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OPTIMAL CONTROL OF STOCHASTIC DISCRETE-TIME SYSTEMS UNDER MARKOV DISTURBANCES DEPENDING ON CONTROL PARAMETERS

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The paper deals with a stochastic maximum principle for discrete-time systems under Markov disturbances depending on control parameters. For the case of controlled Markov chains the principle is also discussed.

1. Introduction. Stochastic discrete-time systems under Markov disturbances depending on control parameters were considered in [1]. The optimal control problem for them includes as special cases the dynamic systems with jump Markov disturbances (see [2, 3, 4]) and the controlled Markov chains (see [5, 6, 7]). In [1] we gave one example to show the motivation for studying such optimization problems.

Note that the dependence of disturbances on control parameters arises in the situation when the change of the system structure is influenced by acting of control strategies. On the other hand, disturbances can be regarded, e. g., as reactions in economic processes. Thus we come to a more complicated problem in economic dynamics: finding an optimal strategy in the case when the disturbances are able to "recognize" the control strategies and to change the probabilities of jumping to more or less expensive models to prevent attainment of gains accordingly.

For such systems we followed in [1] Bellman's dynamic programming approach. It is therefore of general to establish Pontriagin's maximum principle for them. In Section 2 we formulate the optimization problem with non-anticipating control strategies and then consider Pontryagin's maximum principle. In Section 3 we discuss this principle for the case of controlled Markov chains.

2. The stochastic maximum principle. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a basic probabilistic space. We are given a family of transition probabilities depending on the parameter v:

$$p(i, v), \quad p(i, j, v), \quad v \in U \subset R^m, \quad i, j, \in I = \{1, 2, \ldots, s\},$$
$$0 \leqq p(i, v) \leqq 1, \quad 0 \leqq p(i, j, v) \leqq 1,$$

and functions

$$f_t = (f_t^1, f_t^2, \ldots, f_t^n): R^n \times U \times I \to R^n,$$
$$\varphi_t = (\varphi_t^1, \varphi_t^2, \ldots, \varphi_t^k): R^n \times U \times I \to R^k,$$
$$g_t: R^n \times U \times I \to R, \quad 0 \leqq t \leqq N-1.$$

By an admissible control process we mean here a sequence of measurable functions

$$(x, u, q^u) = (\{x_0, u_0\}, \ \{x_t(\omega), \eta_{t-1}(\omega), u_t(\omega) \quad 1 \leqq t \leqq N-1,$$
$$\{x_N(\omega), u_{N-1}(\omega)\}; \quad p(\eta_0(\omega), u_0), \ p(\eta_{t-1}(\omega), \eta_t(\omega), u_t(\omega)),$$
$$1 \leqq t \leqq N-1)$$

with $x_t \in R^n$, $u_t \in U$, $\eta_t \in I$ and satisfying the following relations:

(2.1) (i) $x_{t+1} = f_t(x_t, u_t, \eta_t)$

$x_0(\omega) = x_0 \in R^n, \quad 0 \leqq t \leqq N-1.$

(ii) $\sum_{i=1}^{s} p(i, u_0) = 1$ and for every function

$l: I \to [0, +\infty)$ we have

$$\mathbf{E}\{l(\eta_{t+1}) \mid \eta_t, \eta_{t-1}, \dots, \eta_0\} = \sum_{j=1}^{s} l(j) p(\eta_t, j, u_{t+1}). \tag{2.2}$$

(iii) For every $1 \leqq t \leqq N-1$, $u_t(\omega)$ is $\sigma(\eta_0, \eta_1, \dots, \eta_{t-1})$-measurable, that means the control parameter at time t depends only on the past information until time $t-1$.

(iv) For every $0 \leqq t \leqq N$ one holds

$$\varphi_0(x_0, u_0) \leqq 0 \tag{2.3}$$

$$\varphi_t(x_t, u_t, \eta_{t-1}) \leqq 0 \quad (q_t^u\text{—a. s.}),$$

where the probability measure $q^u = (q_t^u(i_0, i_1, \dots, i_{t-1}))$ is defined on the set $I^t = I \times I \times \dots \times I$ (t times) by the following relations:

$$q_{t+1}^u = p(i_{t-1}, i_t, u_t) q_t^u, \quad 1 \leqq t \leqq N-1 \tag{2.4}$$

$$q_1^u = p(i_0, u_0) q_0, \quad q_0 = 1.$$

The optimization problem consists in finding an optimal admissible control process (x^*, u^*, q^*) such that the performance function

$$J(u, x_0) = \mathbf{E}^u \sum_{t=0}^{N-1} g_t(x_t, u_t, \eta_{t-1}) \tag{2.5}$$

attains its minimum value at (x^*, u^*, q^*). Here by $\mathbf{E}^u$ we denote the expectation with respect to the probability measure q^u.

Remark 1. Relations (i)—(iii) show a symmetry role of x_t and η_t for the control problem (see [1]) in the sense that both state variables $\{x_t\}$ and disturbance variables $\{\eta_t\}$ depend on the control variables $\{u_t\}$.

Nevertheless, it is interesting to see that if we introduce the new state variable $y_t = \{x_t, \eta_t\}$, then the first part of y_t is defined by the difference system (2.1) while the second part is defined by the distributions (2.2). Thus this seems to be the natural type of situation to consider for a generalization of statement of control problems. In this situation the variable q_t^u (see (2.4)) will play an important role in what follows.

