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ON THE TANGENTIAL OBLIQUE DERIVATIVE PROBLEM FOR A CLASS OF ELLIPTIC SECOND ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

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The tangential oblique derivative problem for an elliptic second order differential operator having a monotone type nonlinearity is studied and theorems concerning the existence, the unicity and the smoothness of the solution are shown.

1. Introduction. At the end of the sixties the theory of second order linear elliptic differential operators having Neumann type boundary data was developed. Since the first order boundary operator can be considered as a real nondegenerate smooth vector field l on $\partial\Omega$ two different cases appear:

(i) l is not tangential to $\partial\Omega$;

(ii) l is tangential to $\partial\Omega$ at the points of a subvariety $S\subset\partial\Omega$.

There are no difficulties in case (i) since then the well-known Agmon condition is fulfilled and the corresponding boundary value problem (b. v. p.) turns out to be of Fredholm type. On contrast with this the situation in case (ii) is rather complicated.

In the present work we shall investigate the b. v. p. (ii) in the nonlinear case. This non-classical problem could possibly be applied in geometry, physics etc. [3] which justifies the interest in it. Our study is based on the well-known paper of Egorov-Kondratiev [3] where the tangential oblique derivative problem is investigated in Sobolev spaces for linear elliptic equations. Here is considered the same problem (ii) but unlike [3] it is reffered to a class of elliptic second order p. d. e. having a nonlinear right-hand side $f(x, u)$. The results obtained by us in Hölder spaces are of the same type as these established in Sobolev spaces in [3].

As far as we know, this is the first work where existence, uniqueness and smoothness of the solutions of non-linear elliptic equations with boundary conditions (ii) are studied in Hölder or Sobolev spaces.

To prove our main results the following methods will be used: the maximum principle, barrier functions, Leray-Schauder's principle and elliptic regularization. We shall make use of the classical Hölder space $C^{k,\alpha}$, where k is an integer and $0<\alpha<1$.

In Section 2 we state the main problems and formulate the main results. Section 3 deals with the proof of Egorov-Kondratiev's result [3] in the linear case by means of the Hölder technique as well as with the proof of Theorem 1. In Section 4 Theorem 2 is proved.

A short communication for the results presented here was published in [5].

2. Statement of the problem and formulation of the main results. Let Ω be a bounded domain in R^n, $n\geq 2$, with smooth boundary and S be a smooth subvariety of $\partial\Omega$ of codimension 1 in $\partial\Omega$. When studying the tangential oblique derivative problem two different cases stand out for the smooth nondegenerate vector field l which is tangential to $\partial\Omega$ at the points of S and is transversal to S:

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1. l preserves its sign in a neighbourhood (ngbh) of S (the orientation with respect to the outer normal $\nu=(\nu^1, \nu^2, \dots \nu^n)$ is implied);

2. l changes its sign near S.

First we will consider case 1 i. e. the following b. v. p.

$$(1) \quad \sum_{ij=1}^{n} a^{ij}(x)\partial^2 u/\partial x_i\partial x_j+\sum_{i=1}^{n} b^i(x)\partial u/\partial x_i=f(x, u) \text{ in } \Omega \quad \partial u/\partial l+\sigma(x)u=\varphi(x) \text{ on } \partial\Omega,$$

where $\partial u/\partial l=\sum_{k=1}^{n}\sigma^k(x)\partial u/\partial x_k$,

$$(2) \qquad \sum_{ij=1}^{n} a^{ij}(x)\xi^i\xi^j\geqq\lambda\,|\,\xi\,|^2, \quad \lambda=\text{const}>0$$

for every $x\in\bar\Omega$, $\xi=(\xi^1, \xi^2, \dots, \xi^n)\in \mathrm{R}^n$, $a^{ij}=a^{ji}$. Moreover, we suppose that the vector field l is tangential to $\partial\Omega$ at the points of S, l is transversal to S and l preserves its sign, i. e.

$$(3) \qquad \sum_{k=1}^{n}\sigma^k\nu^k=0 \text{ on } S; \ \sigma\sum_{k=1}^{n}\sigma^k\nu^k>0 \text{ on } \partial\Omega\setminus S.$$

Unlike the classical Neumann problem, here we need additional smoothness for the coefficients of the equation and the boundary operator. More precisely, we assume that

$$(4) \qquad \begin{aligned} &a^{ij},\ b^i\in C^{1,\alpha}(\bar\Omega);\ f(x, u)\in C^{1,\alpha}(\bar\Omega\times\mathrm{R});\\ &\sigma^k,\ \sigma,\ \varphi\in C^{2,\alpha}(\bar\Omega);\ \partial\Omega,\ S\in C^{3,\alpha} \text{ and}\\ &f_u(x, u)\geqq 0 \text{ in } \bar\Omega\times\mathrm{R},\ |\,\sigma\,|\geqq\sigma_0=\text{const}>0 \text{ on } \partial\Omega.\end{aligned}$$

Thus we can formulate the following result.

