Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

CHARACTERISTIC CURVATURES IN THE ALMOST HERMITIAN GEOMETRY

G. STANILOV

Using some proper biquadratic functions we determine the curvature tensor in the almost Hermitian geometry. One of them — the so-called Kaehlerian defect, we generalize for higher dimensions and give some usefull relations for Kaelerian defects of different orders.

It is well known that in the Riemannian geometry the curvature tensor R can be determined by the sectional curvature of the manifold. This means the following: if $k(x, y)=R(x, y, x, y)$ then the value $R(x, y, z, u)$ can be determined by the values of the biquadratic function k [1]. Of course this assertion is also true for the curvature tensor in the almost Hermitian geometry. But it is not a proper result for this geometry. Because of this our main purpose is to find curvatures which are proper quantities for the almost Hermitian geometry and to expeess R by them.

Now we shall use essentially the paper [2] and also our papers [3] and [4].

1. Determination of the curvature tensor in the almost Hermitian geometry. According to [2] the curvature tensor R in the almost Hermitian geometry can be decomposed in four orthogonal components:

$$R=R_1+R_1^{\perp}+R_2^{\perp}+R_3^{\perp}, \tag{1}$$

where the components are geometrically determined. Then the determination of R is equivalently with the determination of its components.

In [3] we have defined the notion mutual curvature of two arbitrary tangent subspaces at a point of a Riemannian manifold and as a special case — curvature of arbitrary tangent subspace. This give us a possibility to introduce the notion Kaehlerian defect $\Delta_R(E^4)=\Delta_R(x, y)$ of every 4-dimensional holomorphical tangent subspace E^4 spanned by the orthonormal quadruple x, y, Jx, Jy. By definition we have

$$\Delta_R(E^4)=K_R(E^4)-K_{R^*}(E^4),$$

where

$$R^*(x, y, z, u)=R(x, y, Jz, Ju).$$

We have

$$\Delta_R(x, y)=R(x, y, x, y)+R(Jx, Jy, Jx, Jy)+R(x, Jy, x, Jy)$$
$$+R(Jx, y, Jx, y)-2R(x, y, Jx, Jy)-2R(x, Jy, y, Jx). \tag{2}$$

Now we can define the biquadratic function Δ_R by (2) for arbitrary x, y. Then for the component

$$R_1^{\perp}=(p_4+p_5+p_6)(R)$$

we have

Partially supported by the Committe of Science in Bulgaria under Contract № 46.

Lemma 1. *The equality*

$$16\,R_1^{\perp}(x, y, x, y)=3\Delta_R(x, y) \tag{3}$$

holds for arbitrary x, y.

We omit the long calculations. It follows

$$K_{R_1^{\perp}}(x\wedge y)=\frac{3}{16}\Delta_R(x, y) \tag{3'}$$

for a plane $\alpha=x\wedge y$ spanned by an orthonormal pair x, y. In (3') $K_{R_1^{\perp}}(x\wedge y)$ is the sectional curvature of α in respect to $R_1^{\perp}$. If α is an antiholomorphical plane, i. e. $\alpha\perp J\alpha=Jx\wedge Jy$ then $\Delta_R(x, y)$ is the Kaehlerian defect of $E^4=x\wedge y\wedge Jx\wedge Jy$.

In [4] we have considered also the difference of the mutual curvatures in respect to R and R^* of the planes $\alpha=x\wedge y$ and $J\alpha=Jx\wedge Jy$ when α is an antiholomorphical plane:

$$(K-K^*)_R(x\wedge y;\ Jx\wedge Jy)=R(x, Jy, x, Jy)+R(Jx, y, Jx, y)-2R(x, Jy, y, Jx). \tag{4}$$

Now we define the biquadratic function $(K-K^*)_R$ by (4) for arbitrary vectors x, y. Then for the components $R_2^{\perp}=p_7(R)$ we can prove the following

Lemma 2. *The equality*

$$R_2^{\perp}(x, y, x, y)=\frac{1}{8}\Delta_R(x, y)-\frac{1}{4}(K-K^*)_R(x\wedge y;\ Jx\wedge Jy) \tag{5}$$

holds for arbitrary x, y.

We remark that for x, y — orthonormal pair

$$K_{R_2^{\perp}}(x\wedge y)=R_2^{\perp}(x, y, x, y).$$

In respect to tensor R^* the sectional curvature K_{R^*} can be defined by

$$K_{R^*}(x\wedge y)=R(x, y, Jx, Jy), \tag{6}$$

where x, y is an orthonormal pair. But we can consider $R(x, y, Jx, Jy)$ also for arbitrary vectors x, y.

For the first component $R_1=(p_1+p_2+p_3)(R)$ by a long but not difficult calculation we can prove the following

Lemma 3. *The equality*

$$R_1(x, y, x, y)=\frac{3}{16}\Delta_R(x, y)+R(x, v, Jx, Jy)-\frac{1}{4}(K-K^*)_R(x\wedge y;\ Jx\wedge Jy) \tag{7}$$

holds for arbitrary vectors.

If x, y is an orthonormal pair of vectors then

$$K_{R_1}(x\wedge y)=R_1(x, y, x, y)$$

is the sectional curvature of $x\wedge y$ in respect to R_1.

The component $R_3^{\perp}=(p_8+p_9+p_{10})(R)$ is equal to $R-L_3R$ and can be determined by the biquadratic function

$$R_3^{\perp}(x, y, x, y)=K_{R_3^{\perp}}(x\wedge y).$$

Now we can formulate the main result.