We shall introduce the following assumptions on the functions f_t, φ_t, g_t and the set $U \subset R^m$.

(A) The functions g_t, f_t, φ_t are continuous in (x, u), differentiable with respect to x and such that their derivatives are continuous in x.

(B) For every $x \in R^n$, $u' \in U$, $u'' \in U$, $\alpha \in [0, 1]$, $i \in I$, there exists an element $u = u(x, u', u'', \alpha, i) \in U$ such that the following relations hold:

$$g_t(x, u, i) \leqq \alpha g_t(x, u', i) + (1-\alpha) g_t(x, u'', i)$$

$$f_t(x, u, j) = \alpha f_t(x, u', j) + (1-\alpha) f_t(x, u'', j), \quad 1 \leqq j \leqq s,$$

$$p(i, j, u) = \alpha p(i, j, u') + (1-\alpha) p(i, j, u''), \quad 1 \leqq j \leqq s,$$

$$\varphi_t(x, u, i) \leqq \alpha \varphi_t(x, u', i) + (1-\alpha) \varphi_t(x, u'', i).$$

(C) There exist functions $\bar{x}_t: I^t \to R^n$, $\bar{q}_t: I^t \to R$, $\bar{u}_t: I^t \to U$ such that for every

$$i_0^{t-1} = (i_0, i_1, \ldots, i_{t-1}) \in I^t, \quad 1 \le t \le N,$$

we have

$$\bar{x}_{t+1} = \frac{\partial}{\partial x} f_t(x_t^*, u_t^*, i_t)\bar{x}_t + f_t(x_t^*, \bar{u}_t, i_t) - f_t(x_t^*, u_t^*, i_t) \tag{2.6}$$

$$\bar{x}_0 = 0, \quad 0 \le t \le N-1,$$

$$\bar{q}_{t+1} = p(i_{t-1}, i_t, u_t^*)\bar{q}_t + p(i_{t-1}, i_t, \bar{u}_t) q_t^* - p(i_{t-1}, i_t, u_t^*) q_t^* \tag{2.7}$$

$$\bar{q}_0 = 0, \quad 0 \le t \le N-1,$$

and for every $r = 1, 2, \ldots, k$, $i_0^{t-1} \in I^t$ with $q_t^* \varphi_t^r(x_t^*, u_t^*, i_{t-1}) = 0$,

$$q_t^* \frac{\partial}{\partial x} \varphi_t^r(x_t^*, u_t^*, i_{t-1})\bar{x}_t + q_t^* \varphi_t^r(x_t^*, \bar{u}_t, i_{t-1}) + \varphi_t^r(x_t^*, \bar{\bar{u}}_t^*, i_{t-1})\bar{q}_t < 0.$$

Here we denote by $(\{x_t^* = x_t^*(i_0^{t-1})\}, \{u_t^* = u_t^*(i_0^{t-1})\}, q^* = q^{u^*})$ the optimal control process.

To formulate the main result we introduce the following Hamiltonian functions:

$$H_{t+1}(i_0^{t-1}, i_t, \psi, \chi, \lambda, (x, q), u) = 1/s\, q \cdot g_t(x, u, i_{t-1}) - \psi' \cdot f_t(x, u, i_t) \tag{2.8}$$

$$-\chi \cdot p(i_{t-1}, i_t, u) q + 1/s\, q \cdot \lambda' \cdot \varphi_t(x, u, i_{t-1}),$$

where $i_0^{t-1} = (i_0, i_1, \ldots, i_{t-1}) \in I^t$, $i_t \in I$, $q: I^t \to [0, 1]$,

$$\psi: I^t \to R^n, \quad \chi: I^t \to R, \quad \lambda: I^t \to (R^+)^k, \quad R^+ = [0, +\infty),$$

s is the number of the elements of the set I and a prime "$'$" denotes the transpose.

Theorem 1. *If Assumptions* (A)—(C) *hold and*

$$(x^*, u^*, q^*) = (\{x_t^*\}, \{u_t^*\}, \{q_t^{u^*}\})$$

is an optimal admissible control process in Problem (2.1)—(2.5), *then there exist functions* $\psi_t: I^t \to R^n$, $\chi_t: I^t \to R$, $\lambda_t' = (\lambda_t^1, \lambda_t^2, \ldots \lambda_t^k): I^t \to (R^+)^k$, $0 \le t \le N-1$, *such that for all* $i_0^{t-1} \in I^t$ *the function of variable* u

$$\mathcal{H}_t(i_0^{t-1}, u) = \sum_{i_t=1}^{s} H_{t+1}(i_0^{t-1}, i_t, \psi_{t+1}(i_0^{t-1}, i_t), \chi_{t+1}(i_0^{t-1}, i_t), \lambda_t, x_t^*, q_t^*, u), \quad 1 \le t \le N,$$

$$\mathcal{H}_0(x_0, u) = \sum_{i_0=1}^{s} H_1(i_0, \psi_1, \chi_1, x_0, q_0, u)$$

attains its minimum value at $u_t^* = u_t^*(i_0^{t-1})$, i. e.

$$\min_{u \in U} \mathcal{H}_t(i_0^{t-1}, u) = \mathcal{H}_t(i_0^{t-1}, u_t^*(i_0^{t-1})).$$