Theorem 1. *Suppose Ω is a bounded domain in R^n and the coefficients of the equation and the boundary operator satisfy* (2)—(4). *Then b. v. p.* (1) *has a unique solution $u\in C^{2,\alpha}(\bar\Omega)\cap C^{3,\alpha}(K)$ for every compact $K\subset\bar\Omega$, $K\cap S=\varnothing$.*

Remark 1. The uniqueness result in Theorem 1 remains valid for the wider class of functions $C^2(\Omega)\cap C^1(\bar\Omega)$.

Remark 2. There are no difficulties to obtain the following regularity theorem. If we assume, in addition, that a^{ij}, $b^i\in C^{k,\alpha}(\bar\Omega)$, $k\geqq 1$ is an integer, $f\in C^{k,\alpha}(\bar\Omega\times\mathrm{R})$; $\partial\Omega$ $S\in C^{k+2,\alpha}$; σ^p, σ, $\varphi\in C^{k+1,\alpha}(\bar\Omega)$ then every $C^2(\Omega)\cap C^1(\bar\Omega)$ solution of b. v. p. (1) belongs to the class $C^{k+1,\alpha}(\bar\Omega)\cap C^{k+2,\alpha}(K)$ for every compact $K\subset\bar\Omega$, $K\cap S=\varnothing$.

In order to formulate the result in case 2 let us recall that for linear elliptic equations the corresponding b. v. p. posseses either a kernel or a cokernal of infinite dimension (see [3], [4]). To avoid this difficulty an extra boundary condition is prescribed on S (see [3]). This new effect arises also for nonlinear equations as will be illustrated in a simple example. More precisely, we shall examine a special b. v. p. for the Laplace equation in a ball i. e.

$$(5) \qquad \begin{aligned} &\Delta u=f(x, u) \text{ in } B=\{x\in\mathrm{R}^n;\ |\,x\,|<R\}\\ &u_{x_n}=u^0(x) \text{ on } \partial B,\ u=u^1(x') \text{ on } E=\partial B\cap\{x_n=0\},\end{aligned}$$

where $x'=(x_1, x_2, \dots, x_{n-1})$.

Theorem 2. *Suppose that $f\in C^{2,\alpha}(\bar B\times\mathrm{R})$, u^0, $u^1\in C^{3,\alpha}(\bar B)$ and $f_u(x, u)\geqq 0$ for $(x, u)\in\bar B\times\mathrm{R}$. Then b. v. p.* (5) *has a unique solution $u\in C^{3,\alpha}(\bar B)\cap C^{4,\alpha}(K)$ for every compact $K\subset\bar B$, $K\cap E=\varnothing$.*

Remark 3. The above theorem remains valid in a bounded domain $\Omega\subset\mathbb{R}^n$, $\partial\Omega\in C^{3,\alpha}$, provided that $\partial/\partial x_n$ is an operator tangential to $\partial\Omega$ at the points of the set $\{x_n=\text{const}\}\cap\partial\Omega$, and transversal to that set.

Remark 4. As in Theorem 1 we can formulate a similar regularity result for b. v. p. (5). If we assume that $f\in C^{k+1,\alpha}(\bar{B}\times\mathbb{R})$, $k\geqq 1$ is an integer, u^0, $u^1\in C^{k+1,\alpha}(\bar{B})$ then every $C^2(B)\cap C^1(\bar{B})$ solution of (5) belongs to the class $C^{k+2,\alpha}(\bar{B})\cap C^{k+3,\alpha}(K)$ for every compact $K\subset\bar{B}$, $K\cap E=\emptyset$.

3. Proof of Theorem 1. First we shall prove the result of Egorov—Kondratiev [3] in the linear case by means of the Hölder technique. For this purpose let us consider the following linear b. v. p.

$$(6)\qquad \begin{aligned} &L_1u=\sum_{ij=1}^{n} a^{ij}(x)\,\partial^2 u/\partial x_i\partial x_j+\sum_{i=1}^{n} b^i(x)\partial u/\partial x_i+c(x)u=f(x) \text{ in } \Omega\\ &\partial u/\partial l+\sigma(x)\,u=\varphi(x) \text{ on } \partial\Omega,\end{aligned}$$

where the operator L_1 and the vector field l satisfy (2) and (3).

Without loss of generality we assume that $\sigma>0$ and $(l,\nu)>0$ on $\partial\Omega\setminus S$. Then we have the following result.

Lemma 1. *Suppose that* a^{ij}, b^i, c, $f\in C^{1,\alpha}(\bar{\Omega})$, σ^k, σ, $\varphi\in C^{2,\alpha}(\bar{\Omega})$; $\partial\Omega$, $S\in C^{3,\alpha}$ *and* $\sigma\geqq\sigma_0>0$, $c(x)\leqq 0$. *Then under the assumptions* (2), (3) *the b. v. p.* (6) *has a unique solution* $u\in C^{2,\alpha}(\bar{\Omega})\cap C^{3,\alpha}(K)$ *for every compact* $K\subset\bar{\Omega}$, $K\cap S=\emptyset$.

