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ON CURVATURE THEORY OF DIRECTION DEPENDENT HERMITE CONNECTION ON COMPLEX MANIFOLDS

P. TAMIA-DIMOPOULOU

This paper is concerned with the curvature theory of the direction dependent Hermite connection and with the study of generalized Hermite spaces with special curvature properties.

1. Let us consider a $2n$-dimensional manifold V_{2n} and a n-dimensional complex structure on it. We denote by $U(V_{2n})$, W, W^c the spaces of non-zero tangent vectors of V_{2n}, of orientated directions on V_{2n} and of tangent complex lines of V_{2n}, respectively. If $(z^{\alpha}, z^{\alpha^*})$ is a complex coordinate system at $z \in V_{2n}$, the corresponding complex coordinate system at $\bar{u} \in U(V_{2n})$ is the system $(z^{\alpha}, z^{\alpha^*}, u^{\alpha}, u^{\alpha^*} = u^{-\alpha})$, where $(u^{\alpha}, u^{\alpha^*})$ are the components of the tangent vector $\bar{u}$ at $z \in V_{2n}$ with respect to the natural frames of local complex coordinate systems on V_{2n}. The space $U(V_{2n})$ is a fibre bundle over V_{2n} with structure group $GL(n, C)$ and standard fibre C^n. The space W is a fibre bundle over V_{2n} with structure group the unitary group $U(n-1)$ and standard fibre the complex projective space M_{n-1}. Similarly, the space W^c is a fibre bundle over V_{2n} with structure group $U(n-1)$ and standard fibre M_{n-1}. Let $T^c_X(V_{2n})$ be the complex tangent space at $x \in V_{2n}$ and J_X the map which maps $T^c_X(V_{2n})$ into itself $(\forall x \in V_{2n})$. This field of maps is difined on V_{2n} by the complex structure of V_{2n}. Let $E^c(V_{2n})$ be the set of bases (R^c_X) of $T^c_X(V_{2n})$ such as the set $(R^c_X, \bar{R}^c_X)$ is an adapted base to the almost complex structure of $T^c_X(V_{2n})$ [2].

We regard the following diagram:

$$\begin{array}{ccccccc}
\pi^{-1}(E^c(V_{2n})) & & E^c(V_{2n}) & & v^{-1}(E^c(V_{2n})) & & q^{-1}(E^c(V_{2n})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
U(V_{2n}) & \xrightarrow{\ \pi\ } & V_{2n} & \xleftarrow{\ p\ } & W & \xrightarrow{\ h\ } & W^c \\
 & & \uparrow & & \underleftarrow{\qquad\quad q \qquad\quad} & &
\end{array}$$

Definition 1.1. *We call* a) *almost complex vector connection every connection which is defined on the induced by* π *fibre bundle* $\pi^{-1}(E^c(V_{2n}))$ *over* $U(V_{2n})$; b) *almost complex direction connection every connection which is defined on fibre bundle* $p^{-1}(E^c(2V_{2n}))$ *over* W *and* c) *almost complex line connection every connection which is defined on fibre bundle* $q^{-1}(E^c(V_{2n}))$ *over* W^c.

Definition 1.2. *We call tensor field on* $U(V_{2n})$ *in a large sense a map* t, *which maps an element of tensor algebra on* $T^c_{\pi z}(V_{2n})$ *at* $z \in U(V_{2n})$.

If $z_1, z_2 \in U(V_{2n})$, $z_1 = \lambda z_2$, $\lambda \in R^t$ and $t(z_2) = t(z_1)$, then we can regard that t is a tensor field in a large sense on W. Similarly, if $z_2 = \mu z_1$, $\mu \in C - \{0\}$ and $t(z_2) = t(z_1)$, we say that t is a tensor field in a large sense on W^c.

There is a canonical vector field in a large sense on $U(V_{2n})$, which maps the tangent vector $\overline{V}(v^{\alpha}, v^{\alpha^*}) \in T_{\pi z}(V_{2n})$ at $z(z^{\alpha}, z^{\alpha^*}, v^{\alpha}, v^{\alpha^*})$ of $U(V_{2n})$. We denote this field by $\overline{V}$.

Let (π^{α}_{β}) be an almost complex vector connection. Then the covariant differential of canonical vector field is defined by

$$\theta^{\alpha} = \nabla v^{\alpha} = dv^{\alpha} + \pi^{\alpha}_{\beta} v^{\beta},$$

$$\theta^{\alpha^*} = \nabla v^{\alpha^*} = dv^{\alpha^*} + \pi^{\alpha^*}_{\beta^*} v^{\beta^*}.$$

If $(dz^{\gamma}, dz^{\gamma^*}, dv^{\gamma}, dv^{\gamma^*})$ are the natural coframes on $U(V_{2n})$, then an almost complex vector connection is given by the relation:

$$\pi^{\alpha}_{\beta} = \Gamma^{\alpha}_{\beta\gamma}\, dz^{\gamma} + 1^{\alpha}_{\beta\gamma^*}\, dz^{\gamma^*} + T^{\alpha}_{\beta\gamma}\, dv^{\gamma} + T^{\alpha}_{\beta\gamma^*} dv^{\gamma^*}$$

and its conjugate.