Moreover, the functions ψ_t, χ_t *satisfy the following adjoint systems:*

$$\psi_t = -\frac{\partial}{\partial x} \mathcal{H}_t(i_0^{t-1}, u_t^*(i_0^{t-1})) = -q_t^* \frac{\partial}{\partial x} g_t(x_t, u_t, i_{t-1}$$

$$+\sum_{i_t=1}^{s}\psi'_{t+1}\frac{\partial}{\partial x}f_t(x_t^*,u_t^*,i_t)-q_t^*\lambda'_t\frac{\partial}{\partial x}\varphi_t(x_t^*,u_t^*,i_{t-1}),\quad 1\leq t\leq N-1,$$

$$\psi_N=0,$$

$$q_t^*\chi_t=-q_t^*g_t(x_t^*,u_t^*,i_{t-1})+\sum_{i_t=1}^{s}\chi_{t+1}(i_0^{t-1},i_t)p(i_{t-1},i_t,u_t^*)q_t^*,$$

$$\chi_N=0,\ 1\leq t\leq N-1,$$

and the functions λ_t *satisfy the condition* $q_t\lambda'_t\varphi_t(x_t^*,u_t^*,i_{t-1})=0$, $1\leq t\leq N$, *for all* $(i_0,i_1,\dots,i_{t-1})\in I^t$, $1\leq t\leq N$.

Proof. First of all we reduce Problem (2.1)—(2.5) to some equivalent deterministic convex-smooth optimization problems to which the maximum principle given in [8] can be applied.

Problem 1. Denote by $y_t=(x_t\ q_t)'$, $t=1$, $y_0=(x_0,1)$; $\bar{f}_t(y_t,u_t,i_{t-1},i_t)=(f_t(x_t,u_t,i_t)\ p(i_{t-1},i_t,u_t)q_t)'$, $(y,u)=(y_t=y_t(i_0^{t-1}),\ u=u(i_0^{t-1}),\ i_0^{t-1}\in I^t,\ 1\leq t\leq N)$, where $u_t=u_t(i_0^{t-1})\in U$ and $y_t=y_t(i_0^{t-1})$ is defined by the following relation:

$$y_{t+1}=\bar{f}_t(y_t,u_t,i_{t-1},i_t),\quad 0\leq t\leq N-1.$$

Associate with each (y,u) a performance function

$$\bar{J}(y,u)=\sum_{t=0}^{N-1}\sum_{i_0^{t-1}\in I^t}\bar{g}_t(y_t,u_t,i_{t-1})\tag{2.9}$$

and constraints on the pair (y,u)

$$\bar{\varphi}_t(y_t,u_t,i_{t-1})\leq 0,\quad \bar{\varphi}_0(y_0,u_0)\leq 0,\tag{2.10}$$

where $\bar{g}_t(y_t,u_t,i_{t-1})=q_tg_t(x_t,u_t,i_{t-1})$, $\bar{\varphi}_t(y_t,u_t,i_{t-1})=q_t\varphi_t(x_t,u_t,i_{t-1})$, $\bar{\varphi}_0(y_0,u_0)=q_0\varphi_0(x_0,u_0)$.

The first optimization problem is to find among all (y,u) one (y^*,u^*) at which the performance function $\bar{J}$ attains its minimum value.

Problem 2. Introduce the sets

$$A_t^*=\{i_0^{t-1}\in I^t: q_t^{u^*}(i_0^{t-1})>0\},\quad 1\leq t\leq N,\tag{2.11}$$

and consider the following problem which is a restriction of Problem 1 on all $i_0^{t-1}\in A_t^*$:

$$y_{t+1}=\bar{f}_t(y_t,u_t,i_{t-1},i_t),\quad 0\leq t\leq N-1\tag{2.12}$$

$$y_0=(x_0\ 1)',\quad i_0^{t-1}\in A_t^*,\quad (i_0^{t-1},i_t)\in A_{t+1}^*,$$

$$\tilde{J}(y,u)=\sum_{t=0}^{N-1}\sum_{i_0^{t-1}\in A_t^*}\bar{g}_t(y_t,u_t,i_{t-1})\tag{2.13}$$

$$\bar{\varphi}_t(y_t,u_t,i_{t-1})\leq 0,\quad i_0^{t-1}\in A_t^*,\quad 1\leq t\leq N.\tag{2.14}$$

Problem 3. Set $\mathscr{Y}=\mathscr{X}=\Pi_{t=1}^{N}l_1(A_t^*,R^{n+1})$, and

$$\mathscr{U}=\{u=(u_0,u_1,\dots,u_{N-1}),\ u_t=u_t(i_0^{t-1})\in U,\ i_0^{t-1}\in A_t^*,\ u_0=u_0(x_0)\},$$

where the finite dimensional space $l_1(A_t^*, R^{n+1})$ consists of all R^{n+1}-valued sequences $y=(x,q)=(x_t(i_0^{t-1}), q_t(i_0^{t-1}))$, $i_0^{t-1}\in A_t^*$, $1\leq t\leq N)$ with the Euclidean norm.

