practical purposes. So the example shows that either the hypothesis: $W(_1, t_2) \in \mathcal{T}_2$ is to be enriched by additional a priori information, or further search for some (more useful for statistical inference than crude survival probabilities) other characteristics of the contribution of each damaging agent to the combined injurious effect would be necessary. One such attempt will be presented in the subsequent communication.

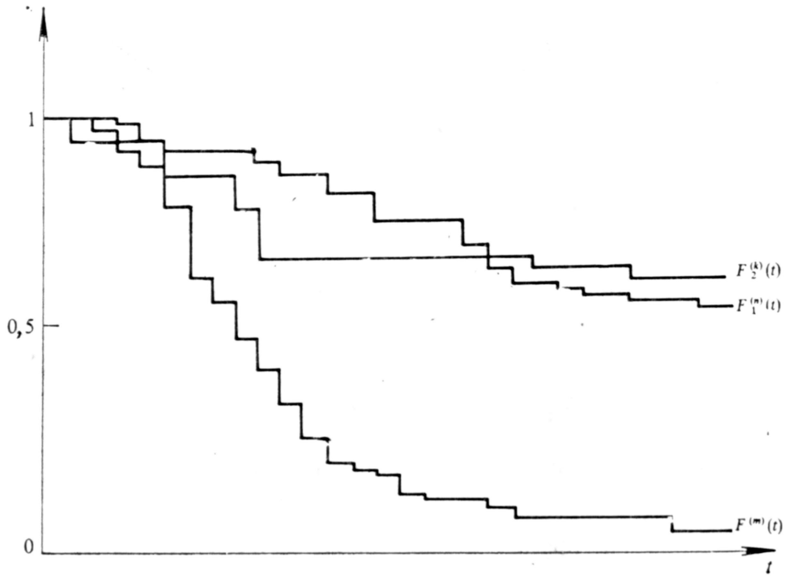

Fig. 1. Empirical counterparts $F_1^{(n)}(t)$, $F_2^{(k)}(t)$, $F^{(m)}(t)$ for the survival functions $F_1(t)$, $F_2(t)$, $F(t)$ constructed by use of censored observations. See text (Example) for explanations.

To take the model more realistic it seems reasonable to replace the hypothesis (i) with the following one:

$$\bar{H}_A(t) = \bar{\psi}_1(t), \quad \bar{H}_B(t) = \bar{\psi}_2(t), \tag{17}$$

where

$$\bar{\psi}_1(t)=\lim_{\Lambda\to\infty} \mathrm{pr}\,(T_1>t \mid T_2>\Lambda), \quad \bar{\psi}_2(t)=\lim_{\Lambda\to\infty} \mathrm{pr}\,(T_2>t \mid T_1>\Lambda).$$

There exist joint distributions for which conditional survival functions $\bar{\psi}_1(t)$ and $\bar{\psi}_2(t)$ do coincide with marginal $\bar{F}_1(t)$ and $\bar{F}_2(t)$. This is the case for the following functional family of distributions:

$$\bar{W}^*(t_1, t_2)=\bar{W}(t_1, t_2)\, I_{t_1, t_2}(\Omega)+\bar{F}_1(t_1)\bar{F}_2(t_2)\,\{1-I(\Omega)\}, \tag{18}$$

where $\bar{W}(t_1, t_2)$ is the arbitrary joint survival function and $I_{t_1,t_2}(\Omega)$ is the step function such that

$$I_{t_1, t_2}(\Omega)=1,\ \max\,(t_1, t_2)<\Omega\,;\ I_{t_1, t_2}(\Omega)=0,\ \max\,(t_1, (t_2)\geqq\Omega.$$

Using the family (18) allows the above results concerning interval estimation of Π_1 and Π_2 to remain valid under the new condition (17), which seems biologically more natural than hypothesis (i.)