P r o p o s i t i o n 1.1. *Let* $(\alpha^{\beta}, \alpha^{\beta^*})$ *be an arbitrary adapted base of* $(T^c_{\pi z}(V_{2n}))^*$. *The set* $(\alpha^{\beta}, \alpha^{\beta^*}, \theta^{\beta}, \theta^{\beta^*})$ *is an adapted base of* $T^c_z(U(V_{2n}))^*$ *if and only if*

$$\det(E^i_j) = \det(\delta^i_j + T^i_{kj} v^k) \neq 0 \quad \textit{and} \quad \Gamma^{\alpha}_{\beta\gamma^*} = T^{\alpha}_{\beta\gamma^*} = 0.$$

Greek indices take the values $1, \ldots, n$ and Latin indices take the values $1, \ldots, 2n$. Moreover, $\alpha^* = \alpha \pm n \leftrightarrow (\alpha^*)^* = \alpha$.

D e f i n i t i o n 1.3. *We call adapted complex vector connection the almost complex vector connection such as: the set* $(\alpha^{\beta}, \alpha^{\beta^*}, \nabla v^{\beta}, \nabla v^{\beta^*})$ *is an adapted base at the almost complex structure of* $T^c_z(U(V_{2n}))^*$, *when* $(\alpha^{\beta}, \alpha^{\beta^*})$ *is an adapted base of* $T^c_{\pi z}(V_{2n})^*$.

We will regard adapted complex vector connection at the following. It is easily to see that an adapted complex vector connection satisfies the relations

$$\pi^{\alpha}_{\beta} = \gamma^{\alpha}_{\beta\gamma}\alpha^{\gamma} + C^{\alpha}_{\beta\gamma}\theta^{\gamma} \quad \text{or} \quad \pi^{\alpha}_{\beta} = L^{\alpha}_{\beta\gamma} dz^{\gamma} + B^{\alpha}_{\beta\gamma}\theta^{\gamma}$$

and their conjugates.

Then we have the following identities:

$$L^{\alpha}_{\beta\gamma} + B^{\alpha}_{\beta\gamma}\Gamma^{\gamma}_{\sigma\delta}v^{\sigma} = \Gamma^{\alpha}_{\beta\delta}, \quad B^{\alpha}_{\beta\delta} + B^{\alpha}_{\beta\gamma}T^{\gamma}_{\sigma\delta}v^{\sigma} = T^{\alpha}_{\beta\delta},$$

$$C^{\alpha}_{\beta\delta} + C^{\alpha}_{\beta\gamma}T^{\gamma}_{\sigma\delta}v^{\sigma} = T^{\alpha}_{\beta\delta} \quad \text{and their conjugates.}$$

P r o p o s i t i o n 1.2. *Let* (π^{α}_{β}) *be an adapted complex vector connection. It can be regarded as adapted complex line connection on* W^c, *if and only if* $\gamma^{\alpha}_{\beta\gamma}(z_1) = \gamma^{\alpha}_{\beta\gamma}(z_2)$, $C^{\alpha}_{\beta\gamma}(z_1) = \mu C^{\alpha}_{\beta\gamma}(z_2)$ *and* $C^{\alpha}_{\beta\gamma} v^{\beta} = 0$, *or* $C^{\alpha}_{0\gamma} = 0$, *where* $z_1, z_2 \in U(V_{2n})$ *and* $z_2 = \mu z_1$, $\mu \in C - \{0\}$.

The torsion tensor of an adapted complex vector connection (π^{α}_{β}) are the tensors

$$S^{\beta}_{\gamma\delta} = -(L^{\beta}_{\gamma\delta} - L^{\beta}_{\delta\gamma}), \tag{1.1}$$

$$h^{\beta}_{\gamma\delta} = -B^{\beta}_{\gamma\delta} \quad \text{and their conjugates.} \tag{1.2}$$

T h e o r e m 1.1. *The torsion tensor of an adapted vector connection of the type* $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ *and their conjugates vanish identically.*

The curvature tensor of (π^{α}_{β}) are defined by the following relation:

$$R^{\alpha}_{\beta\delta j} = \overline{R}^{\alpha}_{\beta\delta j} + B^{\alpha}_{\beta\gamma}E^{\gamma}_{\rho}\overline{R}^{\rho}_{\lambda\delta j}v^{\lambda}, \quad R^{\alpha}_{\beta\delta\sigma} + R^{\alpha}_{\beta\sigma\delta} = 0, \quad R^{\alpha}_{\beta\delta^*\sigma^*} = 0,$$

$$P^{\alpha}_{\beta\delta j}=\overline{P}^{\alpha}_{\beta\delta j}+B^{\alpha}_{\beta\gamma}E^{\gamma}_{\rho}\overline{P}^{\rho}_{\lambda\delta j}\nu^{\lambda},$$