Consider the transformations F: $\mathscr{Y}\times\mathscr{U}\to\mathscr{Z}$, $\Phi^r(i_0^{t-1})$: $\mathscr{Y}\times\mathscr{U}\to R$, $1\leq t\leq N$, $i_0^{t-1}\in A_t^*$ $1\leq r\leq k$, which are defined by the relations

$$F(y, u)=z=(z_t, 1\leq t\leq N),$$

$$z_{t+1}=(x_{t+1}-f_t(x_t, u_t, i_t) \quad q_{t+1}-p(i_{t-1}, i_t, u_t)q_t)',$$

$$\Phi^r(i_0^{t-1})=\Phi^r(i_0^{t-1}, y, u)=\overline{\varphi}_t^r(y_t(i_0^{t-1}), u_t(i_0^{t-1}), i_{t-1}).$$

Thus the third optimization problem is to find the minimum value of the function $\tilde{J}$ on the class of all pairs (y, u) satisfying the constraints $F(y, u)=0$,

$$\Phi^r(i_0^{t-1}, y, u)\leq 0, \; i_0^{t-1}\in A_t^*, \quad 1\leq r\leq k.$$

Our purpose now is to discuss on the relationships between Problem (2.1)—(2.5) and Problems 1—3.

Lemma 1. *The optimal admissible process in Problem* (2.1)—(2.5) *is also optimal admissible for Problems* 1—3.

Proof. First we transform (2.5) into (2.9):

$$J(x, u)=\sum_{t=0}^{N-1} E^u g_t(x_t, u_t, \eta_{t-1})=\sum_{t=0}^{N-1} \sum_{i_0^{t-1}\in I^t} p(i_0, u_0)$$

$$\cdot p(i_0, i_1, u_1)\cdot \ldots \cdot p(i_{t-1}, i_t, u_t)\cdot g_t(x_t(i_0^{t-1}), u_t(i_0^{t-1}), i_{t-1})$$

$$=\sum_{t=0}^{N-1} \sum_{i_0^{t-1}\in I^t} q_t\cdot g_t(x_t, u_t, i_{t-1})=\overline{J}(y, u).$$

The constraint (2.3) is equivalent to (2.10) and in its turn (2.10) is reduced to (2.14).

Hence the optimal admissible process in Problem (2.1)—(2.5) is also optimal admissible for Problem 1.

Obviously Problems 2—3 are equivalent and therefore in order to complete the proof of Lemma 1, it is sufficient to show its assertion for Problem 2.

Let (x_t^*, u_t^*, q_t^*) be an optimal admissible process in Problem (2.1)—(2.5) and let $(y, u)=(y_t(i_0^{t-1}), u_t(i_0^{t-1}), i_0^{t-1}\in A_t^*)$ be any admissible process in Problem 2. Put

$$\widehat{u}_t(i_0^{t-1})=u_t(i_0^{t-1}), \quad i_0^{t-1}\in A_t^*, \quad \widehat{u}_t(i_0^{t-1})=u_t^*(i_0^{t-1}), \quad i_0^{t-1}\in I^t\setminus A_t^*.$$

From (2.11)—(2.12) it follows

$$\widehat{y}_t=(\widehat{x}_t, \widehat{q}_t)=\begin{cases} y_t=(x_t, q_t), & \text{if } i_0^{t-1}\in A_t^* \\ (\widehat{x}_t, 0), & \text{if } i_0^{t-1}\in I^t\setminus A_t^*. \end{cases}$$

Then $(\widehat{y}_t, \widehat{u}_t)$ is admissible for Problem 1 and (see (2.13))

$$\tilde{J}(y, u)=\overline{J}(\widehat{y}, \widehat{u})\geq\overline{J}(y^*, u^*)=J(u^*, x_0),$$

where (y_t^*, u_t^*) is admissible process in Problem 2 corresponding to the optimal admissible process (x_t^*, u_t^*, q_t^*).

The proof of Lemma 1 is completed.

The second step of the proof of Theorem 1 is to show that Problem 3 with Assumptions (A), (B), (C) is a convex-smooth problem satisfying the assumptions of Theorem 3 in [8, p. 79].

By a neighbourhood $V \subset \mathcal{Y}$ of the point y^* we mean all possible sequences $y=((x_t, q_t), i_0^{t-1} \in A_t^*, 1 \leq t \leq N)$ with $x_t \in R^n$, $q_t > 0$, $1 \leq t \leq N$, $i_0^{t-1} \in A_t^*$.

The restrictions of the mappings F and Φ on V have the following properties.

Lemma 2. (a) *For every* $u \in \mathcal{U}$, *the mapping* $y \to F(y, u)$ *and the functions* $y \to \Phi^r(i_0^{t-1}, y, u)$ *belong to the class* C^1 *at the point* y^*.

(b) *For every* $y \in V$ *the mapping* $u \to F(y, u)$ *and the functions* $u \to \tilde{J}(y, u)$, $u \to \Phi$, $\times (i_0^{t-1}, y, u)$ *satisfy the convexity condition: for every* $u' \in \mathcal{U}$, $u'' \in \mathcal{U}$ *and* $0 \leq \alpha \leq 1$ *there exists* $u \in \mathcal{U}$ *such that*

$$F(y, u) = \alpha F(y, u') + (1-\alpha) F(y, u')$$

$$\tilde{J}(y, u) \leq \alpha \tilde{J}(y, u') + (1-\alpha) \tilde{J}(y, u'')$$

$$\Phi^r(i_0^{t-1}, y, u) \leq \alpha \Phi^r(i_0^{t-1}, y, u') + (1-\alpha) \Phi^r(i_0^{t-1}, y, u'')$$

(c) $\operatorname{Im} F_y'(y^*, u^*) = \mathcal{X}$.