Proof of Lemma 1. The uniqueness in Lemma 1 follows from the uniqueness proved in Theorem 1 and we omit it.

In order to prove the existence result we consider the regularized b. v. p.

$$(7)\qquad L_1u=f(x) \text{ in } \Omega,\ \partial u/\partial l+\varepsilon\partial u/\partial\nu+\sigma v=\varphi(x) \text{ on } \partial\Omega$$

for all sufficiently small positive ε. From th. 6.31 in [1] it follows that (7) has a unique solution $u^\varepsilon\in C^{3,\alpha}(\bar{\Omega})$.

Our main purpose is to prove $C^{2,\alpha}(\bar{\Omega})$ estimates for u^ε with a constant independent of ε. For convenience we will omit the index ε in u^ε and we will call the constants C_i, $i=1,2,\dots$, independent of ε and τ, constants under control.

3.1. Global estimates for u^ε. Suppose that the origin does not belong to $\bar{\Omega}$. We consider the auxiliary function $h=\pm u^\varepsilon+N_1(\exp(a\,|\,x\,|^2)-\exp(aR^2)-N)$, where R is the radius of a ball containing $\bar{\Omega}$ and a, N, N_1 are positive constants. We choose the constants a, N_1 sufficiently large, independent of ε and N, so that the inequality $L_1h>0$ holds in Ω. A simple computation shows that $\partial h/\partial l+\varepsilon\partial h/\partial\nu+\sigma h<0$ on $\partial\Omega$ when N is sufficiently large, N is independent of ε. Consequently h does not attain a positive maximum in $\bar{\Omega}$, i. e.

$$(8)\qquad \sup|\,u^\varepsilon\,|\leqq N_1(N+\exp(aR^2)).$$

From (8), the interior Schauder estimates (see th. 6.2 in [1]) and the local Schauder estimates up to the boundary (see th. 7.3 in [2]) we immediately obtain the estimate

$$(9)\qquad \|\,u^\varepsilon\,\|_{C^{3,\alpha}(K)}\leqq C_1(K)$$

for every compact $K\subset\bar{\Omega}$, $K\cap S=\emptyset$, where the constant C_1 is under control.

3.2. Local Schauder estimates near S. There exists a finite covering D_k of S, $k=1,2,\dots,m$, $D_k\subset\bar{\Omega}$ and a positive number $\varepsilon_0>0$ such that the family of $C^{2,\alpha}$ vector fields $l+\varepsilon\nu$, $0\leqq\varepsilon\leqq\varepsilon_0$ can be straightened in D_k by means of a diffeomorphism smoothly depending on ε and having $C^{2,\alpha}(\bar{D}_k)$ norms under control. It is important to note that the integral curves of $-(l+\varepsilon\nu)$, $0\leq\varepsilon\leq\varepsilon_0$, starting from the points of $\Gamma_k=\partial\Omega\cap\partial D_k$ intersect T_k where $T_k\cup\Gamma_k=\partial D_k$, $T_k\cap\Gamma_k=\emptyset$, $k=1, 2, \dots m$ (see [3]). Let the functions ζ^k be partition of unity in a ngbh of S subordinated to D_k, i. e. $\zeta^k\in C^\infty(\bar{D}_k)$, $\Sigma\zeta^k\equiv 1$ near S and $\zeta^k\equiv 0$ in a ngbh of T_k. As was shown in [3] the diffeomorphism mentioned above transforms the vector field $l+\varepsilon\nu$ into $(1, 0, \dots 0)$. Let $\tilde{\Gamma}_k$ be the translation of Γ_k in $-x_1$ direction at a distance $\delta>0$. For convenience we will use the same notations for the transformed domain, as well as for the transformed equation. Further on, any lower index i will denote differentiation with respect to x_i whereas the upper index i is a summation index and summation convention is understood, i. e. $a^{ij}u_{ij}=\Sigma_{ij=1}^n a^{ij}u_{x_ix_j}$.

A simple computation shows that $u^k=\zeta^k u$ satisfies the following b. v. p. in D_k

$$L_1u^k=f\zeta^k+a^{ij}u\zeta^k_{ij}+2a^{ij}\zeta^k_i u_j+b^i u\zeta^k_i \text{ in } D_k$$

$$u^k_1+\sigma u^k=\varphi\zeta^k+u\zeta^k_1 \text{ on } \Gamma_k,\ u^k=0 \text{ on } T_k$$

and the function $v^k=u^k_1$ satisfies the Dirichlet problem

$$L_1v^k=F^k(x) \text{ in } D_k$$

$$v^k=\varphi\zeta^k+u\zeta^k_1-\sigma u\zeta^k\equiv\psi^k \text{ on } \Gamma_k,\ v^k=0 \text{ on } T_k,$$

where $F^k=a^{ij}u\zeta^k_{1ij}+a^{ij}u_1\zeta^k_{ij}+2a^{ij}u_j\zeta^k_{1i}+2a^{ij}\zeta^k_i u_{1j}-a^{ij}_1\zeta^k u_{ij}+b^i\zeta^k_i u_1+b^i u\zeta^k_{1i}-b^i_1\zeta^k u_i-c_1\zeta^k u+f_1\zeta^k+f\zeta^k_1$.