$$P^{\alpha}_{\beta\delta^*\sigma}=\overline{P}^{\alpha}_{\beta\delta^*\sigma}+B^{\alpha}_{\beta\gamma}E^{\gamma}_{\rho}\overline{P}^{\rho}_{\lambda\delta^*\sigma}\nu^{\lambda},\quad P^{\alpha}_{\beta\delta^*\sigma^*}=0,$$

$$Q^{\alpha}_{\beta\delta j}=\overline{Q}^{\alpha}_{\beta\delta j}+B^{\alpha}_{\beta\gamma}E^{\gamma}_{\rho}Q^{\rho}_{\lambda\delta j}\nu^{\lambda},\ Q^{\alpha}_{\beta\delta\sigma}+Q^{\alpha}_{\beta\sigma\delta}=0,\quad Q^{\alpha}_{\beta\sigma^*\delta^*}=0,$$

where $E^{\gamma}_{\rho}=(\delta^{\gamma}-B^{\gamma}_{\rho\beta}v^{\beta})^{-1}$,

$$\overline{R}^{\alpha}_{\beta\delta\sigma}=D_{\delta}L^{\alpha}_{\beta\sigma}-D_{\sigma}L^{\sigma}_{\beta\delta}+L^{\alpha}_{\rho\beta}L^{\rho}_{\beta\sigma}-L^{\alpha}_{\rho\sigma}L^{\rho}_{\beta\delta},$$

$$\overline{R}^{\alpha}_{\beta\delta\sigma^*}=-D_{\sigma^*}L^{\alpha}_{\beta\delta},$$

$$\overline{P}^{\alpha}_{\beta\delta\sigma}=D_{\delta}B^{\alpha}_{\beta\sigma}-D_{\dot{\sigma}}L^{\alpha}_{\beta\delta}-B^{\sigma}_{\beta\gamma}L^{\gamma}_{\sigma\delta}+L^{\alpha}_{\rho\delta}B^{\rho}_{\beta\sigma}-B^{\alpha}_{\rho\sigma}B^{\rho}_{\beta\delta},$$

$$\overline{P}^{\alpha}_{\beta\delta\sigma^*}=-D_{\dot{\sigma}^*}L^{\sigma}_{\beta\delta},$$

$$\overline{P}^{\alpha}_{\beta\delta^*\sigma}=D_{\delta^*}B^{\alpha}_{\beta\sigma},$$

$$\overline{Q}^{\alpha}_{\beta\delta\sigma}=D_{\delta}B^{\alpha}_{\beta\sigma}-D_{\dot{\sigma}}B^{\alpha}_{\beta\delta}+B^{\alpha}_{\beta\gamma}(B^{\gamma}_{\sigma\delta}-B^{\gamma}_{\delta\sigma})+B^{\alpha}_{\rho\delta}B^{\rho}_{\beta\sigma}-B^{\alpha}_{\rho\sigma}B^{\rho}_{\beta\delta}.$$

$\overline{Q}^{\alpha}_{\beta\delta\sigma^*}=-D_{\dot{\sigma}^*}B^{\alpha}_{\beta\delta}$, where D denotes the partial derivatives with respect to the coframes of the form $(dz^{\gamma}, dz^{\gamma^*}, \theta^{\gamma}, \theta^{\gamma^*})$.

Thus we have:

Theorem 1.2. *The curvature tensor of an adapted complex vector connection of the type* $\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$ *and their conjugates vanish identically.*

Theorem 1.3. *If an adapted complex vector connection is an holomorphic connection, then the curvature tensors of the type* $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ *are zero.*

If Σ^{β} and Ω^{α}_{β} denote the torsion form and the curvature of an adaped complex vector connection respectively, then from the Bianchi identities

$$d\Sigma^{\beta}=d\pi^{\beta}_{\gamma}\wedge\alpha^{\gamma}-\pi^{\beta}_{\gamma}\wedge d\alpha^{\gamma},$$

$$d\Omega^{\alpha}_{\beta}=d\pi^{\alpha}_{\rho}\wedge\pi^{\rho}_{\beta}-\pi^{\alpha}_{\rho}\wedge d\pi^{\rho}_{\beta}\quad\text{and their conjugates,}$$

we have the following relations:

$$\nabla_{\mu^*}R^{\alpha}_{\beta\gamma\delta}=\nabla_{\dot{\mu}^*}R^{\alpha}_{\beta\gamma\delta}=0,\tag{1.3}$$

$$\nabla_{\mu^*}P^{\alpha}_{\beta\gamma\delta}=\nabla_{\dot{\mu}^*}P^{\alpha}_{\beta\gamma\delta}=0,\tag{1.4}$$

$$\nabla_{\mu^*}Q^{\alpha}_{\beta\gamma\delta}=\nabla_{\dot{\mu}^*}Q^{\alpha}_{\beta\gamma\delta}=0,\tag{1.5}$$