(d) *The image of the set* $\mathcal{Y} \times \mathcal{U}$ *through the mapping* $(y, u) \to F_y(y^*, u^*) y + F(y^*, u)$ *contains in it a neighbourhood of the point* $0 \in \mathcal{X}$ *and there exists a point* $(\bar{y}, \bar{u})$ *such that*

$$F_y(y^*, u^*) \bar{y} + F(y^*, \bar{u}) = 0, \quad \langle \Phi_y^r(i_0^{t-1}, y^*, u^*), \bar{y} \rangle + \Phi^r(i_0^{t-1}, y^*, \bar{u}) < 0$$

for all r, i_0^{t-1} *for which* $\Phi^r(i_0^{t-1}, y^*, u^*) = 0$.

Proof. The convexity condition (b) and the regularity condition (d) for the mappings F, $\tilde{J}$, $\Phi^r(i_0^{t-1}, y, u)$ follow from Assumptions (B) and (C) of Theorem 1.

Condition (A) implies Condition (a) and for any $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_N)$ we have

$$F_y(y, u) \bar{y} = \{\bar{y}_{t+1} - \frac{\partial}{\partial y} \bar{f}_t(y_t, u_t, i_t) \bar{y}_t, \quad 0 \leq t \leq N-1\}$$

$$\bar{J}_y(y, u) \bar{y} = \sum_{t=0}^{N-1} \sum_{i_0^{t-1} \in A_t^*} \frac{\partial}{\partial y} \bar{g}_t(y_t, u_t, i_{t-1}) \bar{y}_t$$

$$\Phi_y^r(i_0^{t-1}, y, u) \bar{y} = \frac{\partial}{\partial y} \bar{\varphi}_t^r(y_t(i_0^{t-1}), u_t, i_{t-1}) \bar{y}_t(i_0^{t-1}),$$

where $\bar{y}_0 = 0$, $0 \leq t \leq N-1$, $1 \leq r \leq k$.

Moreover, the above mentioned formula for F implies Assertion (c) of Lemma 2 and thus the proof is completed.

We are now in a position to prove Theorem 1.

From Lemmas 1—2 and the extremal principle in [8, p. 79] for the convex-smooth Problem 3 it follows that there exist elements $(\psi_t, \chi_t) \in l_1(A_t^*, R^{n+1})$, $\psi_t = \psi_t(i_0^{t-1})$, $i_0^{t-1} \in A_t^*)$, $\chi_t = (\chi_t(i_0^{t-1}), i_0^{t-1} \in A_t^*)$ and sequences of nonnegative numbers $\lambda_t = (\lambda_t^r(i_0^{t-1}), i_0^{t-1} \in A_t^*, 1 \leq r \leq k)$ such that

$$\lambda_t' \cdot q_t^* \varphi_t(x_t^*, u_t^*, i_{t-1}) = 0, \quad i_0^{t-1} \in A_t^* \tag{2.15}$$

and the Lagrange function

$$\mathscr{L}(\{y_t\}, \{u_t\}, \{\psi_t\}, \{\chi_t\}, \lambda_t(i_0^{t-1})) = \sum_{t=0}^{N-1} \sum_{i_0^{t-1} \in A_t^*} \bar{g}_t(y_t, u_t, i_{t-1})$$

$$+ \sum_{t=0}^{N-1} \langle \psi_{t+1}, x_{t+1} - f_t(x_t, u_t, i_t) \rangle + \sum_{t=0}^{N-1} \langle \chi_{t+1}, q_{t+1} - p(i_{t-1}, i_t, u_t) q_t \rangle$$

$$+ \sum_{t=0}^{N-1} \sum_{i_0^{t-1} \in A_t^*} \langle \lambda_t(i_0^{t-1}), \bar{\varphi}_t(y_t, u_t, i_{t-1}) \rangle = \sum_{t=0}^{N-1} \sum_{i_0^{t-1} \in A_t^*} [\bar{g}_t(y_t, u_t, i_{t-1})$$

$$+ \sum_{i_t=1}^{s} \psi'_{t+1}(i_0^{t-1}, i_t)(x_{t+1}(i_0^{t-1}, i_t) - f_t(x_t, u_t, i_t))$$

$$+ \sum_{i_t=1}^{s} \chi_{t+1}(i_0^{t-1}, i_t)(q_{t+1}(i_0^{t-1}, i_t) - p(i_{t-1}, i_t, u_t) q_t(i_0^{t-1}) + \lambda'(i_0^{t-1}) . \bar{\varphi}_t(y_t, u_t, i_{t-1})]$$

satisfies the following conditions:

(I) the Lagrange function as a function of the variable $u \in \mathscr{U}$ attains its minimum value at $u^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*)$

(II) the Lagrange function as a function of the variable $y \in \mathscr{Y}$ satisfies the loca minimum condition

$$\mathscr{L}_y = 0,$$

where

$$\mathscr{L}_y = \bar{J}_y(y^*, u^*) + \langle \psi_t, F_y(y^*, u^*) \rangle + \sum_{i_0^{t-1} \in A_t^*} \Phi_y(i_0^{t-1}, y^*, u^*).$$

Putting $(\psi_t(i_0^{t-1}), \chi_t(i_0^{t-1})) = 0$ for all $i_0^{t-1} \in I^t \setminus A_t^*$, from (I) and (II) (for the case $\mathscr{L}_x = 0$) we obtain the maximum principle and the adjoint equations for ψ_t described in Theorem 1.