From the Schauder estimates for the Dirichlet problem (10) (see th. 6.6 in [1]) we obtain the estimates

$$\text{(11)}\quad \|v^k\|_{C^{2,\alpha}(\bar{D}_k)}\leq C_2(\|F^k\|_{C^\alpha(\bar{D}_k)}+\|\psi^k\|_{C^{2,\alpha}(\bar{D}_k)}+\|v^k\|_{C^0(\bar{D}_k)})$$

$$\leq C_3(\|u\|_{C^{2,\alpha}(\bar{D}_k)}+1)\leq mC_3\sum_{j=1}^n\|u^j\|_{C^{2,\alpha}(\bar{D}_j)}+C_3\|(1-\sum_{j=1}^m\zeta^j)u\|_{C^{2,\alpha}(\bar{D}_k)}$$

$$\leq mC_3\sum_{j=1}^m\|u^j\|_{C^{2,\alpha}(\bar{D}_j)}+C_4.$$

Let $\delta=1/(8m^2C_3)$ and E_k be the bounded domain surrounded by Γ_k, $\tilde{\Gamma}_k$ and the corresponding parts of T_k. Note that $\tilde{\Gamma}_k$, E_k depend on δ which is fixed and under control.

Since $u^k(x)=\int_{y_1}^{x_1}v^k(s,x'')\,ds+u(y_1,x'')$ for $x=(x_1,x'')\in E_k$, $(y_1,x'')\in\tilde{\Gamma}_k$ we have the estimates

$$\text{(12)}\quad \|u^k\|_{C^{2,\alpha}(\bar{E}_k)}\leq\delta\|v^k\|_{C^{2,\alpha}(\bar{E}_k)}+\|v^k\|_{C^2(\bar{E}_k)}+C_5$$

$$\leq 2\delta\|v^k\|_{C^{2,\alpha}(\bar{E}_k)}+\frac{1}{2}\|u^k\|_{C^{2,\alpha}(\bar{E}_k)}+C_6.$$

In (12) we twice used the standard interpolation inequality (see Lemma 6.35 in [1]) as well as (9). From (11) and (12) we immediately obtain the estimates

$$\|u^k\|_{C^{2,\alpha}(\bar{E}_k)}\leq 4\delta mC_3\sum_{j=1}^m\|u^j\|_{C^{2,\alpha}(\bar{D}_j)}+C_7 \quad \text{for} \quad k=1, 2, \dots m, \quad \text{i. e.}$$

$$\sum_{k=1}^{m} \| u^k \|_{C^{2,\alpha}(\overline{E}_k)} \leq \frac{1}{2} \sum_{j=1}^{m} \| u^j \|_{C^{2,\alpha}(\overline{D}_j)} + mC_7 \leq \frac{1}{2} \sum_{j=1}^{m} \| u^j \|_{C^{2,\alpha}(\overline{E}_j)} + C_8.$$

Consequently we have

$$\sum_{k=1}^{m} \| u^k \|_{C^{2,\alpha}(\overline{E}_k)} \leq 2C_8 \quad \text{and}$$

$$\| u \|_{C^{2,\alpha}(\cup \overline{E}_j)} \leq m \sum_{j=1}^{m} \| u^j \|_{C^{2,\alpha}(\overline{E}_j)} + \| (1 - \sum_{j=1}^{m} \zeta^j) u \|_{C^{2,\alpha}(\cup \overline{E}_j)} \leq C_9.$$

The above estimate as well as (9) give us the estimate

(13) $$\| u^\varepsilon \|_{C^{2,\alpha}(\overline{\Omega})} \leq C_{10},$$

where the constant C_{10} is under control.

By means of (13), Cantor diagonal principle and Ascoli-Arzela theorem we obtain a subsequence $\varepsilon_i \to 0$ for which $u^{\varepsilon_i}(x) \to u(x) \in C^{2,\alpha}(\overline{\Omega})$ in the norm of $C^2(\overline{\Omega})$. Letting $\varepsilon_i \to 0$ in (7) we obtain that $u(x)$ is the desired solution of (6).

Applying the already proved Lemma 1 and Leray-Schauder's theorem (see th. 10.4 in [1]) we will prove Theorem 1.