Theorem 1. *The curvature tensor R in the almost Hermitian geometry can be determined by the following biquadratic functions:*

a) $\Delta_R(x, y)$;

b) $(K-K^*)_R(x\wedge y;\ Jx\wedge Jy)$;

c) $R(x, y, Jx, Jy)$;

d) $R_3^{\perp}(x, y, x, y)$.

More precisely:

1. $\Delta_R(x, y)$, $R(x, y, Jx, Jy)$, $(K-K^*)_R(x\wedge y;\ Jx\wedge Jy)$ *determine the components* R_1;
2. $\Delta_R(x, y)$ *determines the component* $R_1^{\perp}$;
3. $\Delta_R(x, y)$, $(K-K^*)_R(x\wedge y;\ Jx\wedge Jy)$ *determine the component* $R_2^{\perp}$;
4. $R_3^{\perp}(x, y, x, y)$ *determines the component* $R_3^{\perp}$.

From the theorem we can conclude the following

Corollary. *The curvature tensor R in the almost Hermitian geometry can be determined by the following curvatures:*

I. $\Delta_R(x, y)$ (*now* $x\wedge y$ *is an antiholomphic section*);

II. $(K-K^*)_R(x\wedge y;\ Jx\wedge Jy)$ (*now* $x\wedge y$ *is also an antiholomorphic section*);

III. $R(x, y, Jx, Jy)$ (*now* x, y *is an orthonormal pair of vectoı*);

IV. $R_3^{\perp}(x, y, x, y)$ (*now* x, y *is also an orthonormal pair*);

The functions I–IV *we call characteristic functions of the curvature tensor* R.

2. The Kaehlerian defect of higher dimensions. The equality (3′) give us a very important information. Namely the Kaehlerian defect $\Delta_R(x, y)$ is proportional to the antiholomorphic sectional curvature $R_1^{\perp}(x, y, x, y)$. This give us possibility to translate all ideas and results from our paper [3] related with sectional curvatures to the Kaehlerian defects.

We shall do only something. First we define Kaehlerian defect for holomorphical tangent subspace of dimension $2k>4$.

Let $l_1, \dots, l_n, Jl_1, \dots, Jl_n$ is an orthonormal base of the tangent space at a point of an arbitrary almost Hermitian manifold and let k be a fixed number among the numbers $2, 3, \dots n$. The curvature $K_{12\dots k}$ of the holomorphical tangent subspace spanned by the vectors $l_1, \dots, l_k, Jl_1, \dots, Jl_k$ is given by [3]

$$K_{12\dots k}=\sum_{1\le i<j\le k} R(l_i, l_j, l_i, l_j)+\sum_{1\le i<j\le k} R(Jl_i, Jl_j, Jl_i, Jl_j) + \sum_{i=1,\dots,k;\ j=1,\dots,k} R(l_i, Jl_j, l_i, Jl_j).$$

The curvature in respect to thenson R^* of the same subspace is

$$K^*_{12\dots k}=\sum_{1\le i<j\le k} R(l_i, l_j, Jl_i, Jl_j)+\sum_{1\le i<j\le k} R(Jl_i, Jl_j, l_i, l_j) + \sum_{i=1,\dots,k;\ j=1,\dots,k} R(l_i, Jl_j, Jl_i, -l_j).$$

Their difference

$$\Delta_{12\dots k}=K_{12\dots k}-K^*_{12\dots k} \tag{8}$$

we call Kaehlerian defect (in respect to R) of the subspace spanned by the vectors $l_1, \dots, l_k, Jl_1, \dots, Jl_k$. Using the fact that

$$\Delta_{ij}=\Delta(l_i, l_j) \tag{9}$$

and the expressions for $K_{12\ldots k}$ and $K^{*}_{12\ldots k}$ from (8) we get

(10) $$\Delta_{12\ldots k}=\sum_{1\leqq i<j\leqq k}\Delta_{ij}.$$

According to this we can write

(11) $$\Delta_{i_1 i_2\ldots i_k}=\sum_{1\leqq i_1\leqq i_s<i_\sigma\leqq n}\Delta_{i_s i_\sigma}.$$

The method for solving this system in respect to Kaehlerian defects (of order 2) is given in our paper [3]. Because of this we have the following

Theorem 2. *Let* $2\leqq k\leqq n-2$. *From* (11) *follows*

(12) $$\binom{k}{2}\Delta_{12}=\sum_{3\leqq p<q\leqq k+2}\Delta_{12\ldots\widehat{p}\ldots\widehat{q}\ldots k+2}+\binom{k-1}{2}\Delta_{34\ldots k+2}$$
$$-\frac{k-2}{2}\Big(\sum_{3\leqq q\leqq k+2}\Delta_{1\widehat{2}\ldots\widehat{q}\ldots k+2}+\sum_{3\leqq q\leqq k+2}\Delta_{\widehat{1}2\ldots\widehat{q}\ldots k+2}\Big),$$

that is the Kaehlerian defect of order 2 *can be expressed by the Kaehlerian defects of order k (the sign* $\widehat{}$ *denotes that the corresponding index must be omitted).*

REFERENCES

1. D. Gromoll, W. Klingenberg, W. Meyer. Riemannsche Geometrie im Grossen. Berlin, 1968.
2. F. Tricerri, L. Vanhecke. Curvature tensors on almost Hermitian manifolds. *Trans. Amer. Math. Soc.*, **267**, 1981, 365-398.
3. G. Stanilov. Eine Verallgemeinerung der Schnittkrümmung. *Arch. Math.*, **21**, 1970, 424-428.
4. G. Stanilov. Geometric Interpretation of some transformations of curvature-like tensors. *Tensor, N. S.*, **37**, 1982, 155-161.

Centre for Mathematics and Mechanics
Sofia 1090 P. O. Box 373

Received 9. 3. 1988