$$\int\nabla_{\sigma}S^{\beta}_{\gamma\rho}+\int S^{\delta}_{\sigma\gamma}S^{\beta}_{\delta\rho}=\int h^{\beta}_{\gamma\delta}R^{\delta}_{o\rho\sigma}+\int R^{\beta}_{\gamma\rho\sigma},\tag{1.6}$$

where $\int$ denotes the circle permutation of $\sigma.\gamma.\rho$

$$1/2\Delta_{\sigma^*}S^{\beta}_{\rho\gamma}+h^{\beta}_{\gamma\delta}R^{\delta}_{o\rho\sigma^*}+R^{\beta}_{\gamma\rho\sigma^*}=0,\tag{1.7}$$

$$1/2\,\nabla_{\dot{\sigma}}S^{\beta}_{\gamma\delta}-S^{\beta}_{\rho\delta}h^{\rho}_{\gamma\sigma}-\nabla_{\sigma}h^{\beta}_{\gamma\delta}-1/2\,h^{\beta}_{\rho\delta}S^{\rho}_{\sigma\gamma}+h^{\beta}_{\gamma\rho}P^{\rho}_{o\sigma\delta}-P^{\beta}_{\gamma\sigma\delta}=0,\tag{1.8}$$

(1.9) $$1/2\,\nabla_{\dot\sigma^*}S^{\beta}_{\gamma\rho}=h^{\beta}_{\gamma\delta}P^{\delta}_{o\rho\sigma^*}+P^{\beta}_{\gamma\rho\sigma^*},$$

(1.10) $$\nabla_{\dot\sigma}h^{\beta}_{\gamma\rho}-h^{\beta}_{\delta\rho}h^{\delta}_{\gamma\sigma}=1/2\,h^{\beta}_{\gamma\delta}Q^{\delta}_{o\rho\sigma}+Q^{\beta}_{\gamma\rho\sigma},$$

(1.11) $$\nabla_{\dot\sigma^*}h^{\beta}_{\gamma\rho}=h^{\beta}_{\gamma\delta}Q^{\delta}_{o\rho\sigma^*}+Q^{\beta}_{\gamma\rho\sigma^*},$$

(1.12) $$\nabla_{\rho^*}h^{\beta}_{\gamma\sigma}=-h^{\beta}_{\gamma\delta}P^{\delta}_{o\rho^*\sigma}-P^{\beta}_{\gamma\rho^*\sigma}$$

and their conjugates.
From (1.3), (1.4), (1.5) we have

Theorem 1.4. *The curvature tensors of an adapted complex vector connection* $R^{\alpha}_{\beta\gamma\delta}$, $P^{\alpha}_{\beta\gamma\delta}$, $Q^{\alpha}_{\beta\gamma\delta}$, *and their conjugates are holomorphic functions of the local coordinates of* $U(V_{2n})$.

2. Let L be a real-valued, positive function on $U(V_{2n})$ which satisfies the strong homogeinity conditions

$$L(z^{\alpha},\ z^{\alpha^*}=\bar z^{\alpha},\ \mu u^{\alpha},\ \mu^* u^{\alpha^*}=\overline{\mu u^{\alpha}})=|\mu|\,L(z^{\alpha},\ z^{\alpha^*},\ u^{\alpha},\ u^{\alpha^*}),\quad \mu\in C-\{0\}.$$

The norm of a vector $u(u^{\alpha},\ u^{\alpha^*}=\bar u^{\alpha})\in T_x(V_{2n})$ is defined by the relation

$$L(z^{\alpha},\ z^{\alpha^*},\ u^{\alpha},\ u^{\alpha^*})=|\bar u|.$$

Then the metric tensor $h_{ij}={}^*o^{cb}_{ij}\,g_{cb}=\begin{pmatrix}0 & g_{\alpha\beta^*}\\ g_{\alpha^*\beta} & 0\end{pmatrix}$ of L,

where

$$g_{cb}=\delta_{\dot c}\,{}_{\dot b}F\quad\text{and}\quad 2F=L^2,$$

defines a Hermite structure on $T_{qw}(V_{2n})$, $\forall w\in W^c$, when $\det(\delta_{\dot\alpha}\,{}_{\dot\beta^*}F)\neq 0$. The operator ${}^*o^{cb}_{ji}$ applied to a general covariant tensor gives the hybrid part of it. It is defined by the almost complex structure of V_{2n}.

Definition 2.1. *We call generalized Hermite structure of direction dependent Hermite structure, the metric structure which is defined on* V_{2n} *by the tensor field* h_{ij} *in a large sense on* W^c, *when* $\det(\delta_{\dot\alpha}\,{}_{\dot\beta^*}F)\neq 0$.