P r o o f o f T h e o r e m 1. (i) Existence. Let us consider the Banach space $C^{1,\alpha}(\overline{\Omega})$ and the operator T which for every $v \in C^{1,\alpha}(\overline{\Omega})$ is defined as the unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ of the linear b. v. p.

$$L_1 u = f(x, v) \text{ in } \Omega; \quad \partial u/\partial l + \sigma(x) u = \varphi(x) \text{ on } \partial\Omega.$$

It is easy to check that the operator T is a compact one from $C^{1,\alpha}(\overline{\Omega})$ into $C^{1,\alpha}(\overline{\Omega})$. In order to apply Leray-Schauder's theorem for the operator T we must prove the a priori estimate

(14) $$\| u \|_{C^{1,\alpha}(\overline{\Omega})} \leq C_{11}$$

with a constant C_{11} independent of $\tau \in [0, 1]$ and u for every $C^{2,\alpha}(\overline{\Omega})$ solution $u(x)$ of the b. v. p.

$$L_1 u = \tau f(x, u) \text{ in } \Omega; \quad \partial u/\partial l + \sigma(x) u = \tau\varphi(x) \text{ on } \partial\Omega.$$

The proof of (14) will be carried on in several steps.

3.3. G l o b a l a p r i o r i e s t i m a t e f o r u. We introduce the operator

$$L_2 = \sum_{ij=1}^{n} a^{ij}(x) \partial^2/\partial x_i \partial x_j + \sum_{i=1}^{n} b^i(x) \partial/\partial x_i - c(x),$$

where $c(x) = \tau \int_0^1 f_u(x, su)\, ds \geq 0$ and for which the equality $L_2 u = \tau f(x, 0)$ holds in Ω. Further on, we will use the notations stated in the linear case.

For the auxiliary function $h = u + N_1(\exp(a|x|^2) - \exp(aR^2) - N)$ we have the inequality $L_2 h > 0$ in Ω when a, N_1 are sufficiently large, independent of τ and N. When N is sufficiently large the inequality $\partial h/\partial l + \sigma(x) h < 0$ holds on $\partial\Omega$ i. e. h does not attain a positive maximum in $\overline{\Omega}$. Consequently $u \leq N_1(N + \exp(aR^2))$ in $\overline{\Omega}$. Repeating the above procedure for $-u$ we obtain the estimate

(15) $$\sup_{\overline{\Omega}} |u| \leq C_{12}$$

with a constant C_{12} under control.

By means of the standard technique for the oblique derivative problem for quasilinear elliptic equations and using (15) (see for instance [6, 7]) we obtain the estimate

$$\sup_K |Du| \leq C_{13} \tag{16}$$

for every compact $K \subset \overline{\Omega}$, $K \cap S = \emptyset$ with a constant C_{13} depending on K which is under control. Now from (15), (16), the interior Schauder estimates and the local estimates up to the boundary, as in the linear case, we have the estimate

$$\| u \|_{C^{3,\alpha}(K)} \leq C_{14} \tag{17}$$

for every compact $K \subset \overline{\Omega}$, $K \cap S = \emptyset$ and a constant C_{14} depending on K which is under control.

3.4. Local a priori estimates near S. In order to prove $C^{2,\alpha}(\overline{\Omega})$ a priori estimates in a ngbh of S we straighten the vector field l in ngbhs D_k, $k=1, 2, \ldots m$ of S. Using the notations in Lemma 1 we obtain that $v^k = (\zeta^k u)_{x_1}$ satisfies the b. v. p.

$$a^{ij}(x) v^k_{ij} + b^i(x) v^k_i = F^k(x) \quad \text{in } D_k$$

$$v^k = \tau\varphi\zeta^k + u\zeta^k_1 - \sigma u \zeta^k \quad \text{on } \Gamma_k; \ v^k = 0 \text{ on } T_k,$$

where

$$F^k = a^{ij} u \zeta^k_{1ij} + a^{ij} u_1 \zeta^k_{ij} + 2a^{ij} u_j \zeta^k_{1i} + 2a^{ij} \zeta^k_i u_{1j} - a^{ij}_1 \zeta^k u_{ij}$$
$$+ b^i \zeta^k_i u_1 + b^i u \zeta^k_{1i} - b^i_1 \zeta^k u_i + \zeta^k f_1 + \zeta^k f_u u_1 + f \zeta^k_1 .$$

We will repeat the procedure in the linear case, the only difference being the appli cation of the additional estimate

$$\| \zeta^k f_u(x, u) u_1 \|_{C^\alpha(\overline{D}_k)} \leq C_{15} \| u \|_{C^{2,\alpha}(\overline{D}_k)}. \tag{18}$$

Since $\| \cdot \|_{C^\alpha(\overline{D}_k)} \leq C'_{15} \| \cdot \|_{C^1(\overline{D}_k)}$ we reduce the proof of (18) to the proof of the inequality $\| \zeta^k f_{uu}(x, u) u_1 u_j \|_{C^0(\overline{D}_k)} \leq C''_{15} \| u \|_{C^2(\overline{D}_k)}$. It can be easily seen that the only difficulty in evaluating $\| \zeta^k f_{uu}(x, u) u_j u_1 \|_{C^0(\overline{D}_k)}$ is due to the term $|Du|^2$. The inequality

$$\sup_{\overline{D}_k} |Du|^2 \leq C_{16} \| u \|_{C^2(\overline{D}_k)} \tag{19}$$

is a special case of the Giagliardo—Nirenberg estimate (see [8]) $\sup_{\overline{\Omega}} |Du|^2 \leq \text{const} \| u \|_{C^0(\overline{\Omega})} \| u \|_{C^2(\overline{\Omega})}$ which holds for every $C^2(\overline{\Omega})$ smooth function and every bounded domain with C^2 smooth boundary. Another proof follows from the well-known result for the gradient estimate of the nonnegative functions.