Appendix. Distances between the random variables T_1, T_2 and T. Let us introduce in the space of one-dimensional distributions the metrics l_α and m_α as follows:

$$l_\alpha(\Phi_1, \Phi_2)=\begin{cases} \sup \left| \int f(x)d\Phi_1(x)-\int f(x)d\Phi_2(x) \right|, & \alpha\in(0, 1] \\ \{ \int_0^1 \mid \Phi_1^{-1}(u)-\Phi_2^{-1}(u) \mid^\alpha du \}^{1/\alpha}, & \alpha\in[1, \infty), \end{cases} \tag{A1}$$

$$m_\alpha(\Phi_1, \Phi_2)=\begin{cases} \sup \{ \int f(x)d\Phi_1(x)+\int g(x)d\Phi_2(x)\}, & \alpha\in(0, 1] \\ \{ \int_0^1 \mid \Phi_1^{-1}(u)-\Phi_2^{-1}(1-u) \mid du\}^{1/\alpha}, & \alpha\in[1, \infty), \end{cases} \tag{A2}$$

where the upper bound in (A1) is taken over the space Lip (α) of all measurable functions $f(x)$, for which $|f(x)-f(y)|\leqq |x-y|^\alpha$. $x, y\in R^1$; the functions f and g in (A2) satisfy the inequality $f(x)+g(y)\geqq |x-y|$, x and $y\in R^1$; Φ_1^{-1} and Φ_2^{-1} in expressions (A1) and (A2) are inverse functions in respect to Φ_1 and Φ_2. Next we shall introduce the average distance from T_i to T

$$\mathscr{L}_\alpha(T_i, T)=\{E(\mid T_i-T\mid^\alpha)\}^{\alpha'}, \tag{A3}$$

where

$$0<\alpha<\infty, \quad \alpha'=\min\,(1,1/\alpha), \quad i=1, 2.$$

Holding for distance (A3) is the following statement.

For any finite $\alpha\geqq 0$ there are the inequalities

$$\begin{aligned} l_\alpha(F_1, F)&\leqq \mathscr{L}_\alpha(T_1, T)\leqq m_\alpha(F_1, F) \\ l_\alpha(F_2, F)&\leqq \mathscr{L}_\alpha(T_2, T)\leqq m_\alpha(F_2, F), \end{aligned} \tag{A4}$$

where l_α and m_α are defined by expressions (A1) and (A2). Bounds (A4) are sharp in the class $\mathscr{T}_2$.

Proof. For $0\leqq\alpha\leqq 1$ the lower bound in (A4) follows from the Kantorovich theorem (see review: Rachev, 1984a) and the upper one from Rachev's results (1984b). When $1\leqq \alpha<\infty$ inequalities (A4) are deduced using Frechet's bounds (for greater detail see: Rachev, 1984a).

Remark A1. The probability $\Pi_1=\mathrm{pr}\,(T_2>T_1)=\mathrm{pr}\,(T_2\neq T)=E\{I(T_2\neq T)\}$, where I (.) is the indicator function, may be justly regarded as the limit $\mathscr{L}_\alpha(T_2, T)$ at $\alpha\to 0$.

For any finite $\alpha \geq 0$, $\mathscr{L}_\alpha(T_2, T)=0$ if and only if $\mathrm{pr}\,(T_2>T_1)=0$. Dobruchin (1970) proved that $\lim_{\alpha\to 0} l_\alpha(\Phi_1, \Phi_2)=\sigma(\Phi_1, \Phi_2)$, and thus, setting by definition $\Pi_1=\mathscr{L}_0(T_2, T)$, we directly obtain the lower boundary in (8). In this specific sense (A4) may be regarded as generalization of the theorem from Section 3. When $\alpha=1$, there is evidently the equality $\mathscr{L}_1(T_2, T)=E(T_2)-E(T)$, and in this case $l_1(\Phi_1, \Phi_2)$ is the dissimilarity index of Gini, i. e.:

$$l_1(\Phi_1, \Phi_2)=\int_{-\infty}^{\infty} |\Phi_1(x)-\Phi_2(x)|\, dx.$$

Remark A2. Based on the results of Fortet-Mourier [2] and S. Rachev [14] it is valid to say that if $F_1^{(n)}(t)$ and $F^{(m)}(t)$ are the empirical counterparts of the distribution functions $F_1(t)$ and $F(t)$, then

$$\mathrm{pr}\,\{\mathscr{L}_\alpha(T_1, T) \geq \lim_{\min(n,m)\to\infty} l_\alpha(F_1^{(n)}, F^{(m)})\}=1.$$

A similar statement is, evidently, true for $\mathscr{L}_\alpha(T_2, T)$ as well.

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Received 11. 2. 1987