Theorem 2.1. *If a generalized Hermite structure is defined on* V_{2n}, *then there is only one adapted Hermite connection which is defined by the following relations:*

(2.1) $$L^{\delta}_{\alpha\gamma}=g^{\delta\beta^*}\delta_{\gamma}g_{\alpha\beta^*}-g^{\delta\beta^*}g^{\sigma\rho^*}\delta_{\dot\sigma}\,g_{\alpha\beta^*}\,.\,\delta_{\gamma}g_{\lambda\rho^*}v^{\lambda},$$

(2.2) $$B^{\alpha}_{\beta\gamma}=g^{\alpha\sigma^*}B_{\sigma^*\beta\gamma}=g^{\alpha\sigma^*}\delta_{\dot\gamma}\,g_{\beta\sigma^*}=B^{\alpha}_{\gamma\beta}\ \textit{and their conjugates,}$$

where δ *stands for the partial derivatives with respect to the natural frames. We call this connection generalized Hermite connection of direction dependent Hermite connection.*

Theorem 2.2. *The curvature tensors of the generalized Hermite connection* $R^{\alpha}_{\beta\gamma\delta}$, $P^{\alpha}_{\beta\gamma\delta}$, $Q^{\alpha}_{\beta\gamma\delta}$ *and their conjugates vanish identically* [5].

Then the non-zero curvature tensors of the generalized Hermite connection are the tensors $R^{\alpha}_{\beta\delta\sigma^*}$, $P^{\alpha}_{\beta\delta\sigma^*}$, $P^{\alpha}_{\beta\delta^*\sigma}$, $Q^{\alpha}_{\beta\delta\sigma^*}$ and their conjugates, which are defined by the following relations:

(2.3) $$R^{\alpha}_{\beta\delta\sigma^*}=\bar R^{\alpha}_{\beta\delta\sigma^*}+B^{\alpha}_{\beta\rho}\bar R^{\rho}_{o\delta\sigma^*},\quad\text{where}\quad \bar R^{\alpha}_{\beta\delta\sigma^*}=-D_{\sigma^*}L^{\alpha}_{\beta\delta},$$

(2.4) $$P^{\alpha}_{\beta\delta\sigma^*}=\bar P^{\alpha}_{\beta\delta\sigma^*}+B^{\alpha}_{\beta\rho}\bar P^{\rho}_{o\delta\sigma^*},\quad\text{where}\quad \bar P^{\alpha}_{\beta\delta\sigma^*}=-D_{\dot\sigma^*}L^{\alpha}_{\beta\delta},$$

(2.5) $$P^{\alpha}_{\beta\delta^*\sigma} = \overline{P}^{\alpha}_{\beta\delta^*\sigma} + B^{\alpha}_{\beta\rho}\ \overline{P}^{\rho}_{o\delta^*\sigma}, \quad \text{where} \quad \overline{P}^{\alpha}_{\beta\delta^*\sigma} = D_{\delta^*}B^{\alpha}_{\beta\sigma},$$

(2.6) $$Q^{\alpha}_{\beta\delta\sigma^*} = \overline{Q}^{\alpha}_{\beta\delta\sigma^*} + B^{\alpha}_{\beta\rho}\,\overline{Q}^{\rho}_{o\delta\sigma^*}, \quad \text{where} \quad \overline{Q}^{\alpha}_{\beta\delta\sigma^*} = -D_{\dot{\sigma}^*}B^{\alpha}_{\beta\delta}.$$

Thus the curvature of the direction dependent Hermite connection is given by the form

$$\Omega^{\alpha}_{\beta} = R^{\alpha}_{\beta\gamma\delta^*}dz^{\gamma}\wedge dz^{\delta^*} + P^{\alpha}_{\beta\gamma\delta^*}\,dz^{\gamma}\wedge\theta^{\delta^*} + P^{\alpha}_{\beta\gamma^*\delta}dz^{\gamma^*}\wedge\theta^{\delta} + Q^{\alpha}_{\beta\gamma\delta^*}\,\theta^{\gamma}\wedge\theta^{\delta^*}$$

and its conjugate.

From the proposition (1.2) and the relations (2.5) and (2.6) we derive

(2.7) $$P^{\alpha}_{o\delta^*\sigma} = \overline{P}^{\alpha}_{o\delta^*\sigma} = 0,$$

(2.8) $$Q^{\alpha}_{o\delta\sigma^*} = \overline{Q}^{\alpha}_{o\delta\sigma^*} = 0,$$

and then

(2.9) $$P^{\alpha}_{\beta\delta^*\sigma} = D_{\delta^*}B^{\alpha}_{\beta\sigma}$$

(2.10) $$Q^{\alpha}_{\beta\delta\sigma^*} = -D_{\dot{\sigma}^*}B^{\alpha}_{\beta\delta}.$$

From (2.2), (2.9) and (2.10) we get:

(2.11) $$P^{\alpha}_{\beta\delta^*\sigma} = P^{\alpha}_{\sigma\delta^*\gamma},$$

(2.12) $$Q^{\alpha}_{\beta\delta\sigma^*} = Q^{\alpha}_{\delta\beta\sigma^*}.$$

Thus we have:

Proposition 2.1. *The curvature tensors $P^{\alpha}_{\beta\delta^*\sigma}$ and $Q^{\alpha}_{\beta\delta\sigma^*}$ of the direction Hermite connection are symmetric with respect to β, σ and β, δ, respectively.* The Bianchi identities of direction dependent Hermite connection are given by the relations