Repeating the same procedure as in the linear case we have the estimate $\| u \|_{C^{2,\alpha}(\cup \overline{E}_j)} \leq C_{17}$ with a constant C_{17} under control where $\cup \overline{E}_j$ is a compact ngbh of S. The above estimate and (17) give us the desired result $\| u \|_{C^{2,\alpha}(\overline{\Omega})} \leq C_{18}$.

Thus (14) is proved and so is the existence part of Theorem 1.

(ii) Uniqueness. We will prove the uniqueness of the solution of b. v. p. (1) in the wider class of solutions $C^2(\Omega) \cap C^1(\overline{\Omega})$. Suppose $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ are solutions of (1). Further on, we will use the notations in Lemma 1. Let $\varepsilon > 0$ be an arbitrary fixed number and $w = u - v$. Then w satisfies the b. v. p.

$$L_3 w = \sum_{ij=1}^{n} a^{ij}(x)\, w_{x_i x_j} + \sum_{i=1}^{n} b^i(x)\, w_{x_i} - \tilde{f}(x)\, w = 0 \quad \text{in } \Omega$$

$$\partial w / \partial l + \sigma(x) w = 0 \quad \text{on } \partial\Omega,$$

where $\tilde{f} = \int_0^1 f_u\,(x,\ v + s(u - v))\, ds \geq 0$ in Ω. For the auxiliary function $h = \pm w + \varepsilon(\exp(a|x|^2) - \exp(aR^2) - N)$ the following inequalities hold: $L_3 h > 0$ in Ω when a is sufficiently large and independent of ε, N and $\partial h/\partial l + \sigma(x) h < 0$ on $\partial\Omega$ when N is sufficiently large and independent of ε. From the interior and boundary maximum principle it follows that h does not attain a positive maximum in $\bar{\Omega}$, i. e. $h \leq 0$ in $\bar{\Omega}$. Consequently $\sup_{\bar{\Omega}} |w| \leq \varepsilon(\exp(aR^2) + N)$ and since $\varepsilon > 0$ was chosen to be arbitrary it follows that $w \equiv 0$ in Ω, i. e. $u \equiv v$ in Ω.

4. Proof of Theorem 2. First we will prove the uniqueness of b. v. p. (5) in the wider class of solutions $C^2(B) \cap C^1(\bar{B})$.

(i) Uniqueness. Suppose that $u_1, u_2 \in C^2(B) \cap C^1(\bar{B})$ are two different solutions of (5). Then $u = u_1 - u_2$ satisfies the b. v. p.

$$L_4 u = \Delta u - \tilde{f}(x)\, u = 0 \quad \text{in } B \quad u_{x_n} = 0 \quad \text{on } \partial B,\ u = 0 \quad \text{on } E,$$

where $\tilde{f}(x) = \int_0^1 f_u\,(x,\ u_2 + s(u_1 - u_2))\, ds \geq 0$ in $\bar{B}$. Let $\varepsilon > 0$ be an arbitrary positive constant. For the auxiliary function $h = u + \varepsilon(2|x'|^2 - x_n^2 - 2R^2)$ we obtain that $L_4 h > 0$ in B, $h_{x_n} = -2\varepsilon x_n$ on ∂B, i e. h does not attain a positive maximum in B and a maximum on $\partial B \setminus E$. Since $h = 0$ on E it follows that $u \leq 3\varepsilon R^2$ in $\bar{B}$.

Analogously, by means of the auxiliary function $h_1 = u - \varepsilon(2|x'|^2 - x_n^2 - 2R^2)$ we obtain that h_1 does not attain a negative maximum in $\bar{B}$, i. e. $u \geq -3\varepsilon R^2$ in $\bar{B}$. Consequently $|u| \leq 3\varepsilon R^2$ in $\bar{B}$ and since ε is chosen to be arbitrary it follows that $u \equiv 0$ in $\bar{B}$, i. e. $u_1 \equiv u_2$.

(ii) Existence. Before we prove the existence part of Theorem 2 we need the corresponding linear variant of Theorem 2 in Hölder spaces.

Lemma 2. *Suppose that* $u^0, u^1 \in C^{3,\alpha}(\bar{B})$, $f \in C^{2,\alpha}(\bar{B})$, $0 < \alpha < 1$. *Then the b.v.p.*

(20)
$$\Delta u = f(x) \ \text{in } B$$
$$u_{x_n} = u^0(x) \ \text{on } \partial B,\ u = u^1(x') \ \text{on } E$$

has a unique solution $u \in C^{3,\alpha}(\bar{B}) \cap C^{4,\alpha}(K)$ *for every compact* $K \subset \bar{B}$, $K \cap E = \emptyset$.