Also from (II) (for every element of the variable q) we deduce for all $i_0^{t-1} \in A_t^*$ that

$$\mathscr{L}_{q_t^{(i_0^{t-1})}} = g_t(x_t^*(i_0^{t-1}), u_t^*(i_0^{t-1}), i_{t-1}) + \chi_t(i_0^{t-1})$$

$$+ \sum_{i_t=1}^{s} \chi_{t+1}(i_0^{t-1}, i_t)(-p(i_{t-1}, i_t, u_t^*)) + \lambda'(i_0^{t-1}) \varphi_t(x_t^*(i_0^{t-1}), u_t^*(i_0^{t-1}), i_{t-1}) = 0.$$

Hence

$$\chi_t(i_0^{t-1}) = -g_t(x_t^*(i_0^{t-1}), u^*(i_0^{t-1}), i_{t-1})$$

$$+ \sum_{i_t=1}^{s} \chi_{t+1}(i_0^{t-1}, i_t) p(i_{t-1}, i_t, u_t) - \lambda'(i_0^{t-1}) \varphi_t(x_t^*(i_0^{t-1}), u_t^*(i_0^{t-1}), i_{t-1})$$

for all $i_0^{t-1} \in A_t^*$. By the definition of A_t^* (see (2.11)) and $q_t^*(i_0^{t-1}) > 0$ we deduce that χ_t and χ_{t+1} satisfy the relation

$$q_t^*(i_0^{t-1}) \chi_t(i_0^{t-1}) = -q_t^*(i_0^{t-1}) g_t(x_t^*(i_0^{t-1}), u_t^*(i_0^{t-1}), i_{t-1})$$

$$+ \sum_{i_t=1}^{s} \chi_{t+1}(i_0^{t-1}, i_t) p(i_{t-1}, i_t, u_t^*) q_t^* - \lambda'(i_0^{t-1}) q_t^* \varphi_t(x_t^*(i_0^{t-1}), u_t^*(i_0^{t-1}), i_{t-1}).$$

From (2.15) and the definition of q^*_{t+1}, the last equation becomes

$$q^*_t(i_0^{t-1})\chi_t(i_0^{t-1}) = -q^*_t g_t(x^*_t(i_0^{t-1}), u^*_t(i_0^{t-1}), i_{t-1}) + \sum_{i_t=1}^{s} \chi_{t+1}(i^{t-1}, i_t) q^*_{t+1}(i_0^t).$$

which is also true for $\chi_t(i_0^{t-1}) = 0, i_0^{t-1} \in I^t \setminus A^*_t$.

The proof of Theorem 1 is completed.

3. The case of controlled Markov chains. Consider a controlled Markov chain $\{\eta^u_t, t=0, 1, 2, \dots\}$ with transition probabilities $p(i, j, u)$, $u \in R^m$, $i, j \in I = \{1, 2, \dots, s\}$ and the performance function

$$J(u) = \mathbf{E}^u \sum_{t=1}^{N-1} g_t(\eta_{t-1}, \eta_t, u_t). \tag{3.1}$$

The control u_t is of the form $u_t = u_t(\eta_0, \eta_1, \dots, \eta_{t-1})$ and the constraints are

$$\varphi_t(\eta_{t-1}, u_t) \leqq 0, \quad (q^u_t \text{ — a. s.}). \tag{3.2}$$

Here the functions g_t, φ_t take values in R and q^u_t, $t=0, 1, 2, \dots$ is defined by (2.4).

The optimization problem is to find u^* such that the function $J(u)$ attains its minimum value at u^*.

Taking $f_t = 0$ as a special case of Problem (2.1)—(2.5) we can apply direct Theorem 1 to Problem (3.1)—(3.2) with suitable assumptions on g_t, φ_t and the assumption that $p(i, j, u)$ is linear in u.

If $p(i, j, u)$ is nonlinear in u, but the functions g_t, p, φ_t are differentiable in u we shall apply Theorem 1 to this case as follows (see [4]).

We introduce the variables (z_t, y_t) defined by

$$\begin{aligned} &z_{t+1} = u_t \\ &y_{t+1} = q^u_t = p(\eta_{t-2}, \eta_{t-1}, u_{t-1}) q^u_{t-1} = p(\eta_{t-2}, \eta_{t-1}, z_t) y_t \end{aligned} \tag{3.3}$$

and the functions

$$\tilde{\varphi}_t(\eta_{t-2}, z_t) = \varphi_{t-1}(\eta_{t-2}, u_{t-1}), \tilde{g}_t(\eta_{t-2}, z_t) = \sum_{i=1}^{s} g_{t-1}(\eta_{t-2}, i, z_t) \cdot p(\eta_{t-2}, i, z_t). \tag{3.4}$$