(2.13) $$\int \nabla_{\sigma}S^{\beta}_{\gamma\rho} + \int S^{\delta}_{\sigma\gamma}S^{\beta}_{\delta\rho} = 0,$$

(2.14) $$1/2\nabla_{\sigma^*}S^{\beta}_{\rho\gamma} + \overline{R}^{\beta}_{\gamma\rho\sigma^*} = 0,$$

(2.15) $$1/2\nabla_{\dot{\sigma}}S^{\beta}_{\gamma\rho} - S^{\beta}_{\rho\delta}h^{\rho}_{\gamma\sigma} - \nabla_{\sigma}h^{\beta}_{\gamma\delta} - 1/2h^{\beta}_{\rho\delta}S^{\rho}_{\sigma\gamma} = 0,$$

(2.16) $$1/2\ \nabla_{\dot{\sigma}^*}S^{\beta}_{\gamma\rho} = \overline{P}^{\beta}_{\gamma\rho\sigma^*},$$

(2.17) $$\nabla_{\dot{\sigma}}h^{\beta}_{\gamma\rho} = h^{\beta}_{\delta\rho}h^{\delta}_{\gamma\sigma}, \quad \nabla_{\rho^*}h^{\beta}_{\gamma\sigma} + \overline{P}^{\beta}_{\gamma\rho^*\sigma} = 0$$

(2.18) $$\nabla_{\dot{\sigma}^*}h^{\beta}_{\gamma\rho} = \overline{Q}^{\beta}_{\gamma\rho\sigma^*}.$$

From (2.14) and (2.16) it follows directly by means of (1.1):

(2.19) $$\overline{R}^{\beta}_{\gamma\rho\sigma^*} + \overline{R}^{\beta}_{\rho\gamma\sigma^*} = 0,$$

(2.20) $$\overline{P}^{\beta}_{\gamma\rho\sigma^*} + \overline{P}^{\beta}_{\rho\gamma\sigma^*} = 0.$$

Thus

(2.21) $$R^{\beta}_{oo\sigma^*} = 0,$$

(2.22) $$P^{\beta}_{oo\sigma^*} = 0,$$

which are an immediate consequence of the following calculations:

$$R^{\beta}_{o\gamma\sigma^*}=R^{\beta}_{\alpha\gamma\sigma^*}v^{\alpha}=\overline{R}^{\beta}_{o\gamma\sigma^*},\quad \text{by virtue of (2.3) and}$$

proposition 1.1 and

$$R^{\beta}_{oo\sigma^*}=R^{\beta}_{\alpha\gamma\sigma^*}v^{\alpha}v^{\gamma}=\overline{R}^{\beta}_{oo\sigma^*}=-1/2\nabla_{\sigma^*}S^{\beta}_{\alpha\gamma}v^{\alpha}v^{\gamma},$$

$$\nabla_{\sigma^*}S^{\beta}_{\alpha\gamma}v^{\alpha}v^{\gamma}=0 \text{ by virtue of (1.1).}$$

3. Let X and Y be two vector fields in a large sense on W^c. Then for every $z\in W^c$ the $X, Y\in T_{qz}(V_{2n})$. We assume that $(X^{\alpha}, X^{\alpha^*})$, $(Y^{\alpha}, Y^{\alpha^*})$ are their components with respect to the natural frames on V_{2n}. Then $(X^{\alpha}, 0)$, $(0, Y^{\alpha^*})$ are also components of vector fields in a large sense on W^c. We denote by $\mu_1(z, X_1, Y_1)$ the element plane of $X_1=(X^{\alpha}, 0)$ and $Y_1=(0, Y^{\alpha^*})$ at the point $z\in W^c$. This plane is subspace of $Tqz(V_{2n})$.

Definition 3.1. *We call first sectional curvature of the generalized Hermite space on the element plane μ_1 the function:*

$$k_1(z, \mu_1)=\frac{R_{\beta\alpha^*\gamma\delta^*}X^{\beta}X^{\gamma}Y^{\alpha^*}Y^{\delta^*}}{\langle X_1, Y_1\rangle^2}, \tag{3.1}$$

where

$$R_{\beta\alpha^*\gamma\delta^*}=g_{\alpha^*\rho}R^{\rho}_{\beta\gamma\delta^*}. \tag{3.2}$$

Definition 3.2. *The generalized Hermite space is called isotropic, if k_1 does not depend on μ_1.*

The relation (3.1) may be written as

$$(k_1g_{\alpha^*\gamma}g_{\beta\delta^*}-R_{\beta\alpha^*\gamma\delta^*})X^{\beta}X^{\gamma}Y^{\alpha^*}Y^{\delta^*}=0.$$

If k_1 does not depend on μ_1, we have

$$2k_1(g_{\alpha^*\gamma}g_{\beta\delta^*}+g_{\alpha^*\beta}g_{\gamma\delta^*})=R_{\beta\alpha^*\gamma\delta^*}+R_{\gamma\alpha^*\beta\delta^*}+R_{\beta\delta^*\gamma\alpha^*}+R_{\gamma\delta^*\beta\alpha^*}. \tag{3.3}$$

If we multiply (3.3) by v^{β}, v^{γ}, we get

$$4k_1g_{\alpha^*\gamma}g_{\beta\delta^*}v^{\beta}v^{\gamma}=0, \tag{3.4}$$

by virtue of (3.2) and (2.21).