Sketch of the proof of Lemma 2: We consider the b. v. ps.

(21)
$$\Delta v = f_{x_k}(x) \ \text{in } B \quad v = u^0(x) \ \text{on } \partial B$$

(22)
$$\Delta' w = f(x', 0) - v_{x_n}(x', 0) \ \text{in } T = B \cap \{x_n = 0\}$$
$$w = u^1(x') \ \text{on } E = \partial B \cap \{x_n = 0\},$$

where $\Delta' w = \Sigma_{i=1}^{n-1} w_{x_i x_i}$. From the Schauder theory (see corollary 6.9 in [1]) b. v. p. (21) has a unique solution $v \in C^{3,\alpha}(\bar{B})$ and b. v. p. (22) has a unique solution $w \in C^{3,\alpha}(\bar{T})$. It is easy to check that the function $u = \int_0^{x_n} v(x', s) ds + w(x') \in C^{3,\alpha}(\bar{B})$ is a solution of (20). Thus Lemma 2 is proved.

By means of the already proved Lemma 2 and Leray-Schauder's theorem we will prove Theorem 2. Let us consider the Banach space $C^{2,\alpha}(\bar{B})$ and the operator G which is defined for every $g \in C^{2,\alpha}(\bar{B})$ as the unique solution $u \in C^{3,\alpha^2}(\bar{B})$ of the linear b. v. p.

$$\Delta u = f(x, g) \text{ in } B\ u_{x_n} = u^0 \text{ on } \partial B,\ u = u^1(x') \text{ on } E.$$

It is easy to check that G is a compact operator from $C^{2,\alpha}(\bar{B})$ into itself. In order to apply Leray—Schauder theorem we must prove the following estimate

$$\|u\|_{C^{2,\alpha}(\bar{B})} \leqq M_1 \tag{23}$$

with a constant M_1 independent of $\tau \in [0, 1]$ and u, for every $C^{3,\alpha^2}(\bar{B})$ solution u of the b. v. p.

$$\Delta u = \tau f(x, u) \text{ in } B,\ u_{x_n} = \tau u^0(x) \text{ on } \partial B,\ u = \tau u^1(x') \text{ on } E. \tag{24}$$

We will prove (23) in several subsequent steps.

4.1. Global a priori estimates for u and u_{x_n}. We consider the operator

$$L_5 = \Delta - \tau F(x),\quad F(x) = \int_0^1 f_u(x, su)\,ds \geqq 0$$

in $\bar{B}$ which satisfies the equality $L_5 u = \tau f(x, 0)$ in B. For the auxiliary function $h_2 = \pm u \mp \tau x_n u^0(x', 0) + N(2|x'|^2 - x_n^2 - 2R^2) \mp \tau u^1(x') - \tau \sup_{\bar{B}} |x_n u^0(x', 0)| - \tau \sup_{\bar{B}} |u^1(x')|$ we have the estimate $L_5 h_2 > 0$ in B when N is sufficiently large, independent of τ, and the equality $\partial h_2/\partial x_n = -x_n(2N \mp \tau \Phi(x))$ on ∂B, where $\Phi(x) = \int_0^1 u^0_{x_n}(x', sx_n)ds$. From the maximum principle it follows that h_2 does not attain a positive maximum in B or on $\partial B \setminus E$. Since $h_2 \leqq 0$ on E it follows that $h_2 \leqq 0$ in $\bar{B}$, i. e. $\sup_{\bar{B}} |u| \leqq M_2$ with a constant M_2 under control.

Differentiating (24) with respect to x_n and replacing $u_{x_n} = v$ we obtain that v is a solution of the b. v. p.

$$\Delta v = \tau f_{x_n}(x, u) + \tau f_u(x, u) v \text{ in } B,\ v = \tau u^0 \text{ on } \partial B. \tag{25}$$

For the auxiliary function $h_3 = \pm v \mp \tau u^0 - \tau \sup |u^0| + N(2|x'|^2 - x_n^2 - 2R^2)$ and the operator $L_6 = \Delta - \tau f_u(x, u)$ we have the inequality $L_6 h_3 \geqq \pm \tau f_{x_n}(x, u) \mp \tau L_6 u^0 + N(4n - 4 - 2) > 0$ when N is sufficiently large. Hence h_3 does not attain a positive maximum in B and since $h_3 \leqq 0$ on ∂B it follows that $h_3 \leqq 0$ in $\bar{B}$, i. e.

$$\sup_{\bar{B}} |u_{x_n}| = \sup_{\bar{B}} |v| \leqq M_3 \tag{26}$$

with a constant M_3 under control.