Then the performance (3.1) becomes

$$J(u) = \mathbf{E}^u \sum_{t=0}^{N-1} \sum_{i=1}^{s} g_t(\eta_{t-1}, i, u_t) p(\eta_{t-1}, i, u_t) \tag{3.5}$$

$$= \mathbf{E}^u \sum_{t=1}^{\Lambda} \sum_{i=1}^{s} g_{t-1}(\eta_{t-2}, i, z_t) p(\eta_{t-2}, i, z_t) = \mathbf{E}^u \sum_{t=1}^{\Lambda} \tilde{g}_t(\eta_{t-2}, z_t)$$

Note that in (3.3) we consider the transition probabilities $p(\eta_{t-2}, \eta_{t-1}, z_t)$ depending on the state parameter z_t. Thus Problem (2.1)—(2.5) would be generalized to the case of transition probabilities $p(i, j, u, x)$ in the sense that (see (2.2))

$$\mathbf{E}[l(\eta_{t+1}) \mid \eta_t, \eta_{t-1}, \dots, \eta_0] = \sum_{j=1}^{s} p(\eta_t, j, u_{t+1}, z_{t+1}). \tag{3.6}$$

The inclusion of x in the function p and η_{t-2} in g_t, φ_t (see (3.5), (2.5)) causes no difficulties and all of the results go through with minor modifications.

Theorem 2. *Assume that Assumptions* (A), (B), (C) *hold for the functions* f_t g_t, φ_t, $p(i, j, u, x)$ *and for their derivatives with respect to* x.

Then the maximum principle formulated in Theorem 1 remains true for the optimization problem $\{(2.1), (3.6), (2.3), (2.5)\}$ *in which* ψ_t, χ_t *are defined by the following relations:*

$$\psi_t = -\frac{\partial}{\partial x}\mathscr{H}_t(i_0^{t-1}, u_t^*(i_0^{t-1})) = -q_t^*\frac{\partial}{\partial x}g_t(x_t^*, u_t^*, i_{t-1}) + \sum_{i_t=1}^{s}\psi'_{t+1}\frac{\partial}{\partial x}f_t(x_t^* u_t^*, i_t)$$

$$+\sum_{i_t=1}^{s}\chi'_{t+1}\frac{\partial}{\partial x}p(i_{t-1}, i_t, u_t^*, x_t^*)\,q_t^* - q_t^*\lambda'_t\frac{\partial}{\partial x}\varphi_t(x_t^*, u_t^*\lambda_t^*, i_{t-1}, \quad 1\leq t\leq N-1,$$

$$\psi_N = 0,$$

$$q_t^*\cdot\chi_t = -q_t^* g_t(x_t^*, u_t^*, i_{t-1}) + \sum_{i_t=1}^{s}\chi_{t+1}(i_0^{t-1}, i_t)\,p(i_{t-1}, i_t, u_t^*, x_t^*)q_i^* \quad 1\leq t\leq N-1,$$

$$\psi_N = 0.$$

Applying Theorem 2 to Problem (3.3)—(3.5) we obtain the following maximum principle:

$$\mathscr{H}_t(i_0^{t-2}, i_{t-1}, u_t^*) = y_t^*\tilde{g}_t(i_{t-2}, z_t^*) - \sum_{j=1}^{s}\psi_{t+1}(i_0^{t-1}, j)u_t^*(i_0^{t-1})$$

$$-\sum_{j=1}^{s}\psi_{t+1}(i_0^{t-1}, j)p(i_{t-2}, i_{t-1}, z_t^*)\,y_t^* + y_t^*\lambda_t\varphi_t(i_{t-2}, z_t^*) = \min_u \mathscr{H}_t(i_0^{t-2}, i_{t-1}, u),$$

where $\mathscr{H}_t(i_0^{t-2}, i_{t-1}, u)$ is $\mathscr{H}_t(i_0^{t-2}, i_{t-1}, u_t^*)$ with replacing u_t^* by u.

Hence the principle becomes

$$-\sum_{j=1}^{s}\psi_{t+1}(i_0^{t-1}, j)u \to \min_u, \tag{3.7}$$

where the adjoint variables ψ_t, χ_t satisfy the relations

$$\psi_t(i_0^{t-2}, i_{t-1}) = -y_t^*\frac{\partial}{\partial z}\tilde{g}_t(i_{t-2}, z_t^*) \tag{3.8}$$

$$+\sum_{i=1}^{s}\chi_{t+1}(i_0^{t-1}, j)\frac{\partial}{\partial z}p(i_{t-2}, i_{t-1}, z_t^*)\,y_t^* - q_t^*\lambda_t\frac{\partial}{\partial z}\tilde{\varphi}_t(i_{t-2}, z_t^*);$$

$$y_t^*\chi_t = -y_t^*\tilde{g}_t(i_{t-2}, z_t^*) + \sum_{i_t=1}^{s}\chi_{t+1}(i_0^{t-1}, i_t)p(i_{t-2}, i_{t-1}, z_t^*)\,y_t^*; \tag{3.9}$$

$$y_t^*\lambda_t\tilde{\varphi}_t(i_{t-2}, z_t^*) = 0. \tag{3.10}$$

From (3.8) it follows that $\psi_{t+1}(i_0^{t-1}, j) = -\frac{\partial}{\partial u}\mathscr{H}_{t+1}(i_0^{t-1}, j, q_t^*, u_t^*)$, where

$$\mathscr{H}_{t+1}(i_0^{t-1}, j, q_t^*, u) = q_t^*\sum_{i=1}^{s}g_t(i_{t-1}, i, u)p(i_{t-1}, i, u). \tag{3.11}$$

$$-\sum_{k=1}^{s}\chi_{t+2}(i_0^{t-1}, j, k)p(i_{t-1}, j, u)q_t^* + q_t^*\lambda_{t+1}\varphi_t(i_{t-1}, u).$$