If we multiply (3.4) by v^{α^*}, v^{δ^*}, we have

$k_1|\overline{v}|^2=0$ and then $k_1=0$. Thus we have

Theorem 3.1. *If the generalized Hermite space, is isotropic then the first sectional curvature on μ_1 is zero.*

If the generalized Hermite space is isotropic, then

$$R_{\beta\alpha^*\gamma\delta^*}+P_{\gamma\alpha^*\beta\delta^*}+R_{\beta\delta^*\gamma\alpha^*}+R_{\gamma\delta^*\beta\alpha^*}=0 \tag{3.5}$$

and, according (2.3), (2.19) and (3.2), we have

$$B_{\alpha^*\beta\rho}R^{\rho}_{\alpha\gamma\delta^*}+B_{\alpha^*\gamma\rho}R^{\rho}_{o\beta\delta^*}+B_{\delta^*\beta\rho}R^{\rho}_{o\gamma\alpha^*}+B_{\delta^*\gamma\rho}R^{\rho}_{o\beta\alpha^*}=0, \tag{3.6}$$

where

$$B_{\alpha^*\beta\rho}=q_{\alpha^*\lambda}B^{\lambda}_{\beta\rho^*}.$$

Conversely, from (3.6) we obtain (3.5) and then

$$R_{\beta\alpha^*\gamma\delta^*}X^{\beta}X^{\gamma}Y^{\alpha^*}Y^{\delta^*}=0$$

and k_1 vanish identically.

Thus we have

T h e o r e m 3.2. *A necessary and sufficient condition that a generalized Hermite space is isotropic is that the first curvature tensor satisfies the condition* (3.6).

D e f i n i t i o n 3.3. *We call second sectional curvature on the element plane* μ_2 *of the vectors* $X_2=(X^{\alpha},\ 0)$, $Y_2=(0,\ \dot{Y}^{\alpha^*})$ *the function*

$$k_2(z,\ \mu_2)=\frac{P_{\beta\alpha^*\gamma\delta^*}X^{\beta}X^{\gamma}\dot{Y}^{\alpha^*}\dot{Y}^{\delta^*}}{\langle X_2,\ Y_2\rangle^2},\quad where\quad P_{\beta\alpha^*\gamma\delta^*}=g_{\alpha^*\rho}P^{\rho}_{\beta\gamma\delta^*}\ .$$

T h e o r e m 3.3. *If the second sectional curvature does not depend on* μ_2, *then we have* $k_2=0$.

T h e o r e m 3.4. *A necessary and sufficient condition that a generalized Hermite space has constant second sectional curvature is that the second curvature tensor satisfies the condition*

$$B_{\alpha^*\beta\rho}P^{\rho}_{o\gamma\delta^*}+B_{\alpha^*\gamma\rho}P^{\rho}_{o\beta\delta^*}+B_{\delta^*\beta\rho}P^{\rho}_{o\gamma\alpha^*}+B_{\delta^*\gamma\rho}P^{\rho}_{o\beta\alpha^*}=0.$$

The proofs of the theorems 3.3 and 3.4 are analogous to the proofs of 3.1 and 3.2, respectively.

D e f i n i t i o n 3.4. *We call third sectional curvature on the element plane* μ_3 *of the vectors* $X_3=(0,\ X^{\alpha^*})$, $Y_3=(\dot{Y}^{\alpha},\ 0)$ *the function*

$$k_3(z,\ \mu_3)=\frac{P_{\beta\alpha^*\gamma^*\delta}\ X^{\alpha^*}X^{\delta^*}\dot{Y}^{\beta}\dot{Y}^{\delta}}{\langle X_3,\ Y_3\rangle^2},\ where\ P_{\beta\alpha^*\gamma^*\delta}=g_{\alpha^*\rho}P^{\rho}_{\beta\gamma^*\delta}. \tag{3.7}$$

We assume that k_3 does not depend on μ_3. Then from proposition (2.1) and (3.7) we have

$$2k_3(g_{\alpha^*\gamma}g_{\beta\delta^*}+g_{\alpha^*\beta}g_{\delta^*\gamma})=2P_{\beta\alpha^*\gamma^*\delta}\ +\ 2P_{\beta\gamma^*\alpha^*\delta}. \tag{3.8}$$