4.2. Global a priori estimate for $u_{x_n x_n}$. First we will estimate Dv, $v = u_{x_n}$ on the boundary. Let us consider the auxiliary function $h_4 = \pm v \mp \tau u^0(x) + N(|x|^2 - R^2)$. Then $L_6 h_4 = \pm \tau f_{x_n}(x, u) \mp \tau(\Delta u^0 - \tau f_u(x, u) u^0) + 2nN > 0$ when N is sufficiently large. Since $h_4 = 0$ on ∂B, from the maximum principle it follows that h_4 attains its maximum on ∂B. Consequently, $\partial h_4/\partial \nu_1 \leqq 0$ on ∂B, where ν_1 is the unit inner normal to ∂B, i. e. $|\partial v/\partial \nu_1| \leqq M_4'$ on ∂B. After differentiating the boundary condition $v = \tau u^0(x)$ in the tangential directions on ∂B we obtain the estimate

$$\sup_{\partial B} |Dv| = \sup_{\partial B} |Du_{x_n}| \leqq M_4. \tag{27}$$

Let us differentiate (25) with respect to x_n. The function v_{x_n} satisfies the equation

$$\Delta v_n = \tau f_{nn}(x, u) + 2\tau f_{nu}(x, u) v + \tau f_{uu}(x, u) v^2 + \tau f_u(x, u) v_n \text{ in } B.$$

For the function $h_5 = v_n^2 + v^2 + N(|x|^2 - R^2)$ we have the estimates

$$L_6 h_5 = 2\sum_{k=1}^{n} v_{nk}^2 + 2v_n \Delta v_n + 2\sum_{k=1}^{n} v_k^2 + 2v\,\Delta v - \tau f_u v_n^2 - \tau f_u v^2 - \tau f_u (|x|^2 - R^2) + 2nN$$

$$\geq \tau f_u v_n^2 + 2v_n(\tau f_{nn} + 2\tau f_{nu} v + \tau f_{uu} v^2) + 2\sum_{k=1}^{n} v_k^2 + 2v(\tau f_n + \tau f_u v) + 2nN > 0$$

in B when N is sufficiently large since $|u|$ and $|v|$ are under control. From the maximum principle it follows that h_5 does not attain a positive maximum in B. Thus from (27) we obtain the estimate

$$\sup_{\overline{B}} |v_{x_n}| = \sup_{\overline{B}} |u_{x_n x_n}| \leqq M_5. \tag{28}$$

5. Global a priori estimates for $|Du|$. First we will estimate $|Du|$ in $\overline{T}$. For this purpose let us change the variable $z = w - \tau u^1(x') = u(x', 0) - \tau u^1(x', 0)$. It is easy to check that z satisfies the b. v. p.

$$\begin{aligned} &\Delta' z = \tau f(x', 0, u', (x', 0)) - u_{x_n x_n}(x', 0) - \tau \Delta' u^1 \ \text{ in } T \\ &z = 0 \ \text{ on } E = \partial T. \end{aligned} \tag{29}$$

From the Schauder estimates for the solutions of the Poisson equation (see th. 4.3 and formula 4.10 in [1]) and (28) it follows that $\sup_{\overline{T}} |D_{x'} u(x', 0)| = \sup_{\overline{T}} |w| \leqq M_6'$.

The above estimate as well as (26) give us $\sup_{\overline{T}} |Du| \leqq M_6$.

Now we will estimate $|Du|$ in $B_+ = B \cap \{x_n > 0\}$. For the auxiliary function $h_6 = \Sigma_{k=1}^{n-1} u_k^2 e^{-x_n} + Nu^2 + N_1(|x|^2 - R^2) - N_2 x_n$ and for sufficiently large positive constants N, N_1 independent of N_2 we have the inequality $L_6 h_6 > 0$ in B_+. When N_2 is sufficiently large from (27) we obtain the estimate $\partial h_6/\partial x_n = \Sigma_{k=1}^{n-1}(2u_k u_{kn} - u_k^2) e^{-x_n} + 2uu_n + 2N_1 x_n - N_2 < 0$ on $\partial B \cap \{x_n > 0\}$. Thus from the maximum principle we have that h_6 does not attain a positive maximum in B_+ or on $\partial B_+ \cup \{x_n > 0\}$, i. e. either h_6 attains a nonpositive maximum in $B_+ \cup (\partial B \cap \{x_n > 0\})$ or h_6 attains its maximum on $\overline{T}$. Since $|Du|$ is already estimated in $\overline{T}$ we obtain an estimate for $|Du|$ in $\overline{B}_+$. Analogously, we have a gradient estimate for u in $B_- = B \cap \{x_n < 0\}$, i. e.

$$\sup_{\overline{B}} |Du| \leqq M_7. \tag{30}$$

From (30) and the global Schauder estimates for the solutions v, w (or z) of the b. v. p. (25), (29) it follows that $\|v\|_{C^{2,\alpha}(\overline{B})} \leqq M_8$, $\|w\|_{C^{2,\alpha}(\overline{T})} \leqq M_8$. Finally, from the identity $u(x) = \int_0^{x_n} v(x', s)\,ds + w(x')$ we obtain the estimate

$$\|u\|_{C^{2,\alpha}(\overline{B})} \leqq M_9$$

with a constant M_9 under control.

Thus (23) is proved and so is Theorem 2.

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