Then the maximum principle for Problem (3.1)—(3.2) is the form

$$(3.12)\qquad [\sum_{j=1}^{s} \frac{\partial}{\partial u} \mathscr{H}_{t+1}(i_0^{t-1}, j, q_t^*, u_t^*)]\, u_t^* = \min_u [\sum_{j=1}^{s} \frac{\partial}{\partial u} \mathscr{H}_{t+1}(i_0^{t-1}, j, q_t^*, u_t^*)]u.$$

In the case when $\varphi_t=0$ and for some r, $p(r, k, u_t^*)=0$, $k=1, 2, \ldots, s$, we get

$$q_t^*[\frac{\partial}{\partial u} \sum_{j=1}^{s} g_t(r, j, u_t^*)p(r, j, u_t^*)]\, u_t^* = \min_u q_t^*[\frac{\partial}{\partial u} \sum_{j=1}^{s} g_t(r, j, u_t^*)p(r, j, u_t^*)]u.$$

Obviously, we consider only the relations with $q_t^*>0$, that means

$$[\frac{\partial}{\partial u} \sum_{j=1}^{s} g_t(r, j, u_t^*)p(r, j, u_t^*]u_t^* = \min_u [\frac{\partial}{\partial u} \sum_{j=1}^{s} g_t(r, j, u_t^*)p(r, j, u_t^*)]u.$$

Therefore,

$$\frac{\partial}{\partial u}[\sum_{j=1}^{s} g_t(r, j, u_t^*)p(r, j, u_t^*)]=0.$$

If $\Sigma_{j=1}^{s} g_t(r, j, u)p(r, j, u)$ is convex in u, we obtain

$$\min_u \sum_{j=1}^{s} g_t(r, j, u)p(r, j, u) = \sum_{j=1}^{s} g_t(r, j, u_t^*)p(r, j, u_t^*).$$

Thus is the result given in [1].

Instead of a conclusion we summarize the results for Problem (3.1)—(3.2) as follows.

Theorem 3. (a) *If $p(i, j, u)$ is linear in u and Assumptions (B), (C) on the functions $g_t(i, j, u)$, $p(i, j, u)$, $\varphi_t(i, u)$ hold, then*

$$\min_u\{\sum_{i_t=1}^{s} g_t(i_{t-1}, i_t, u)p(i_{t-1}, i_t, u) - \sum_{i_t=1}^{s} \chi_{t+1}(i_0^{t-1}, i_t)p(i_{t-1}, i_t, u) + \lambda_t(i_0^{t-1})\varphi_t(i_{t-1}, u)\}$$

$$= \sum_{i_t=1}^{s} g_t(i_{t-1}, i_t, u_t^*)p(i_{t-1}, i_t, u_t^*) - \sum_{i_t=1}^{s} \chi_{t+1}(i_0^{t-1}, i_t)p(i_{t-1}, i, u_t^*) + \lambda_t(i_0^{t-1})\,\varphi_t(i_{t-1}, u_t^*),$$

where

$$\chi_t(i_0^{t-1}) = -\sum_{i=1}^{s} g_t(i_{t-1}, i, u_t^*)p(i_{t-1}, i, u_t^*) + \sum_{i=1}^{s} \chi_{t+1}(i_0^{t-1}, i)p(i_{t-1}, i, u_t^*),$$

$$\chi_N=0.$$

(b) *If $\Sigma_{j=1}^{s} g(i, j, u)p(i, j, u)$, $p(i, j, u)$ and $\varphi_t(i, u)$ are differentiable in u and their derivatives satisfy Assumption (A), then the maximum principle (3.12) holds for Problem (3.1)—(3.2) in which $\mathscr{H}_{t+1}$, χ_t and λ_t are given by (3.11), (3.6) and (3.10).*

REFERENCES

1. Pham T. Nhu. Feedback under Markov disturbances depending on control parameters. *Rev. Roum. Math. Pures Appl.*, **33**, 1988.
2. Pham T. Nhu. Optimal stabilization of stochastic discrete-time systems. *Rev. Roum. Math. Pures Appl.*, **24**, 1979, 129-151.

3. T. Morozan. Stochastic stability and control for discrete-time systems with jump Markov disturbances. *Rev. Roum. Math. Pures Appl.*, 24, 1979, 111-127.
4. V. I. Arkin, I. V. Evstigneev. Random models of control and economic dynamics. M., 1979. (Russian).
5. P. Mandl. A connection between controlled Markov chains and martingals. *Kybernetika*, 9. 1973, 237-241.
6. E. V. Dunkin, A. A. Yuskevich. Controlled Markov processes and their applications. M., 1975 (Russian).
7. I. I. Ghihman, A. V. Skorohod. Controlled stochastic processes. Kiev, 1977 (Russian).
8. A. D. Ioffe, V. M, Tihomirov. Theory of extremal problems. M., 1974 (Russian).

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