If we multiply (3.8) by ν^{β}, we get

$$2k_3(g_{\alpha^*\gamma}g_{\beta\delta^*}\nu^{\beta}+g_{\alpha^*\beta}g_{\delta\gamma}\nu^{\beta})=0, \tag{3.9}$$

by virtue of (2.7).
It follows from (3.9), that

$$k_3|\overline{\nu}|^2=0\leftrightarrow k_3=0.$$

Thus we have

T h e o r e m 3.5. *If the third sectional curvature* $k_3(z,\ \mu_3)$ *does not depend on* μ_3, *then it is zero.*

If k_3 *does not depend on* μ_3, *then from* (3.8) *we obtain*

$$P_{\beta\alpha^*\gamma^*\delta}=-P_{\beta\gamma^*\alpha^*\delta}. \tag{3.10}$$

Thus we have

T h e o r e m 3.6. *A necessary and sufficient condition that a generalized Hermite space has constant third sectional curvature is that the curvature tensor* $P_{\beta\alpha^*\gamma^*\delta}$ *is skey-symmetric with respect to* α^*, γ^*.

D e f i n i t i o n 3.5. *We call fourth sectional curvature on the element plan* μ_4 *of the vectors* $X_4=(\dot{X}^{\alpha},\ 0)$, $Y_4=(0,\ \dot{Y}^{\alpha^*})$ *the function*

$$K_4(z,\ \mu_4)=\frac{Q_{\beta\alpha^*\gamma\delta^*}\ \dot{X}^{\beta}\dot{X}^{\gamma}\dot{Y}^{\alpha^*}\dot{Y}^{\delta^*}}{\langle X_4\ \ Y_4\rangle^2},\quad where\quad Q_{\beta\alpha^*\gamma\delta^*}=g_{\alpha^*\lambda}Q^{\lambda}_{\beta\gamma\delta^*}\ .$$

If k_4 is independent of μ_4, then we have

$$K_4(g_{\alpha^*\gamma}g_{\beta\delta^*}+g_{\alpha^*\beta}g_{\delta^*\gamma})=Q_{\beta\alpha^*\gamma\delta^*}+Q_{\beta\delta^*\gamma\alpha^*} \tag{3.11}$$

$$K_4(g_{\alpha^*\gamma}g_{\beta\delta^*}\nu^{\beta}+g_{\alpha^*\beta}g_{\delta^*\gamma}\nu^{\beta})=0,$$

and

$$K_4 g_{\alpha^*\gamma} g_{\beta\delta^*} v^\beta v^\gamma = 0 \quad \text{implies} \quad K_4 = |\bar{v}|^2 = 0 \leftrightarrow K_4 = 0.$$

The tensor $Q_{\beta\alpha^*\gamma\delta^*}$ has the property $Q_{\beta\alpha^*\gamma\delta^*} = Q_{\beta\delta^*\gamma\alpha^*}$.

In fact $Q_{\beta\alpha^*\gamma\delta^*} = g_{\rho\alpha^*} Q^{\rho}_{\beta\gamma\delta^*} = g_{\rho\alpha^*}(-D_{\dot{\delta}^*} B^{\alpha}_{\beta\gamma}) = g_{\rho\alpha^*}(-\delta_{\dot{\delta}^*}(g^{\rho k^*} B_{k^*\beta\gamma})$

$$= -g_{\rho\alpha^*}\delta_{\dot{\delta}^*} g^{\rho k^*} B_{k^*\beta\gamma} - g_{\rho\alpha^*} g^{\rho k^*} \delta_{\dot{\gamma}^*} B_{k^*\beta\gamma}$$

$$= -g^{\rho k^*}\delta_{\dot{\delta}^*} g_{\rho\alpha^*} B_{k^*\beta\gamma} - \delta_{\dot{\delta}^*} B_{\alpha^*\beta\gamma}$$

$$= -g^{\rho k^*}\delta_{\dot{\delta}^*\dot{\rho}\,\dot{\alpha}^*} F \,.\, \delta_{\dot{\gamma}\,\dot{\beta}\,\dot{k}^*} F - \delta_{\dot{\delta}^*\dot{\gamma}\,\dot{\beta}\,\dot{\alpha}^*} F$$

$$= -g^{\rho k^*}\delta_{\dot{\alpha}^*\dot{\rho}\,\dot{\delta}^*} F \,.\, \delta_{\dot{\gamma}\,\dot{\beta}\,\dot{k}^*} F - \delta_{\dot{\alpha}^*\dot{\gamma}\,\dot{\beta}\,\dot{\delta}^*} F = Q_{\beta\delta^*\gamma\alpha^*}\,.$$

Then from (3.11) it implies that $Q_{\beta\alpha^*\gamma\delta^*} = 0$ and consequently $Q^{\rho}_{\beta\gamma\delta^*} = 0$. Thus we have

Theorem 3.7. *If the fourth sectional curvature k_4 is constant then it is zero.*

Theorem 3.8. *If a generalized Hermite space has constant fourth sectional curvature, then it is a complex Berwald space.*

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Aristotle Univ. of Thessaloniki
Faculty of Sciences
Department of Mathematics
54006 Thessaloniki, GREECE

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